Math. Z. 214, 179–216 (1993)

Mathematische Zeitschrift

Truncated symmetric products and configuration spaces

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Received 20 November 1991; in final form 10 August 1992

0 Introduction

The spaces studied in this article are all related to the symmetric products $SP^n(Y)=Y^n/\mathcal{S}_n$ of a space Y. The symmetric group $\mathcal{S}_n$ acts on the cartesian product Y^n by permuting coordinates, and a point in $SP^n(Y)$ will be written multiplicatively $\langle y_1,\dots,y_n\rangle = y_1\dots y_n$. If a basepoint $*$ in Y is given, one has inclusions $SP^n(Y)\to SP^{n+1}(Y)$ identifying $y_1\dots y_n$ with $y_1\dots y_n *$. The union over all these spaces is the infinite symmetric product $SP^\infty(Y)$. By the fundamental result about infinite symmetric products, obtained in [DT], for Y connected and of the homotopy type of a CW-complex $SP^\infty(Y)$ is a product of Eilenberg-MacLane-spaces,

(1) $$SP^\infty(Y)=\prod_{i>0} K(H_i(Y;\mathbb{Z}),i).$$

In Part I we study the truncated symmetric product $TP^n(Y)$; this is the quotient of $SP^n(Y)$ by the relation $y^2=*=1$ for all $y\in Y$. The union of all the $TP^n(Y)$ is the infinite truncated product $TP^\infty(Y)$, the topological vector space over $\mathbb{F}_2$ generated by the points of Y modulo the subspace generated by $*$. For Y connected it is also a product of Eilenberg-MacLane-spaces,

(2) $$TP^\infty(Y)=\prod_{i>0} K(H_i(Y;\mathbb{Z}/2\mathbb{Z}),i).$$

The mod(2) homology of $TP^n(Y)$ was determined in [LM] as a functor of the mod(2) homology of Y. Furthermore, there are inclusions $H_*(TP^n(Y);\mathbb{F}_2)\to H_*(TP^{n+1}(Y);\mathbb{F}_2)$ whose image we describe using a bigrading of the homology of the limit $TP^\infty(Y)$. The rational or mod(p) homology for $p>2$ is more complicated, see 4.1 below.

Research partially supported by the NSF and the Sonderforschungsbereich 170, Göttingen

In Part II we study the configuration spaces $C^n(Y)$, the subspace of $SP^n(Y)$ consisting of all $\langle y_1, \ldots, y_n\rangle$ with $y_i \neq y_j$ when $i \neq j$. Recently, as in [Seg, MD, A-J], there has been renewed interest in these spaces and more particularly the associated deleted symmetric products

$$C^k(X) = \{\langle x_1, \ldots, x_k\rangle \mid x_i \neq x_j \text{ for } i \neq j\} \subset SP^k(X).$$

Somewhat before that [C-T1, C-T2, C-T3], there were numerous applications indicated, as well as the work of D. Anderson and P. Trauber reported on in those references which gave ample evidence, already of the importance of these spaces. For some particular manifolds the homology of these configuration spaces was computed in [A, B-C, C, V].

To amalgamate these spaces one introduces labels for the y_i in some other space, e.g. a sphere S^L; a point y with label s is cancelled from a configuration if $y = *$ or $s = \infty \in S^L$. The result is an infinite dimensional space $C(Y; S^L)$. When Y is a compact manifold of dimension m, a basic result in [MD, B1], identifies $C(Y; S^L)$ as the space of sections in a S^{m+L}-bundle over Y. Furthermore, it stably splits into a bouquet of filtration quotients $D_k(Y; S^L) = C_k(Y; S^L)/C_{k-1}(Y; S^L)$, [B2, CT1], and its homology is the homology of a product of loop spaces of spheres [BCT],

$$C(Y; S^L \simeq_s V D_k(Y; S^L), \tag{3}$$

$$H_*(C(Y; S^L); \mathbb{F}) \cong \bigotimes_0^m \otimes^{\beta_q} H_*(\Omega^{m-q} S^{m+L}; \mathbb{F}), \tag{4}$$

where β_q is the q^{th} $\mathbb{F}$-Betti number of Y. (Here $\operatorname{char}(\mathbb{F}) = 2$ or $\operatorname{char}(\mathbb{F}) \neq 2$ and $\dim(Y) + L \cong 1 \bmod (2)$.)

Our point of view in Part I is that $C^n(Y) = TP^n(Y) - TP^{n-2}(Y)$. In Part II the configuration space occurs as the base of an mL-dimensional vector bundle whose Thom space is $D_n(Y; S^L)$. Thus the quotients $TP^n(Y)$ of $SP^n(Y)$, and the subspaces $C^n(Y)$ of $SP^n(Y)$ are homologically dual to each other.

Theorem 3.1 *Let $\mathbb{F}$ be a field, then*
(a) $H^{2nk-r}(DP^k(M^{2n} - *); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-1}(M); \mathbb{F})$.
(b) $H^{2nk-r}(DP^k(M^{2n}); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

Theorem 3.2 *Suppose M^{2n+1} is a compact, oriented, closed manifold. Then we have*
(a) $H_t^{(2n+1)k-r}(C^k(M - *); \mathbb{F}) \cong \mathcal{T}\mathcal{P}_{r,k}(M; \mathbb{F}) = H_r(TP^k(Y), \dot{T}P^{k-1}(Y); \mathbb{Z}/p)$.
(b) $H_T^{(2n+1)k-r}(C^k(M; \mathbb{F}) = H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

(The T subscript appearing in the homology here denotes the fact that the coefficients are twisted by the – action of $\mathbb{Z}/2 = \mathcal{S}_k/\mathcal{A}_k$ on the field $\mathbb{F}$.)

Part I continues with the main computational result, a spectral sequence to compute the bigraded groups $\mathcal{T}\mathcal{P}_{*,*'}(Y) := H_*(TP^{*'}(Y), TP^{*'-1}(Y); \mathbb{Z}/p)$ which taken together form a bigraded ring.

Theorem 4.1 *There is a spectral sequence with E^2-term*

$$E^2_{*,*'} = \operatorname{Tor}^{\mathcal{C}_{*,*'}(Y;\mathbb{F})}(\mathbb{F}, \mathbb{F})$$

converging to $\mathcal{T}\mathcal{P}_{*,*'}(\Sigma Y; \mathbb{F})$.

Here $\mathbb{F}$ is $\mathbb{Q}$, or $\mathbb{F}=\mathbb{F}_p$ with $p>2$.

We then apply this to the case $Y=M_g$, a closed, connected, oriented surface of genus g. Completing the results of [BC] is the following result which combines 5.12 and 5.13 in the body of the text

Theorem. *Let* $p>g$, *and suppose* T_g^2 *is a closed Riemann surface of genus* g. *Let* f *have bidegree* (2, 1), $P^{[i]}$ *bidegree* $(2p^i+1, 2p^i)$, *the* h_j *have bidegree* (2.2), $1 \leqq j \leqq 2g$, *and* $\beta P^{[i]}$ *have bidegree* $(2p^i+2, 2p^i)$, *then the bigraded algebra*

$$V_{*,*'} \cong W_g \otimes \Gamma[f, h_1, \ldots, h_j, \ldots, \beta P^{[i]}, \ldots] \otimes E[\ldots P^{[i]} \ldots]$$

satisfies $H^{2kp-j}(\mathscr{C}^k(T_g^2), \mathbb{F}_p) \cong V_{j,k}$.

Here W_g is a complicated algebra determined as the homology of a certain complex described more precisely in § 5, while $\Gamma[\,]$ denotes the divided power algebra on the stated generators and $E(\,)$ is the exterior algebra on the given generators.

There is an interesting relation between alternating products $AP^n(Y)=Y^n/\mathscr{A}_n$ (where $\mathscr{A}_n$ is the alternating group), symmetric products, and truncated products.

Lemma 7.1 *Let* X *be a simplicial complex, then for all odd primes,* p, *there is a split exact sequence*

$$0 \to H_*(TP^n(X)/TP^{n-2}(X); \mathbb{F}_p^-) \to H_*(AP^n(X); \mathbb{F}_p) \to H_*(SP^n(X); \mathbb{F}_p^+) \to 0$$

where $\mathbb{F}_p^{\pm}$ *denotes the field* $\mathbb{F}_p$ *with* $\mathbb{Z}/2$-*action* $t(a) = \pm a$.

Part II continues the methods of [BCT]. Among the results we mention a short proof of the main result of [BCT].

Corollary 8.4 [B-C-T] (i) *If* $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ *or* m *is odd, then* $H_* C^k(M)$ *depends only on* $H_* M$ *and* m. *Furthermore there is an isomorphism of vector spaces for* $n>>0$ *and* $0<q<mk$ *given by*

$$\begin{aligned} H_q C^k(M) &\cong H_{q+2nk} C(M, S^{2n}) \\ &\cong H_{q+2nk}\left(\prod_{q=0}^{m} \prod^{\beta(q)} \Omega^{m-q} S^{m+2n}\right). \end{aligned}$$

(ii) *if* $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ *or* m *is even, then* $H_*(C^k(M); \mathbb{F}(-1))$ *depends only on* $H_* M$ *and* m. *Furthermore there is an isomorphism of vector spaces for* $n>>0$ *and* $0<q<mk$ *given by*

$$\begin{aligned} H_q(C^k(M); \mathbb{F}(-1)) &\cong H_{q+(2n+1)k} C(M, S^{2n+1}) \\ &\cong H_{q+(2n+1)k}\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right). \end{aligned}$$

Next we discuss certain product decompositions of mapping spaces of surfaces into spheres. These decompositions in turn give information about the configuration space of unordered k-tuples of points in the given surface.

Theorem 8.5 *If $n>1$, there exist principal $\Omega^2 S^{2n}$ fibrations*

$$\Omega^2 S^{2n} \to Y_{g,2n} \to (S^{2n-1})^{2g}$$

and a principal $\Omega^2 S^3$-fibration

$$\Omega^2 S^3 \to Y_{g,2} \to (S^1)^{2g}$$

such that

(1) *there is a homotopy equivalence* $\mathrm{map}^0_*(M_g, S^2) \xrightarrow{\simeq} (\Omega S^3)^{2g} \times Y_{2,g}$ *and*

(2) *there is a homotopy equivalence after inverting* 6

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$$

for $n>1$,

(3) *there is a fibration for $n>1$*

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}$$

which has a section after inverting 2.

As a convenience for the reader we point out that there are several distinct notations used for these spaces, both in the existing literature and in this paper: namely

(1) $\tilde{C}^n(Y) = F(Y, n)$, and
(2) $C^n(Y) = F(Y, n)/\mathcal{S}_n = B(Y, n) = TP^n(Y) - TP^{n-2}(Y)$.

Part I: Truncated symmetric products

1 Preliminaries

[M2] studied the homology and geometry of symmetric products using techniques – and largely reproducing results which had been anticipated by Steenrod [St2]. The homology results in [M2] are complete in that closed formulae are given for $H_*(SP^k(X); \mathbb{F}_p)$ for any locally finite CW-complex X.

The key new ingredient in [M2] was an explicit way of obtaining a combinatorial cell decomposition for $\coprod_k SP^k(\Sigma Y)$ if an appropriate CW-decomposition was given for $SP^\infty(Y)$.

ΣY was regarded as the identification space $I \times Y/\{0 \times Y = 1 \times Y = I \times * = *\}$ and the suspension coordinate was used to attempt to put an ordering on the unordered n-tuple $\langle (t_1, y_1), \ldots, (t_n, y_n) \rangle$. Precisely, we can partially order the points by assuming $t_1 \leqq t_2 \leqq \ldots \leqq t_n$. (Note there that $\{0 \leqq t_1 \leqq \ldots \leqq t_n \leqq 1\} = \sigma^n$ is just the ordinary n-simplex.) Where $t_i = t_{i+1}$ i.e. on the i^{th}-face of σ^n we observed that the resulting point could be identified with

$$\langle (t_1, y_1), \ldots, (t_{i-1}, y_{i-1}), (t_i, y_i \cdot y_{i+1}), \ldots, (t_n, y_n) \rangle$$

where $y_i \cdot y_{i+1}$ is the unordered pair $\langle y_i, y_{i+1} \rangle \in SP^2(Y) \subset SP^\infty(Y)$. Extending this idea the geometric bar construction was defined and explored in [M1].

For any associative H-space Y with unit $*$, set

$$B_Y = \coprod_k \sigma^k \times Y^k$$

with the compactly generated topology, where we make identifications of 3 kinds motivated by the geometry of $SP^\infty(X)$.

(i) On the face $\sigma_i^k = \{t_1, \ldots, t_i = t_{i+1}, \ldots t_k\}$, or when $y_{i+1} = *$ we identify

$$(t_1, \ldots, t_n, y_1, \ldots, y_n) \quad \text{with} \quad (t_1, \ldots, \hat{t}_{i+1}, \ldots, t_n, y_1, \ldots, y_{i-1}, y_i y_{i+1}, \hat{y}_{i+1}, \ldots, y_n).$$

(ii) If $t_1 = 0$ then set $(t_1, \ldots, t_n, y_1, \ldots, y_n) \sim (t_2, \ldots, t_n, y_2, \ldots, y_n)$.
(iii) If $t_n = 1$ set $(t_1, \ldots, t_n, y_1, \ldots, y_n) \sim (t_1, \ldots, t_{n-1}, y_1, \ldots, y_{n-1})$.

If $*$ is an NDR, define $E_Y = \coprod \sigma^k \times Y^k/$*relations of type* (i), (ii) *only*. One sees easily that E_Y is contractible, and there is a principal quasi-fibering

$$Y \to E_Y \to B_Y.$$

Hence B_Y is a model for the classifying space of Y. Moreover, in the case $Y = SP^\infty(X)$ then by the remarks preceeding the construction we see that

$$B_Y = SP^\infty(\Sigma X).$$

Thus, one can relate the homology of the $SP^n(\Sigma X)$ to a specific construction on pieces of $\oplus H_*(SP^n(X))$, creating an inductive situation which was relatively easy to analyze. An interesting extension of these ideas occurs in [Mc].

Recall that Cartan determined the cohomology of the Eilenberg-MacLane spaces $K(\pi, n)$ for π any finitely generated abelian group as follows. Let $\mathcal{A}_n(p)$ be the subset of the mod(p) Steenrod algebra spanned by monomials in the mod(p) Bockstein β and the P^i having excess $< n$. (For $p = 2$ we do not need the Bockstein as it is identified with Sq^1.) Then

$$H^*(K(\mathbb{Z}/p; n); \mathbb{F}_p) = S(\mathcal{A}_n(p)(\iota_n)),$$

where ι_n is the fundamental class in $H^n(K(\mathbb{Z}/p; n); \mathbb{F}_p) = \mathbb{F}_p$, and $S(W)$ is the symmetric algebra generated by W. More exactly $S(W)$ is the tensor product of the polynomial algebra on the even dimensional generators of W and the exterior algebra on the odd dimensional generators.

For $K(\mathbb{Z}, n)$ let $\mathcal{B}_n(p) = \mathcal{A}_n(p)/W$ where W is the subspace spanned by those monomials with rightmost term equal to β. Then $\mathcal{B}_n(p)$ acts on the fundamental class $\iota \in H^n(K(\mathbb{Z}, n); \mathbb{F}_p)$ and

$$H^*(K(\mathbb{Z}; n); \mathbb{F}_p) = S(\mathcal{B}_n(p)(\iota_n)).$$

The explicit structure of $H^*(SP^\infty(X); \mathbb{F}_p)$ is now given by bigrading the results of Cartan above. When X is a locally finite CW-complex:

$$H^*(SP^\infty(X); \mathbb{F}_p) = \prod_n S_{\alpha_i}[\ldots, P^I(\alpha_i), \ldots]$$

where $P^I=\beta^{e_1}P^{i_1}\beta^{e_2}P^{i_2}\dots\beta^{e_r}P^{i_r}$ runs over all admissible monomials in the mod(p) Steenrod algebra $\mathscr{A}(p)$ of excess $<n$, while the α_i run over a selected basis for $H^n(X;\mathbb{F}_p)$. The second degree of P^I is p^r.

In the particular case where $X=S^1$ we obtain that $H^*(SP^\infty(S^1);\mathbb{Z})=E[e_1]$, and indeed, $SP^n(S^1)\simeq S^1$ for all $n\geqq 1$. When $X=S^2$, we have $H^*(SP^\infty(S^2);\mathbb{Z})=\mathbb{Z}[e_2]$, where e_2 has dimension 2 and second degree 1. As a consequence

(1.1) $$H^*(SP^n(S^2);\mathbb{Z})=\mathbb{Z}[e_2]/(e_2^{n+1}).$$

Indeed, it is a standard result that $SP^n(S^2)\simeq\mathbf{CP}^n$. Finally we have

(1.2) $$H^*(SP^\infty(S^3);\mathbb{F}_p)=E[e_3,\dots,P^{[i]}(e_3),\dots]\otimes\mathbb{F}_p[\beta P^1(e_3),\dots,\beta P^{[i]}(e_3),\dots]$$

where $P^{[i]}=P^{p^{i-1}}P^{p^{i-2}}\dots P^1$ runs over the admissible monomials of excess 2 in the Steenrod p-powers. Note that both $P^{[i]}$ and $\beta P^{[i]}$ have second degree p^i. Then a basis for $H^*(SP^n(S^3);\mathbb{F}_p)$ is given by all those monomials in the generators above for which the second degree is $\leqq n$.

2 The functors $TP^n(Y)$

In this section we assume all spaces are based locally finite CW-complexes.

Definition 2.1 $TP^\infty(Y)=SP^\infty(Y)/(y^2=*,y\in Y)$.

$TP^\infty(Y)$ is a topological group – the free abelian group generated by the points of Y (with $*$ as identity) subject to $y^2=*$ for $y\in Y$. $TP^\infty(Y)$ has been studied in [D-T], it is the free topological $\mathbb{F}_2$-vector space generated by the points of Y modulo the subspace generated by the base point. There is a natural projection $SP^\infty(Y)\to TP^\infty(Y)$ and we set

$$TP^n(Y)=\operatorname{im}(SP^n(Y))\subset TP^\infty(Y).$$

Additionally, if $f\colon Y\to Z$ is a based map then the induced map

$$TP(f)\colon TP^\infty(Y)\to TP^\infty(Z)$$

is defined by

$$TP(f)(\langle y_1,\dots,y_n\rangle)=\langle f(y_1),\dots,f(y_n)\rangle.$$

Together these two operations define the TP-functor on based spaces and based, continuous maps.

Properties

(2.2) $TP^\infty(Y)$ is a homotopy functor, i.e. if $Y\simeq Z$ then $TP^\infty(Y)\simeq TP^\infty(Z)$, and $TP^n(Y)$ is a (based) homotopy functor for each n. Indeed, if

$$H_t\colon\ Y\to Z$$

is a based homotopy from f to g, then $TP(H_t)$ defines a homotopy from $TP(f)$ to $TP(g)$ and $TP(H_t)$ preserves the $TP^n(\ )$ functors.

(2.3) If f: $Y \to Z$ is a based map then $TP(f)$: $TP^\infty(Y) \to TP^\infty(Z)$ is a group homomorphism, and consequently is a principal fibering over its image with fiber = kernel = $TP(f^{-1})(*)$. Hence the functor TP^∞ takes cofiberings to (principal) fiberings. In particular

$$TP^\infty(Y_1 \vee Y_2) = TP^\infty(Y_1) \times TP^\infty(Y_2).$$

Additionally, if $S^{n-1} \xrightarrow{f} Y_1 \to Y_1 \cup e^n = Y$ gives Y from Y_1 by adjoining a based cell, then $TP^\infty(Y)$ is the total space of the principal fibering

$$TP^\infty(Y) = TP^\infty(Y_1) \times_{TP^\infty(S^{n-1})} TP^\infty(cS^{n-1}) \to TP^\infty(S^n)$$

where $TP^\infty(cS^{n-1})$ is the (contractible) total space of the fibering

$$TP^\infty(S^{n-1}) \to TP^\infty(cS^{n-1}) \to TP^\infty(S^n).$$

Here cY denotes the reduced cone on Y.

(2.4) More generally, $TP^\infty(\Sigma Y) = B_{TP^\infty(Y)}$ where $B_{(\,)}$ is the topological bar construction of [M1, St1] discussed in § 1.

(2.5) From (2.3) $\pi_*(TP^\infty(Y))$ is a homology theory for Y. Note that $TP^\infty(S^0) = \mathbb{Z}/2$, so by (2.3), $TP^\infty(S^1) = B_{\mathbb{Z}/2} = \mathbf{RP}^\infty$, and more generally $TP^\infty(S^n) = K(\mathbb{Z}/2, n)$. Thus we have the Dold-Thom theorem for TP^∞,

Theorem 2.6 *If Y is a based, locally finite* CW-*complex, then*

$$TP^\infty(Y) \simeq \prod_i K(H_i(Y; \mathbb{Z}/2), i).$$

Corollary 2.7 $TP(f)_*$: $\pi_*(TP^\infty(Y)) \to \pi_*(TP^\infty(Z))$ *is just the map induced in $\mathbb{Z}/2$ homology,*

$$f_*:\ H_*(Y; \mathbb{Z}/2) \to H_*(Z; \mathbb{Z}/2).$$

(2.8) The points of $TP^n(Y) - TP^{n-1}(Y)$ have the form $\langle x_1, \ldots, x_n \rangle$, $x_i \neq x_j \neq *$, if $i \neq j$. The points of $TP^{n-1}(Y)$ embed into $TP^n(Y)$ by adding one $*$, i.e. regarding them as $\langle x_1, \ldots, x_{n-1}, * \rangle$. Let

$$\mathrm{Sing}^n(Y) = \{\langle y_1, \ldots, y_n \rangle \in SP^n(Y) \mid y_i = y_j, \text{ for some } i \neq j\}.$$

Then $TP^n(Y)/TP^{n-2}(Y) = SP^n(Y)/\mathrm{Sing}^n(Y)$. Similarly we have that

$$TP^n(Y)/TP^{n-1}(Y) = SP^n(Y - *)/\{\mathrm{Sing}^n(Y - *) \cup SP^{n-1}(Y)\}.$$

Theorem 2.9 $H_*(TP^n(Y); \mathbb{Z}/2) = \bigotimes_{r \leq n} H_*(TP^r(Y), TP^{r-1}(Y); \mathbb{Z}/2)$ *for Y a locally finite based* CW-*complex.*

Proof. Let

$$D_n:\ TP^n(Y) \to TP^\infty(TP^{n-1}(Y))$$

be defined by the formula

$$(2.10)\qquad D_n(\langle y_1,\ldots,y_n\rangle)=\prod_{i=1}^{n}\langle y_1,\ldots,\hat{y}_i,\ldots,y_n\rangle.$$

D_n is continuous, and, by checking on points we have the following commutative diagram

$$(2.11)\qquad \begin{array}{ccc} TP^{n-1}(Y) & \xrightarrow{i_{n-1}} & TP^n(Y) \\ \Big\downarrow{\scriptstyle id\times D_{n-1}} & & \Big\downarrow{\scriptstyle D_n} \\ TP^{n-1}(Y)\times TP^{\infty}(TP^{n-2}(Y)) & \xrightarrow{j} & TP^{\infty}(TP^{n-1}(Y)) \end{array}$$

where $j=\circ(i_1\times TP(i_{n-2}))$. Now, pass to homology with $\mathbb{Z}/2$-coefficients, i.e. apply the TP^{∞}-functor, and use the retraction

$$r\colon\ TP^{\infty}(TP^{\infty}(Z))\to TP^{\infty}(Z),\qquad r(\langle w_1,\ldots,w_s\rangle)=w_1\cdot w_2\ldots w_s.$$

(2.9) now follows directly (or one can apply [D, Lemma (2)] to 2.11). □

Remark 2.12 The groups $H_*(TP^n(Y),\ TP^{n-1}(Y);\ \mathbb{Z}/p)$ p an odd prime, are more complex. Of course, a class $\alpha\in H_*(TP^n(Y),\ TP^{n-1}(Y);\ \mathbb{Z}/p)$ which comes from the group $H_*(TP^n(Y);\ \mathbb{Z}/p)$ must, sooner or later, be in the image of the ∂-map from

$$H_*(TP^{n+r}(Y),\ TP^{n+r-1}(Y);\mathbb{Z}/p)$$

for some r. Nevertheless, there is an associative pairing

$$\begin{aligned}&H_*(TP^n(Y),\ TP^{n-1}(Y);\mathbb{Z}/p)\otimes H_{*'}(TP^r(Y),\ TP^{r-1}(Y);\mathbb{Z}/p)\\ &\quad\to H_{*+*'}(TP^{n+r}(Y),\ TP^{n+r-1}(Y);\mathbb{Z}/p)\end{aligned}$$

making the groups

$$\mathcal{T}\mathcal{P}_{*,n}(Y;\mathbb{Z}/p)=\bigoplus_{*,n} H_*(TP^n(Y),\ TP^{n-1}(Y);\mathbb{Z}/p)$$

into a commutative, *bigraded* ring with unit. Note that $\mathcal{T}\mathcal{P}_{*,*'}(Y;\mathbb{Z}/p)$ is a homotopy invariant of the space Y.

Example 2.13 $S^1=\Sigma S^0$, consequently $TP^{\infty}(S^1)=B_{TP^{\infty}(S^0)}=B_{\mathbb{Z}/2}$. But then

$$TP^n(S^1)=(B_{\mathbb{Z}/2})_n=\sigma^n\times(\mathbb{Z}/2)^n/\text{identifications}.$$

These identifications collapse σ^n to its i^{th}-face if the i^{th}-coordinate in $(\mathbb{Z}/2)^n$ is 1, or if we are on the i^{th}-face directly. Thus, there is a single n-cell in $TP^n(S^1)$, $\sigma^n\times(T)^n$, and it is a straightforward induction to show that $TP^n(S^1)\simeq\mathbf{RP}^n$ for all n.

In particular, $H_*(TP^n(S^1), TP^{n-1}(S^1); \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & * = n \\ 0 & \text{otherwise} \end{cases}$ and we have

$$\mathcal{T}\mathcal{P}_{**'}(S^1; \mathbb{Z}/p) = E(e_{1,1}) \otimes \Gamma(b_{2,2})$$

where $E(e)$ is the exterior algebra on a one dimensional generator and $\Gamma(b)$ is the divided power algebra on a single two dimensional generator $\left(\Gamma(b)_{2i,2i} = \mathbb{Z}/p \text{ with generator } b_i \text{ and } b_i b_j = \binom{i+j}{j} b_{i+j}\right)$.

Note also that the above description of $\mathcal{T}\mathcal{P}_{**'}(S^1; \mathbb{F})$ remains valid even when $\mathbb{F} = \mathbb{Z}$, and consequently is valid for all coefficients. In fact we have the following (formal) ring isomorphism

(2.14) $$\mathcal{T}\mathcal{P}_{*,*'}(S^1; \mathbb{F}) \cong H_*(K(\mathbb{Z}, 1); \mathbb{F}) \otimes H_*(K(\mathbb{Z}, 2); \mathbb{F}).$$

Remark 2.15 Let $Y = \Sigma Z$, then the coproduct map

$$e\colon \Sigma Z \to \Sigma Z \vee \Sigma Z, \qquad (t, z) \mapsto \begin{cases} (2t, z), * & t \leqq \frac{1}{2} \\ *, (2t-1, z) & t \geqq \frac{1}{2} \end{cases}$$

induces a bigraded coproduct map

$$M(e)\colon\ TP^\infty(Y) \to TP^\infty(Y \vee Y) = TP^\infty(Y) \times TP^\infty(Y)$$

which, on passing to homology makes $\mathcal{T}\mathcal{P}_{*,*'}(Y; \mathbb{F})$ into a bigraded Hopf algebra whenever $\mathbb{F}$ is a field. Such a structure is not generally available to us when Y is not a suspension, however, since

(1) the diagonal $\Delta\colon TP^\infty(Y) \to TP^\infty(Y) \times TP^\infty(Y)$ is not filtration preserving, and
(2) $TP^\infty(Y \times Y)$ is not $TP^\infty(Y) \times TP^\infty(Y)$ so the map $TP(\Delta)$ does not help either.

Of course, the filtration of $TP^\infty(Y)$ by the $TP^n(Y)$ gives rise to a spectral sequence converging to $H_*(TP^\infty(Y); \mathbb{F})$, with E^1-term $E^1_{*,n} = \mathcal{T}\mathcal{P}_{*,n}(Y; \mathbb{F})$. With respect to the derivations, d_r, every term $E^r_{*,*'}$ is a differential graded algebra, and when Y is a suspension is even a differential graded Hopf algebra.

Remark 2.16 Each term $E^r_{*,*'}$ in this spectral sequence is a homotopy invariant for Y.

3 Connection with deleted symmetric products

Suppose that M^{2n} is a compact, oriented manifold with base point $*$, and empty boundary. We have

Theorem 3.1 *Let $\mathbb{F}$ be a field, then*
(a) $H^{2nk-r}(C^k(M^{2n} - *); \mathbb{F}) \cong \mathcal{T}\mathcal{P}_{r,k}(M^{2n}; \mathbb{F})$.
(b) $H^{2nk-r}(C^k(M^{2n}); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

Proof. This is just Alexander-Poincaré duality. Indeed, if $N = N(\mathrm{Sing}^k(M^{2n}))$ is any regular neighborhood of the singular set, then ∂N is a closed manifold of dimension $2nk-1$ and $SP^k(M^{2n}) - \mathrm{int}(N)$ is an oriented manifold with boundary $=\partial N$. (Unless char$(\mathbb{F})=2$ the fact that this manifold is oriented is crucial when we apply duality but $C^n(M)$ is only oriented if $\dim(M)$ is even.) Thus, we have

$$\begin{aligned} H_*(SP^k(M), \mathrm{Sing}^k(M)) &= H_*(SP^k(M), N) = H_*(SP^k(M) - \mathrm{int}\, N, \partial N) \\ &= H^{2nk-*}(SP^k(M) - \bar{N}) = H^{2nk-*}(SP^k(M) - \mathrm{Sing}^k(M)) \\ &= H^{2nk-*}(C^k(M^{2k})). \end{aligned}$$

Hence, (b) follows from (2.8).

To prove (a) note that if N_1 is a regular neighborhood of

$$V_k = \{\mathrm{Sing}^k(M^{2n}) \cup SP^{k-1}(M^{2n})\},$$

thern the same argument works here since

$$SP^k(M^{2n})/V = TP^k(M^{2n})/TP^{k-1}(M^{2n}),$$

and $SP^k(M^{2n}) - V = C^k(M^{2n} - *)$.

When M is odd dimensional we must use twisted coefficients. Thus let

$$\mathbb{R} \to \xi \to C^k(M)$$

be the orientation line bundle (with $w_1(\xi) = w_1(C^k(M))$, the first Stiefel-Whitney class of $C^k(M)$). The cohomology of $C^k(M)$ with orientation twisted coefficients,

$$H^i_T(C^k(M); \mathbb{F}) \cong \tilde{H}^{i+1}(T(\xi); \mathbb{F}),$$

is defined as the ordinary reduced cohomology in one higher dimension of the Thom complex of ξ. □

Theorem 3.2 *Suppose M^{2n+1} is a compact, oriented, closed manifold. Then we have*

(a) $H_t^{(2n+1)k-r}(C^k(M - *); \mathbb{F}) \cong \mathcal{TP}_{r,k}(M; \mathbb{F}) = H_r(TP^k(Y), TP^{k-1}(Y); \mathbb{F})$.
(b) $H_t^{(2n+1)k-r}(C^k(M; \mathbb{F}) = H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

(This is the way Poincaré duality works for non-oriented manifolds. Otherwise, the proof is as above.)

In [B-C-T], Cohen et al. show that if M^{2n+1} is an odd dimensional, oriented, manifold, then $H_*(C^k(M); \mathbb{F})$ depends only on $H_*(M; \mathbb{F})$. But they also show, in the same paper, that $H_*(C^k(M^{2n}); \mathbb{F}^-)$ depends only on $H_*(M^{2n}, \mathbb{F})$ when M^{2n} is even dimensional, and oriented. Hence, we now have effective methods for identifying both the twisted and untwisted homology of the deleted symmetric products of manifolds in all dimensions. Short proofs of these results are given in Part II.

4 Calculational techniques

In this section we start by introducing an important spectral sequence which allows us to understand $\mathcal{TP}_{*,*}(\Sigma Y)$. The case of more general spaces is handled in 4.6 where we identify $TP^n(Y)$ up to homology with a much larger space. This larger space, however, has a natural filtration and associated spectral sequence which will be our main calculational tool in the remainder of Part I of this paper.

For $TP^\infty(\Sigma X)$, we have, exactly as in § 1, an identification $TP^\infty(\Sigma X)\cong B_{TP^\infty(Y)}$. In particular, this allows us to apply the Eilenberg-Moore spectral sequence with E^2-term $\mathrm{Tor}^{H_*(TP^\infty(Y))}(\mathbb{F},\mathbb{F})$ to study $H_*(TP^\infty(\Sigma Y);\mathbb{F})$. With a little care we can even extend this to the study of $\mathcal{T}_{*,*}(\Sigma Y)$.

For a bigraded algebra $A_{*,*}$, the $\mathrm{Tor}^{A_{*,*}}(\mathbb{F},\mathbb{F})$ groups are naturally *trigraded.* The terms of tridegree (l,m,n) come from terms of the form $|\theta_1|\dots|\theta_l|$ with θ_i of bidegree (m_i, n_i) and $\sum m_i=m$, $\sum n_i=n$. The associated bigraded Tor groups are then given as $\mathrm{Tor}_{s,t}=\sum_{l+m=s}\mathrm{Tor}_{l,m}$. When A is a commutative, unitary algebra Tor is a bigraded commutative ring.

Theorem 4.1 *There is a spectral sequence with E^2-term*

$$E^2_{*,*'}=\mathrm{Tor}^{\mathcal{T}_{*,*'}(Y;\mathbb{F})}(\mathbb{F},\mathbb{F})$$

converging to $\mathcal{T}_{*,*'}(\Sigma Y;\mathbb{F})$.

Proof. Filter $TP^n(\Sigma Y)$ by setting

$$TP^n(\Sigma Y)_j=TP^n(\Sigma Y)\cap(B_{TP^\infty(Y)})_j.$$

Note that $(B_{TP^\infty(Y)})_j=\mathrm{im}\{\sigma^j\times(TP^\infty(Y))^j\to B_{TP^\infty(Y)}\}$, and

$$\begin{aligned}&TP^n(\Sigma Y)\cap(B_{TP^\infty(Y)})_j\\ &\quad=\{(t_1,y_1),\dots,(t_r,y_r)\,|\,\text{there are at most } j \text{ distinct values among } t_1,\dots,t_r\}.\end{aligned}$$

Then $TP^n(\Sigma Y)_n=TP^n(\Sigma Y)$, so the filtration is complete and the associated spectral sequence converges. Next consider the induced filtration on

$$TP^n(\Sigma Y)/TP^{n-1}(\Sigma Y).$$

The E^1_j term is obviously the direct summand of the bar construction on $C_{*,*'}(Y;\mathbb{F})$ defined as

$$\sum_{r=1}^{j}\sigma^r\times\prod_{\sum_1^r l_k=n}\mathcal{T}_{*,l_k}(Y;\mathbb{F})=\{|\theta_1|\dots|\theta_r|\,|\,\theta_r\in\mathcal{T}_{*,l_i}(Y;\mathbb{F})\}$$

and the E^2-term is the associated bigraded Tor-term as claimed. □

Corollary 4.2

$$\mathcal{T}_{*,*'}(S^n;\mathbb{F})\cong\bigoplus_{\substack{r+s=*\\ r'+2s'=*'}}H_{r,r'}(SP^\infty(S^n);\mathbb{F})\otimes H_{s,s'}(SP^\infty(S^{n+1});\mathbb{F})$$

for any field $\mathbb{F}$.

Proof. We proceed by induction. The corollary amounts to the asertion that the spectral sequence of 4.1 collapses when $Y=S^n$. This is implicit in Example 2.13 (2.14) for the case S^1. For S^n we simply use the standard techniques of [Sem] to embed the requisite *cycles* in the chain complex of $TP^m(S^n)/TP^{m-1}(S^n)$ corresponding to all the classes in the Tor-groups. □

Remark 4.3 Something like 4.2 should be expected since

$$TP^\infty(S^{n+1}) \simeq SP^\infty(S^n) \times_T SP^\infty(cS^n)$$

where T: $SP^\infty(S^n) \to SP^\infty(S^n)$ is the squaring map. Since the squaring map doubles degrees, a consistent bigrading on $SP^\infty(S^n) \times_T SP^\infty(cS^n)$ can only be realized if the bidegree in $SP^\infty(cS^n)$ is doubled.

Example 4.4 From 4.2

$$\mathcal{T}_{*,*'}(S^2; \mathbb{F}) = \bigotimes_{\substack{r'=2s'=*' \\ r+s=*}} H_{r,r'}(SP^\infty(S^2); \mathbb{F}) \otimes H_{s,s'}(SP^\infty(S^3); \mathbb{F}).$$

Also, $H_{r,r'}(SP^\infty(S^2); \mathbb{F}) = \Gamma(b_{2,1})$, while (if we let $P^{[r]} = P^{p^r} P^{p^{r-1}} \dots P^p P^1$ denote the monomial in the p^{th}-power operations in the Steenrod algebra $\mathcal{A}(p)$)

$$H^{s,s'}(SP^\infty(S^3); \mathbb{Z}/p) = E[\iota_3, P^{[0]}(\iota_3), \dots, P^{[r]}(\iota_3) \dots] \otimes \mathbb{Z}/p[\dots \beta P^{[r]}(\iota_3) \dots]$$

where the bidegree of $P^{[r]}(\iota_3)$ in $\mathcal{T}_{*,*'}(S^2; \mathbb{Z}/p)$ is $(2p^{r+1}+1, 2p^{r+1})$, and is $(2p^{r+1}+2,\ 2p^{r+1})$ for $\beta P^{[r]}(\iota_3)$. For the rationals $\mathbb{Q}$ the calculation reduces to $\mathcal{T}_{*,*'}(S^2; \mathbb{Q}) = \Gamma(b_{2,1}) \otimes E(\iota_{3,2})$, so

$$\mathcal{T}_{i,*}(S^2; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 2i \quad \text{generator} \quad b_{2i,i} \\ \mathbb{Q} & * = 2i-1 \quad \text{generator} \quad b_{2(i-2),i-2}\, \iota_{3,2} \\ 0 & \text{otherwise} \end{cases}$$

as long as $i > 1$.

Example 4.5 For $TP^\infty(S^2)$ we have that $d^1(b_{2,1}) = \iota_{3,2}$, and more generally

$$d^{p^i}(b^{p^i}) = P^{[i-1]}(\iota_{3,2}), \qquad d^{p^i}(b^{p^{i-1}(p-1)}\, P^{[i-2]}(\iota_{3,2})) = \beta P^{[i-1]}(\iota_{3,2})$$

for odd p, after comparing with more standard sequences, or, alternately, using the Hopf algebra structure, and the fact that for odd p the E^∞ term is 0, to force the successive differentials.

To analyze the structure of $\mathcal{C}_{*,*'}(Y)$ for more general spaces we have the following result which leads to a very useful spectral sequence.

Theorem 4.6 *Let* $\mathcal{U}_n(Y) = \bigcup_{i+2j \leq n} SP^i(Y) \times_T SP^j(cY) \subset SP^\infty(Y) \times_T SP^\infty(cY)$, *then* $H_*(\mathcal{U}_n(Y); A) \cong H_*(TP^n(X); A)$ *for all untwisted coefficients* A.

Proof. There is a map

$$p: SP^\infty(Y) \times_T SP^\infty(xY) \to TP^\infty(Y)$$

defined on points by $p(\{a, b\}) = \{a\} \in TP^\infty(Y)$. This map is well defined since first it is a homomorphism, and second, $p(\{a, (0, x)\}) = p(\{ax^2, *\}) \in TP^\infty(Y)$. Note that the image of p when restricted to $\mathcal{U}_n(Y)$ is exactly $TP^n(Y)$, and set

$$\mathcal{U}_{n,j}(Y) = p^{-1}(TP^j(Y)) \cap \mathcal{U}_n(Y)$$

so $\mathscr{U}_{n,1}(Y)\subset\mathscr{U}_{n,2}(Y)\subset\ldots\subset\mathscr{U}_{n,n}(Y)=\mathscr{U}_n(Y)$. Then $w\in\mathscr{U}_{n,j}(Y)$ implies that w is the image of a point of the form $(\langle x_1,\ldots,x_j,\lambda_1^2,\ldots,\lambda_s^2\rangle,(t_1,y_1),\ldots,(t_l,y_l))$ where $l\leqq[n-j/2]-s$. Moreover any such point is equivalent to the image of

$$(\langle x_1,\ldots,x_j\rangle,(0,\lambda_1),\ldots,(0,\lambda_s),\ldots,(t_l,y_l)).$$

Thus, $w\in\mathscr{U}_{n,j}(Y)$ if and only if $w\in\mathrm{im}(SP^j(Y)\times SP^{[(n-j)/2]}(cY))$.

In particular $w\in\mathscr{U}_{n,j}(Y)-\mathscr{U}_{n,j-1}(Y)$ if and only if

$$\omega\in\mathrm{im}((Sp^j(Y)-(\mathrm{Sing}_j(Y)\cup SP^{j-1}(Y)))\times SP^{[(n-j)/2]}(cY)),$$

and

$$\begin{aligned}\mathscr{U}_{n,j}(Y)/\mathscr{U}_{n,j-1}(Y)&=\frac{SP^j(Y)\times SP^{[(n-j)/2]}(cY)}{SP^{j-1}(Y)\cup\mathrm{Sing}_j(Y)\times SP^{[(n-j)/2]}(cY)}\\&\cong SP^j(Y)/\mathrm{Sing}_j(Y)\\&=TP^j(Y)/TP^{j-1}(Y)\end{aligned}$$

since $SP^l(cY)$ is contractible for all l. Moreover, the map

$$\hat{p}:\mathscr{U}_{n,j}(Y)/\mathscr{U}_{n,j-1}(Y)\to TP^j(Y)/TP^{j-1}(Y)$$

is directly seen to be a homotopy equivalence for each j. Consider now the diagram

$$\begin{array}{ccccccc}\mathscr{U}_{n,1}(Y)&\subset&\mathscr{U}_{n,2}(Y)&\subset&\ldots&\subset&\mathscr{U}_{n,n}(Y)\\ \downarrow{\scriptstyle p_1}&&\downarrow{\scriptstyle p_2}&&&&\downarrow{\scriptstyle p_n}\\ TP^1(Y)&\subset&TP^2(Y)&\subset&\ldots&\subset&TP^n(Y).\end{array}$$

The first map p_1 is a homotopy equivalence, and so are the relative maps $\hat{p}_j$. Hence by iterate applications of the 5-lemma the homology maps p_{j*} are all isomorphisms and 4.6 follows. □

Example 4.7 $V_4(X)=\mathscr{U}_{4,1}(X)$ is given as the double mapping cylinder

$$SP^2(X)\times X\xleftarrow{\Delta\times 1}X\times X\xrightarrow{p}SP^2(X),$$

where the middle term is identified with the points of the form $\langle(t,x_1),(1,x_2)\rangle$, the first identification occurs when $t=0$, and the second when $t=1$. Then $\mathscr{U}_{4,2}(X)$ is the mapping cylinder of $p\colon X\times X\to SP^2(X)$, and thus has the homotopy type of $SP^2(X)$. Similarly, $V_4(X)/\mathscr{U}_{4,2}(X)\simeq SP^2(X)\times X/(\Delta X\times X)$ and these are the same homotopy types as in the corresponding decomposition of the singular locus of $SP^4(X)$.

Consider the Serre spectral sequence of the map

$$p\colon SP^\infty(Y)\times_T SP^\infty(cY)\to SP^\infty(\Sigma Y).$$

We find

Lemma 4.8 *For each prime p, we have*

(1) $E^2_{i,j}(SP^\infty(X) \times_T SP^\infty(cX))$

$$\cong \bigoplus_k H_i(SP^k(\Sigma X), SP^{k-1}(\Sigma X); \mathbb{F}_p) \otimes H_j(SP^\infty(X); \mathbb{F}_p).$$

(2) $E^2_{i,j}(\mathscr{U}_n(X))$

$$\cong \bigoplus_{k \leqq [\frac{n}{2}]} H_j(SP^k(\Sigma X), SP^{k-1}(\Sigma X); \mathbb{F}_p) \otimes \bigoplus_{k+2v \leqq n} H_j(SP^v(X), SP^{v-1}(X); \mathbb{F}_p).$$

(3) *The map of spectral sequences induced from* 4.9,

$$E^2_{i,j}(\mathscr{U}_n(X)) \to E^2_{i,j}(SP^\infty(X) \times_T SP^\infty(cX)),$$

is an injection.

(This is clear.)

The structure of differentials in the Serre spectral sequence for the quasi-fibering

$$\begin{array}{ccc} SP^\infty(Y) & \longrightarrow & SP^\infty(cY) \\ & & \downarrow \\ & & SP^\infty(\Sigma Y) \end{array}$$

is studied in [Sem, M2]. On the other hand the quasi-fibering of 4.6 is induced from this quasi-fibering via the squaring map

$$\times 2: SP^\infty(\Sigma Y) \to SP^\infty(\Sigma Y), \qquad \times 2(\theta) = \theta^2,$$

and so we have a natural map of spectral sequences which determines the differentials in the spectral sequence for the total space $SP^\infty(Y) \times_T SP^\infty(cY)$. However, this observation does not necessarily determine the differentials when we use the spectral sequence of 4.8 on restricting the filtration of the Serre spectral sequence to the spaces $\mathscr{U}_n(Y)$ or the quotients $\mathscr{U}_n(Y)/\mathscr{U}_{n-1}(Y)$. This problem occurs typically when, for one reason or another, the first differential on an element, (which is predicted by comparing spectral sequences), is zero. Then we need a method of determining that all further differentials on that element will be zero as well. For this the following results will be quite useful.

Let M^{2n} be a closed, oriented, manifold. Suppose $* \in M^{2n}$ is chosen. Define an embedding

$$e: \; C^k(M^{2n} - *) \to C^{k+1}(M^{2n} - *)$$

by first deforming away from $*$, so as to assume all points of the k-tuples in $C^k(M^{2n} - *)$ lie outside a small neighborhood of $*$, then adjoin a new point in this small neighborhood.

Theorem 4.9 *The induced map in homology*

$$e_*: \; H_*(C^k(M^{2n} - *); \mathbb{Z}) \to H_*(C^{k+1}(M^{2n} - *); \mathbb{Z})$$

is an injection onto a direct summand.

Proof. Define a map

$$\phi_{k+1}\colon C^{k+1}(M^{2n}-*)\to SP^{\infty}(C^k(M^{2n}-*))$$

by $\langle x_1\ldots x_{k+1}\rangle\mapsto\prod_{i=1}^{k+1}\langle x_1\ldots\hat{x}_i\ldots x_{k+1}\rangle$. Then

$$\phi_{k+1}\circ e_k(\langle x_1\ldots x_k\rangle)=\langle x_1\ldots x_k\rangle\circ(SP^{\infty}(e_k)\,\phi_k(\langle x_1\ldots x_k\rangle))$$

and the result follows by an easy induction. □

Remark 4.10 4.9 is a special case of many results already in the literature. In particular there is a stable section for e_k and the relevant spaces stably split.

Corollary 4.11 (a) *M^{2n} as above, then the E^2-term of the induced spectral sequence (from* 4.11*) for $TP^k(M^{2n})/TP^{k-1}(M^{2n})$ is given by projecting onto the terms in* 4.11(2) *of exactly degree k.*
(b) *In the restricted spectral sequence, elements of the form $\theta\circ\gamma_i([M])$ cannot be in the images of differentials on elements not of the form $\theta'\,\gamma_i([M])$.*

(The first statement is clear. The second statement follows by duality from 4.9 when we realize that the dual of the inclusion is $\cup[M^{2n}]^*$.)

5 Closed, oriented surfaces

We consider, first the case $X=M_g^2$, the closed, oriented, surface of genus g. We have

$$(5.1)\qquad H_i(M_g^2;\mathbb{Z})=\begin{cases}\mathbb{Z}^{2g} & \text{generators } e_1,\ldots,e_{2g},\ te_i=2e_i\\ \mathbb{Z} & \text{generator } f,\ tf=2(f+\sum_{i<g} e_i\,e_{g+i}).\end{cases}$$

Here t is the map in homology induced from the squaring map $M_g^2\to SP^2(M_g^2)$ sending x to x^2 for all $x\in M_g^2$.

Next, note that if we are interested in only $\mathcal{T}_{*,n}(M_g^2)$, we need only study the restriction of the differentials in the Leray-Serre spectral sequence of 4.11 to the terms of exact filtration n,

$$(5.2)\quad E^2_{j,s}=\sum_{k\leq\left[\frac{n}{2}\right]} H_j(SP^k(\Sigma X),SP^{k-1}(\Sigma X);\mathbb{F}_p)\otimes H_s(SP^{n-2k}(X),SP^{n-2k-1}(X);\mathbb{F}_p).$$

To begin the explicit calculations we check first when $\mathbb{F}=\mathbb{Q}$, the rationals. Then $H_{i,j}(SP^{\infty}(\Sigma M_g^2);\ \mathbb{Q})=\mathbb{Q}[h_1,\ldots,h_{2g}]\otimes E[\sigma f]$. Here, $\partial h_i=2e_i$ has lower second filtration, hence is not seen in (5.2). On the other hand, $\partial\sigma f=2(f+\sum_{i<g} e_i\circ e_{g+i})$, and this gives the reduced differential

$$\partial(\sigma f)=2\sum_{i<g} e_i\otimes e_{g+i}=\mu_g$$

for (5.2). We thus obtain

Lemma 5.3 *Let* $A_i \subset E_i[e_1, \ldots, e_{2g}]$, *be the subgroup* $\mathrm{Ann}(\mu\circ)$, *and*

$$B_i = \mathrm{im}(\mu\circ) \cap E_i[e_1, \ldots, e_{2g}],$$
$$C_{2i} = \mathbb{Q}[h_1, \ldots, h_{2g}],$$
$$D_i = E_i[e_1, \ldots, e_{2g}]/B_i.$$

Then

$$\mathcal{T}_{*,n}(M_g^2) = \sum_{s+2i+j+2=n} f^s C_{2i} A_j \sigma f \oplus \sum_{s+2i+j=n} f^s C_{2i} D_j.$$

Proof. This is the result of the d^2-differentials, i.e. the E^3-term of the spectral sequence. To see that there are no further differentials, note that we can imbed the homology of $SP^\infty\left(\bigvee_1^{2g} S^1\right)$ in its chain complex as a commutative ring. Moreover, the generators appear in the correct bidegrees, and the differential $d^2(\sigma f) = 2(f + \sum_{i<g} e_i \circ e_{g+i})$, also embeds. Thus, the elements $A_j \sigma f$, D_j are infinite cycles, being represented by actual cycles. Also, the generators h_i are non-zero in $H_2(TP^2(M_g^2), M_g^2; \mathbb{Q})$, hence their products are infinite cycles in the spectral sequence. Similarly for $f \in H_2(M_g^2; \mathbb{Q})$. But this counts all the elements occuring at E^3, and 5.3 follows. □

Example 5.4 In the case of the torus M_1^2, we have $D_* = D_1 = \langle e_1 \rangle + \langle e_2 \rangle$, while $A_* = H_*(M_1^2, \mathbb{Q})$. We thus have

$$\mathcal{T}_{*,*'}(M_1^2; \mathbb{Q}) = \mathbb{Q}[f, h_1, h_2](D_1, \tilde{H}_*(M_1^2; \mathbb{Q})\, \sigma f)$$

where the bidegree of f is (2, 1), that for h_i is (2, 2), that for the elements of D_1 is (1, 1), and that for $A_1\, \sigma f$ is (4,3), while that for $A_2\, \sigma f$ is (5, 4).

Thus determining the $E^2_{*,*}$ term above, and consequently the groups $H_*(C^n(M_g^2 - \infty); \mathbb{Q})$ reduces to analyzing the differential algebra $E[e_1, \ldots, e_{2g}, \sigma f]$ with differential $d(\sigma f) = \mu$. We write

$$E[e_1, \ldots, e_{2g}, \sigma f] = E[e_1, \ldots, e_{2g}] \oplus E[e_1, \ldots, e_{2g}]\, \sigma f$$

and let $E_k[e_1, \ldots, e_{2g}, \sigma f]$ denote the submodule of elements of degree k in the generators $e_1, \ldots, e_{2g}$. Note that E_k has dimension exactly $\binom{2g}{k}$, 1 generates E_0, while the volume form $\omega_g = e_1 e_2 \ldots e_{2g}$ generates E_{2g}.

Since μ is homogeneous of degree 2, the differential now takes the form

$$(5.5) \qquad d = \bigoplus_k d_k(g) \quad \text{with} \quad d_k(g)\colon E_k[e_1, \ldots, e_{2g}] \to E_{k+2}[e_1, \ldots, e_{2g}]$$

and $d_k(x) = x\mu$.

We will study $d_k(g)$ by using induction on both k and g. It turns out that the (co)kernels of $d_k(g)$ are determined by the (co)kernels of $d_l(g-1)$ and $(d_l(g-1))^2$; the (co)kernels of $(d_l(g-1))^2$ are in turn determined by the (co)kernels of $(d_m(g-2))^2$ and $(d_m(g-2))^3$; and so on. To facilitate the induction write $\mu = \sum_1^g x_i$ where $x_i = e_i \circ e_{g+i}$, so $\mu_g = \mu_{g-1} + x_g$, and

$$\begin{aligned}\mu_g^r &= \mu_{g-1}^r + r\mu_{g-1}^{r-1} x_g \\ &= r! \sum_{(l_1,\ldots,l_r)} x_{l_1} \ldots x_{l_r}\end{aligned}$$

where the sum runs overall all $L=(l_1, \ldots, l_r)$ with $1 \leqq l_1 < \ldots < l_r \leqq g$. In particular, for $r=g$ we have $\mu_g^g = g!\,\omega_g$.

For each $E_k[e_1, \ldots, e_{2g}]$ the products $e_I = e_{i_1} \ldots e_{i_k}$ with $I=(i_1, \ldots, i_k)$ and $1 \leqq i_1 < i_2 < \ldots < i_k \leqq 2g$ form the cannonical basis. There are 4 types of basis elements particularly adapted to the proposed induction

(5.6)
(1) $(i_1, \ldots, i_k)$ with no $i_l = g$ or $2g$,
(2) $(i_1, \ldots, i_k)$ with g but not $2g$ in the list,
(3) $(i_1, \ldots, i_k)$ with $2g$ but not g in the list,
(4) $(i_1, \ldots, i_k)$ with both g and $2g$ in the list,

so $E_k[g] = E_k[g-1] \oplus E_{k-1}[g-1]\; e_g \oplus E_{k-1}[g-1]\; e_{2g} \oplus E_{k-2}[g-1]\, x_g$. It is direct now to write out the operator $\wedge \mu^r$ with respect to this decomposition. In matrix form we have

$$(5.7) \qquad d_k(g)^r = \begin{pmatrix} \mu_{g-1}^r & 0 & 0 & r\mu_{g-1}^{r-1} \\ 0 & \mu_{g-1}^r & 0 & 0 \\ 0 & 0 & \mu_{g-1}^r & 0 \\ 0 & 0 & 0 & \mu_{g-1}^r \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & A' \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix}.$$

Lemma 5.8 [BC] *For $0 \leqq k \leqq 2g$ and $0 \leqq r \leqq g$ the differential $d_k(g)^r$: $E_k[g] \to E_{k+2r}[g]$ is a*

(I) *monomorphism for $k < g-r$,*
(II) *isomorphism for $k = g-r$,*
(III) *epimorphism for $k > g-r$.*

Proof. For $g=1$, the only non-trivial differential, $d_0(1)$: $E_0[1] \to E_2[1]$, is an isomorphism. For $g \geqq 2$ and $k=0$, we have $d_0(g)^r(1) = \mu_g^r$, and hence $d_0(g)^r$ is monic.

Case I. If $k < g-r$, then A, A', B as well as C in (5.7) are all monomorphisms by hypothesis. Hence, from

$$0 = d_k(g)^r(a, b_1, b_2, c) = (A(a), B(b_1), B(b_2), A'(a) + C(c))$$

we conclude that $a = b_1 = b_2 = 0$, and so $c=0$ as well. Thus the operator is a monomorphism in this case.

Case II. If $k = g-r$, then A is an epimorphism, A' and B are isomorphisms, and C is a monomorphism. Assume that $d_k(g)^r(a, b_1, b_2, c) = (A(a),\ B(b_1),\ B(b_2),\ A'(a) + C(c)) = 0$. First, $b_1 = b_2 = 0$. We now have $A(a) = \mu_{g-1}^r\, a = 0$ and $\mu_{g-1}^r\, c = -r\mu_{g-1}^{r-1}\, a$ so $\mu_{g-1}\, c = -ra$ since μ_{g-1}^{r-1} is an isomorphism in this degree. Hence $\mu_{g-1}^{r+1}\, c = \mu_{g-1}^r(-ra) = A(-rA) = 0$, but by hypothesis μ_{g-1}^{g+1} is monic here. Thus,

$c=0$. Hence, $-r\mu_{g-1}^{r-1}a=0$, and therefore $a=0$ since μ_{g-1}^{r-1} is an isomorphism here. Thus the map is a monomorphism, but the dimensions of both range and domain are equal, so it is actually an isomorphism.

Case III. It $k>g-r$, then by hypothesis A, B and C are epimorphisms, and A' is a multiple of an epimorphism. Given $(\bar{a},\bar{b}_1,\bar{b}_2,\bar{c})\in E_{k+2r}[g]$, we can find first a, b_1, b_2 satisfying $A(a)=\bar{a}$, $B(b_1)=\bar{b}_1$, and $B(b_2)=\bar{b}_2$. We then choose c so that $C(c)=\bar{c}-A'(a)$, and we see that we have an epimorphism in this case. □

Corollary 5.9 *The ranks of the homology groups $H_i(E[e_1,\dots,e_2g,\sigma f]; d)$ are given as follows,*

$$\begin{cases} \binom{2g}{i}-\binom{2g}{i-2} & \textit{for } i=0,1,\dots,g,\\ \binom{2g}{i}-\binom{2g}{i+2} & \textit{for } i=g,g+1,\dots,2g,\\ 0 & \textit{in all other dimensions.}\end{cases}$$

Note the apparent duality between H_i and H_{2g-i+1}.

Next, turning to the field $\mathbb{F}_p$ for p an odd prime, we see that the differentials discussed above are still present, but the situation is more complex for the new generators, $\beta^\varepsilon P^{[i]}(\sigma f)^*$, $P^{[i]}(\sigma f)^*$. We have that $P^{[i]}(\sigma f)^*$ transgresses to the fiber, where its differential is the mod p reduction of the class $\gamma_p(f)$, a class characterized by the property that $p!\,\gamma_p(f)=f^p$ in the multiplication induced from the product pairing $SP^\infty(X)\times SP^\infty(X)\to SP^\infty(X)$. We have

$$(f+\mu)^p=\sum\binom{p}{k}f^k\mu^{p-k}=\sum\frac{p!}{k!}\gamma_k(f)\,\mu^{p-k},$$

and $\mu^{p-k}=(p-k)!\,\gamma_{p-k}(\mu)$, where

$$\gamma_{p-k}(\mu)=\sum_{i_1<i_2<\dots<i_{p-k}\leqq g}\chi_{i_1}\otimes\chi_{i_2}\otimes\dots\otimes\chi_{i_{p-k}},\qquad \chi_i=e_i\otimes e_{i+g}$$

so, with no indeterminacy, since $H_*(SP^\infty(M_g^2);\mathbb{Z})$ is torsion free, we have that

$$\gamma_p(f+\mu)=\sum\gamma_k(f)\,\gamma_{p-k}(\mu),$$

and factoring out by terms of lower filtration, we have that in the spectral sequence for $TP^{2p}(M_g^2)/TP^{2p-1}(M_g^2)$,

$$d_{2p+1}(P^{[1]}(\sigma f))=\gamma_p(\mu)\bmod(\text{the ideal generated by }\mu).$$

We have

Lemma 5.10 *The element $\gamma_p(\mu_g)\not\equiv 0$ mod (the ideal generated by μ), for $2p-1\leqq g$ and $p\geqq 3$, but $\gamma_p(\mu_{p+1})\in$ this ideal.*

Proof. There are $\frac{(2p-1)(2p-2)}{2}$ distinct monomials in the χ_i in $\gamma_p(\mu)$ for M^2_{2p-1}. On the other hand a basis for the part of the ideal spanned by the χ_j's in this dimension is given by all elements of the form

$$\chi_{i_1} \otimes \dots \otimes \chi_{i_{p-1}} \mu,$$

and each of these consists of a sum of exactly p monomials. But since p and $\left(\frac{2p-1}{2}\right)$ are relatively prime we see that the best that could happen in terms of writing a multiple of $\gamma_p(\mu)$ as an element in the ideal, is that the multiple be divisible by p. Thus the result is true for this genus. Now, for any $g > 2p-1$, project $(S^1)^{2g} \to (S^1)^{4p-2}$, by projecting onto the first $2p-1$ coordinates and the coordinates from $g+1$ to $g+2p$ project to the next $2p-1$ coordinates. This map is a homomorphism, and, in homology, satisfies $\mu_g \mapsto \mu_{2p-1}$. Hence it preserves the ideals, and takes $\gamma_k(\mu_g)$ to $\gamma_k(\mu_{2p-1})$. This proves the first statement of 5.10.

The second statement is similar. The generators for the ideal

$$(\mu_{p+1}) \cap E_p(\chi_1, \dots, \chi_{p+1})$$

are given by the classes $\chi_{i_1} \otimes \dots \otimes \chi_{i_{p-1}} \otimes \mu$, and these give you monomials 2 at a time. On the other hand $\gamma_p(\mu_{p+1}) = \sum_1^{p+1} \chi_1 \otimes \dots \hat{\chi}_i \dots \otimes \chi_{p+1}$. Now, note that

$$\gamma_p(\mu_{p+1}) = \sum_{i=1}^{\frac{p+1}{2}} \chi_1 \otimes \dots \hat{\chi}_{2i-1} \otimes \hat{\chi}_{2i} \dots \otimes \chi_{p+1} \otimes \mu.$$

5.10 follows. □

We do not know what happens in the range between $p+1$ and $2p-1$, nor what happens to the elements $\gamma_{p^i}(\mu)$ relative to the ideal generated by

$$\mu, \gamma_p(\mu), \dots, \gamma_{p^{i-1}}(\mu).$$

The next classes to study are the $\beta P^{[i]}(\sigma f)^*$. In the general spectral sequence the Kudo transgression theorem shows that

$$d_{2p^{i-1}(p-1)+1}(\beta P^{[i]}(\sigma f)^*) = (d_{2p^{i-1}+1}(P^{[i-1]}(\sigma f)^*)^{p-1} \otimes (P^{[i-1]}(\sigma f)^*).$$

The first of these is thus given by

$$\partial_{2p-1}(\beta P^1(\sigma f)^*) = \mu^{p-1} \sigma f. \tag{5.11}$$

This differential is non-zero exactly when $p-1 \leqq g$.

But we also have that the bidegree of $\beta P^{[i]}(\sigma f)^*$ is $(2p^i+2, 2p^i)$, and, among those terms of second degree exactly n, the largest dimension which can occur without the dual of a power of f^* being present is $n+2g+1$. But the differentials raise the part of the dimension involved with fiber terms while lowering dimension involved with h_j, σf, etc. on the base. And this raise can be at most $2g$

on the fiber until we involve powers of f^*. Thus, there can be no non-trivial differential of degree higher than $2g$, and we have

Lemma 5.12 *Let $p \geqq g+1$, then $E^3 = E^\infty$ for the spectral sequence converging to $H_*(TP^k(M_g^2)/TP^{k-1}(M_g^2); \mathbb{F}_p)$. Also, for $p^i \geqq g+1$ the terms $\beta P^{[i]}(\sigma f)^*$ and $P^{[i]}(\sigma f)^*$ are infinite cycles.*

Hence

$$\mathscr{T}_{*,*'}(M_g^2; \mathbb{F}_p) = V \otimes \Gamma_{i > \log_p(g)}[\ldots, P^{[i]}(\sigma f) \ldots] \otimes E_{i > \log_p(g)}[\ldots \beta P^{[i]}(\sigma f), \ldots]$$

where V is obtained from the differential $\partial \sigma f = 2 \sum_1^g e_i \circ e_{i+g}$ and the higher differentials discussed above when $p < g$.

Remark 5.13 Using the results above the homology structure of the spaces $C^n(M_g^2 - \infty)$ seems not too tedious, at least for special cases. However, to obtain the groups $H_*(C^n(M_g^2); \mathbb{F}_p)$, we must consider the portion of the spectral sequence involving (5.2) for both n and $n-1$. Here, the full differential

$$\partial \sigma f = 2\left(f + \sum_1^g e_i \circ e_{i+g}\right), \qquad \partial h_i = 2 e_i,$$

is required. This changes the result somewhat, (in particular all possible differentials now occur), but, once more, it is easy to handle the necessary modifications in specific cases.

6 $m-1$ connected $2m$-manifolds

The next class of spaces we will look at are $m-1$-connected, closed manifolds X of dimension $2m$, $m \geqq 2$, so for some finite $r > 0$

$$\text{(6.1)} \quad \tilde{H}_*(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}^r & * = m, \text{ generators } h_1, \ldots, h_r, \Delta h_j = h_j \otimes 1 + 1 \otimes h_j \\ \mathbb{Z} & * = 2m, \text{ generator } L, \Delta L = L \otimes 1 + 1 \otimes L + \sum a_{k,s}(h_k \otimes h_s) \\ 0 & \text{otherwise.} \end{cases}$$

We can always assume we are working over a ring containing $\frac{1}{2}$, so we have

Lemma 6.2 *In* (6.1) *if m is odd, then r is even, and for both even and odd m we can change variables h_i over $\mathrm{GL}_i(\mathbb{Z}(\frac{1}{2}))$ so that*

$$a_{k,s} = \begin{cases} 0 & k \neq s \text{ if } m \text{ is even} \\ 0 & (k,s) \neq (2j, 2j-1) \text{ or } (2j-1, 2j),\ 1 \leqq j \leqq \left(\frac{i}{2}\right) \text{ if } m \text{ is odd} \\ -1 & (k,s) = (2j, 2j-1) \text{ if } m \text{ is odd} \\ 1 & (k,s) = (2j-1, 2j) \text{ if } m \text{ is odd.} \end{cases}$$

Moreover, in case m is even the terms $\alpha_{j,j}$ are all units in $\mathbb{Z}(\frac{1}{2})$.

Proof. Since X is a closed Poincaré duality complex the cup product from

$$H^m(X;\mathbb{Z}(\tfrac{1}{2}))\otimes H^m(X;\mathbb{Z}(\tfrac{1}{2}))\to\mathbb{Z}(\tfrac{1}{2})$$

defined by $\langle a,b\rangle=\langle a\cup b,[X]\rangle$ is non-singular, and symmetric for m even, skew-symmetric for m odd. It is now standard reduction theory for bilinear forms to reduce to the forms in 6.2. That is, when m is even the form is diagonalizable, while when m is odd it becomes

$$\begin{pmatrix} J & 0 & 0 & \dots & 0\\ 0 & J & 0 & \dots & 0\\ \cdot & \cdot & \cdot & \dots & \cdot\\ 0 & 0 & 0 & \dots & J\end{pmatrix}\quad \text{where } J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The fact that when m is even each term in the diagonalized form is a unit follows since the determinant for the original cup product form is ± 1. Hence, the determinant of the diagonalized form is a unit over $\mathbb{Z}(\tfrac{1}{2})$.

Corollary 6.3 *When m is even, the forms which occur can be chosen so that $\alpha_{j,j}=\pm 1$, $1\leqq j\leqq i$.*

Proof. First, the units of $\mathbb{Z}(\tfrac{1}{2})$ have the form $\pm(2^s)$, $-\infty<s<\infty$, so $U\mathbb{Z}(\tfrac{1}{2})=\mathbb{Z}/2\times\mathbb{Z}$, and $U/U^2=\mathbb{Z}/2\times\mathbb{Z}/2$ with generators ± 1, $\tfrac{1}{2}$. Hence, we can suppose every entry $\alpha_{j,j}$ is ± 1 or $\dfrac{\pm 1}{2}$. By the change of variables $e\mapsto e+f$, $f\mapsto e-f$ the form $\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ becomes $\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$. Similarly we can change $\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. This cuts us down to forms with all ± 1's along the diagonal except perhaps for a single $\dfrac{\pm 1}{2}$. But the latter situation cannot occur, since the original discriminant was ± 1. □

In the situation of 6.2, 6.3 the rational calculation becomes quite simple. We have that the E_2 terms are given by

$$(6.4)\qquad E_2=\begin{cases}\mathbb{Q}[h_1,\dots,h_i,L)\otimes E[\sigma h_1,\dots,\sigma h_i,\sigma L] & m\text{ even}\\ \mathbb{Q}[L,\sigma h_1,\dots,\sigma h_i]\otimes E[\sigma L,h_1,\dots,h_i] & m\text{ odd.}\end{cases}$$

There is only one differential which does not lower bidegree since the h_i are spherical in X. It is given as

$$(6.5)\qquad \partial(\sigma L)=\begin{cases}\sum_k \alpha_{k,k}h_k^2 & m\text{ even,}\\ 2\sum h_{2j-1}h_{2j}\quad 1\leqq j\leqq\left(\dfrac{i}{2}\right) & \text{if } m \text{ is odd.}\end{cases}$$

When m is even the homology of the complex given by (6.4), (6.5) is additively isomorphic to the (graded) ring

$$(6.6)\qquad \mathbb{Q}[h_2,\dots,h_i]\otimes E[h_1]\otimes E[\sigma h_1,\dots,\sigma h_i]$$

with the multiplication given by $h_1^2 = \pm h_2^2 + \ldots + \pm h_i^2$. On the other hand, the complex in (6.4), (6.5) for m odd is exactly the one which has already appeared and been analyzed in § 5.

Remark 6.7 The situation at odd primes is similar to that for surfaces. But it is actually quite a bit worse from two respects. First, the ideals are much harder to manage, and second, there are now an infinite number of non-zero differentials.

7 Alternating products and deleted products of manifolds

In this section we give an elementary proof of the main results of [B-C-T], in case M is an orientable, closed manifold without boundary by using simple properties of alternating products, Poincaré duality, and some fundamental results of Dold, [D2]. The case where $\partial M \neq \emptyset$ can also be analyzed in this way, but the method does not work at all if M is not orientable.

Set $AP^n(X) = X^n/\mathscr{A}_n$, where $\mathscr{A}_n$ is the alternating group on n-letters. We also explicitly calculate the homology with coefficients $\mathbb{F}_p$, p an odd prime, of the alternating products $AP^n(X)$ for all connected spaces X having the homotopy types of CW-complexes. Let $\mathbb{F}^{\pm}$ denote the field $\mathbb{F}$ with an action of $\mathbb{Z}/2$ given by $a \mapsto \pm a$. The connection between our previous work and the $AP^n(X)$ is given by

Lemma 7.1 *Let X be a simplicial complex, then for all odd primes, p, there is a split exact sequence*

$$0 \to H_*(TP^n(X)/TP^{n-2}(X); \mathbb{F}_p^-) \to H_*(AP^n(X); \mathbb{F}_p) \to H_*(SP^n(X); \mathbb{F}_p^+) \to 0.$$

Proof. First, a map $e: SP^n(X) \to SP^2(AP^n(X)) \to SP^\infty(AP^n(X))$ is defined by

$$\langle x_1, \ldots, x_n \rangle \mapsto \langle [x_1, x_2, x_3, \ldots, x_n][x_2, x_1, x_3, \ldots, x_n] \rangle$$

where $[x_1, \ldots, x_n]$ denotes the equivalence class of $(x_1, \ldots, x_n)$ in $AP^n(X)$. The composition

$$AP^n(X) \xrightarrow{\ p\ } SP^n(X) \xrightarrow{\ e\ } SP^\infty(AP^n(X))$$

is $(i + Ti)$, where T: $AP^n(X) \to AP^n(X)$ is the evident $\mathbb{Z}/2$-action, and

$$i: AP^n(X) \to SP^\infty(AP^n(X))$$

is the usual inclusion. Since $T^2 = 1$ on $AP^n(X)$ we have a direct sum splitting

$$H_*(AP^n(X); \mathbb{F}_p) = H_*(AP^n(X); \mathbb{F}_p)^+ \oplus H_*(AP^n(X); \mathbb{F}_p)^-,$$

and on $H_*(AP^n(X); \mathbb{F}_p)^+$ the composite is just multiplication by 2, while the composite $SP^\infty(p) \circ e$: $H_*(SP^n(X); \mathbb{F}_p) \to H_*(SP^\infty(SP^n(X)); \mathbb{F}_p)$ is also just multiplication by 2. The identification of $H_*(AP^n(X); \mathbb{F}_p)^+$ with $H_*(SP^n(X); \mathbb{F}_p)$ follows.

It remains to study $H_*(AP^n(X); \mathbb{F}_p)^-$. First a simplicial decomposition of X^n is easily given (see e.g. [M2]), so that the simplices σ_i are either pointwise fixed by an element $\alpha \in \mathscr{S}_n$ or $\alpha\sigma_i \cap \sigma_i = \emptyset$. This gives rise to a cellular decomposition of $AP^n(X)$ so the singular set is a subcomplex. Since $T\equiv 1$ precisely on the cells of the singular set we see that $H_*(AP^n(X); \mathbb{F}_p)^-$ is given as the homology of the subcomplex $\frac{(T-1)}{2}\,\mathscr{C}_\#(AP^n(X))\otimes\mathbb{F}_p$, where by $\mathscr{C}_\#$ we mean the chain complex, and this is exactly

$$\mathscr{C}_\#(SP^n(X)/\mathrm{Sing}^n(X))\otimes\mathbb{F}_p^-.$$

7.1 follows. □

On the other hand, the main result of [D2] shows that $H_*(AP^n(X); \mathbb{F}_p)$ depends only on $H_*(X; \mathbb{F}_p)$, and we have immediately from 3.2

Corollary 7.2 [B-C-T] *Let X be a compact, connected and oriented manifold without boundary, then $H_*(C^n(X); \mathbb{F}_p)$ is a functor only of $H_*(X; \mathbb{F}_p)$ if X is odd dimensional and p is an odd prime, but also, $H_*(C^n(X); \mathbb{F}_p^-)$ is a functor only of $H_*(X; \mathbb{F}_p)$ if the dimension is even.*

It remains to describe the functors in 7.2. Here, again, [D2] reduces the problem to a much simpler one. In fact, we have

Lemma 7.3 (a) *For any odd prime p, to calculate $H_*(AP^n(X); \mathbb{F}_p)$ it suffices to calculate $H_*(AP^n(Y); \mathbb{F}_p)$ where Y is a wedge of spheres.*
(b) *If $Y=Z\vee W$ then*

$$\begin{aligned} H_*(AP^n(Y); \mathbb{F}_p) \cong & \sum_{j=0}^{n} H_*(SP^j(Z); \mathbb{F}_p)\otimes H_*(SP^{n-j}(W); \mathbb{F}_p) \\ & \oplus \sum_{j=0}^{n} H_+(TP^j(Z)/TP^{j-2}(Z); \mathbb{F}_p^-) \\ & \otimes H_*(TP^{n-j}(W)/TP^{n-j-2}(W); \mathbb{F}_p^-). \end{aligned}$$

(c) *If Y is a sphere S^n, then*

$$H_*(C^m(S^n); \mathbb{F}^\varepsilon)\cong H_*(C^m(\mathbb{R}^n); \mathbb{F}^\varepsilon)\oplus \Sigma^n H_*(C^{m-1}(\mathbb{R}^n); \mathbb{F}^\varepsilon)$$

where, $\varepsilon=+$ if n is odd $\varepsilon=-$ for n even.

Proof. The first statement is direct from [D2]. Specifically, Dold shows that any two CW-complexes, L, M, with the same $\mathbb{F}_p$ homology will satisfy $H_*(AP^n(L); \mathbb{F}_p)\cong H_*(AP^n(M); \mathbb{F}_p)$. To see (b), note that

$$(7.4)\qquad \begin{aligned} AP^n(Z\vee W) = & \bigvee_{j=1}^{n-1} AP^j(Z)\times AP^{n-j}(W)/\{(x,y)\sim(Tx,Ty)\} \\ & \vee AP^n(Z)\vee AP^n(W) \end{aligned}$$

where T is the involution on $AP^j(\ )$. Next, apply 8.1 to each of the summands. Tensoring the homology groups and taking the invariant subgroup under $T\times T$ gives the result.

Finally, to show (c), we again use a result from [D2]. In [D2, §9, especially p. 76], Dold shows that if we filter $AP^n(X)$, by saying that

$\{x_1, \ldots, x_n\} \in \mathbb{F}_r(AP^n(X))$ if and only if there are at least r copies of the base point $*$ in $\{x_1, \ldots, x_n\}$, then

$$\tilde{H}_*(AP^n(X); \mathbb{F}_p) = \sum_0^{n-1} \tilde{H}_*(\mathbb{F}_r(AP^n(X))/\mathbb{F}_{r+1}(AP^n(X)); \mathbb{F}_p).$$

In particular, the filtering pieces for $AP^n(X)$ are given explicitly as

$$\begin{cases} SP^{n-r}(X)/SP^{n-r-1}(X) & r \geqq 2 \\ AP^{n-1}(X)/\mathbb{F}_1(AP^{n-1}(X)) & r = 1 \\ AP^n(X)/\mathbb{F}_1(AP^n(X)) & r = 0. \end{cases}$$

Now, applying 7.1 and our usual duality techniques, it follows that, for n odd, and p an odd prime, $H_*(AP^m(S^n)/\mathbb{F}_1(AP^m(S^n); \mathbb{F}_p)^- = H^{nm-*}(C^m(S^n - *); \mathbb{F})$, while for n even we have $H_*(AP^m(S^n)/\mathbb{F}_1(AP^m(S^n); \mathbb{F}_p)^- = H^{nm-*}(C^m(S^n - *); \mathbb{F}_p^-)$. This completes the proof of 7.3. □

Thus, it suffices to determine the groups $H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^{\pm})$. Let $\mathbb{F}(X, k)$ denote the configuration space of ordered k-tuples of distinct points in X. Probably the easiest way to make this calculation is to use the well known calculations of the cohomology of the spaces

$$D_{n,m}(S^l) = \mathbb{F}(\mathbb{R}^n, m)_+ \wedge_{\mathscr{S}_m} (S^l)^{(m)}$$

as split summands of the homology of the loop space $\Omega^n S^{n+l}$. In [May] it was shown that a model for $\Omega^n S^n(X)$ was given as a filtered object by the construction

$$J_n(X) = \coprod_1^\infty \mathbb{F}(\mathbb{R}^n, k) \times_{\mathscr{S}_k} X^k/(z_1, \ldots, z_k, x_1, \ldots, x_k)$$
$$\sim (z_1, \ldots, \hat{z}_i, \ldots, z_k, x_1, \ldots, \hat{x}_i, \ldots, x_k) \quad \text{whenever } x_i = *,$$

for connected X homotopy equivalent to CW-complexes and [Sn] showed that stably $J_n(X) \simeq \bigvee_1^\infty D_{n,m}(X)$. $H_*(\Omega^n \Sigma^n X; \mathbb{F})$ was calculated in [M3] for any connected CW-complex X. On the other hand, $D_{n,m}(S^l)$ is just the Thom complex of the vector bundle

$$\mathbb{R}^{ml} \to \mathbb{F}(\mathbb{R}^n, m) \times_{\mathscr{S}_m} (\mathbb{R}^l)^m \to C^m(\mathbb{R}^n),$$

and as a result we have

$$(7.5) \qquad H_{lm+*}(D_{n,m}(S^l); \mathbb{F}_p) \cong \begin{cases} H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^+) & l \text{ even} \\ H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^-) & l \text{ odd}, \end{cases}$$

so the calculation is complete.

Remark 7.6 The arguments in [D2] are quite elementary, involving, almost entirely, formal properties of semi-simplicial complexes. As the exposition given there can hardly be improved on, we will not summarize the proofs here, but urge the interested reader to read [D 2].

Remark 7.7 The splitting in 7.3(c) does not hold when n is even and ε is $+$, since, in the spectral sequence of type 4.11 converging to $H_*(TP^m(S^{2n})/TP^{m-2}(S^{2n}); \mathbb{F}_p)$, the differential $d_2(\sigma f) = 2f$ is non-trivial. This is already seen in the remarks after (5.2). When we are looking at $H_*(TP^m(S^{2n})/TP^{m-1}(S^{2n}); \mathbb{F}_p)$, we do not see this differential, and $E^2 = E^\infty$.

Remark 7.8 In some ways the calculation indicated above for $H_*(C^m(\mathbb{R}^n); \mathbb{F}_p)$ is unaesthetic. The point of view, here, is that the structure of symmetric products should be logically prior to the homology of iterated loop spaces, although the two points of view are essentially equivalent. (In hindsight we can see that the essential techniques for both approaches lie in the results of [D-T] on the criteria for a map to be a quasi-fibration.) In any case, in the next few paragraphs we give a direct method for obtaining the twisted homology groups above.

Lemma 7.9 *There is a natural injection*

$$e_n: H_*(SP^n(X)/\{\mathrm{Sing}^n(X) \cup SP^{n-1}(X)\}; \mathbb{F}_p^-) \to H_{*+n}(SP^n(\Sigma X)/SP^{n-1}(\Sigma X); \mathbb{F}_p^+)$$

for X any connected CW-*complex.*

Outline proof. One begins by replacing X by a simplicial complex, then replaces X^n by a simplicial complex for which each $\alpha \in \mathscr{S}_n$ satisfies either $\alpha\sigma \cap \sigma = \emptyset$ or $\alpha\sigma = \sigma$ and σ fixes σ pointwise for each simplex σ in X^r. In particular, the images of these simplices give cell decompositions for X^n/Γ, for all subgroups $\Gamma \subset \mathscr{S}_n$.

Next a *cellular* decomposition of $(\Sigma X)^n$ is given. Its cell are (generally) the products of the cells of X^n with I^n, but when the cells of X^n are in the singular set (necessarily a subcomplex by our assumptions above), more care must be taken. A simplicial decomposition of I^r is given, with the r-cells indexed by $\mathscr{S}_r$ as follows

$$\sigma_\alpha^r = \{(t_1, \dots, t_r) \mid t_{\alpha(1)} \leqq t_{\alpha(2)} \leqq \dots \leqq t_{\alpha(r)}\}.$$

Then, over the singular set, note the ordering of the suspension coordinates, so there the cells have the form $\sigma \times I^{n-r} \times \sigma_{\alpha_1}^{r_1} \times \dots \times \sigma_{\alpha_v}^{r_v}$ with $r = r_1 + \dots + r_v$, σ a simplex in the decomposition of X^n, and the simplexes $\sigma_{\alpha_i}^{r_i}$ occuring where the coordinates in σ are equal.

We now take the images of the cells above in $SP^n(\Sigma X)$, and study the resulting chain complex. A direct calculation shows that if $\sigma \notin \mathrm{Sing}^n(X)$, but a face $F_i(\sigma) \in \mathrm{Sing}^n(X)$ then in $\partial(\sigma \times I^n)$ the coefficient involving $F_i(\sigma) \times (?)$ is always 0. Moreover, the remainder of the boundary, is $(\partial_-(\sigma)) \times I^n$, where ∂_- denotes the boundary with twisted coefficients. It follows that the chain complex of $SP^n(X)/\mathrm{Sing}^n(X)$ with twisted coefficients, but suspended up n dimensions, is a direct summand of a chain complex for $SP^n(\Sigma X)$.

Finally, we should note the collapsing which occurs when we are at base point. This lowers dimension by at least 1 and forces us to include the basepoint conditions in 7.9. The result follows. □

As an example, we easily check that $AP^n(S^1)/AP^{n-1}(S^1) \simeq S^n$, so

$$H_*(SP^n(S^1)/\{\mathrm{Sing}^n(S^1) \cup SP^{n-1}(S^1)\}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise,} \end{cases}$$

and this class suspends up to the generator for $H_{2n}(SP^n(S^2); \mathbb{Z}) = \mathbb{Z}$.
Next, an easy induction, based on the bar construction shows

Lemma 7.10 *When X is S^n the injection e_n of 7.9 is an isomorphism.*

Thus, we obtain a complete calculation of the twisted homology groups needed in 7.3, entirely within the context of symmetric products and Eilenberg-MacLane spaces.

Part II: Configuration spaces and mapping spaces

The purpose of this part is to continue the observations of [B-C-T] giving the homology of some configuration spaces. As at the end of § 7 we write $\mathbb{F}(M, k)$ for the subspace of M^k given by $\{(m_1, \ldots, m_k) | m_i \neq m_j \text{ if } i \neq j\}$ and as usual $C^k(M)$ for the orbit space $\mathbb{F}(M, k)/\mathscr{S}_k$ where the symmetric group $\mathscr{S}_k$ acts by permuting coordinates. Throughout Part II M is assumed to be a smooth (not necessarily closed) manifold and all homology is taken with coefficients in the field $\mathbb{F}$. With $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and M closed, the homology of $C^k(M)$ was first given in [L-M].

Among the main results here is a short proof of the results in [B-C-T] giving the homology of $C^k(M)$ if at least one of the following is satisfied: (1) $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, (2) dimension$(M)$ is odd, or (3) dimension(M) is even and $\mathscr{S}_k$ acts on $\mathbb{F}$ by the sign representation. Also product decompositions for certain function spaces give results for surfaces which, in rational cohomology, generalize to $(n-1)$-connected $2m$-manifolds. These product decompositions give the cohomology of certain other configuration spaces.

8 Results

Let V denote a graded vector space over $\mathbb{F}$ with V concentrated in degrees at least one. Write

$$V^{\otimes k} = V \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} V$$

as a $\mathscr{S}_k$-module where $\mathscr{S}_k$ acts by permutation of coordinates with the usual sign conventions. If $v_i \in V$ write V_α for the cyclic $\mathscr{S}_k$-module generated by $\alpha = v_1 \otimes \ldots \otimes v_k$. There are 3 important examples of $\bar{V}_\alpha$ here:

(i) $V = \mathbb{F}$ concentrated in an even degree. Thus $\bar{V}_\alpha$ is a trivial $\mathscr{S}_k$-module.
(ii) $V = \mathbb{F}$ concentrated in an odd degree. Thus $\bar{V}_\alpha$ is given by the sign representation of $\mathscr{S}_k$, and we write $\bar{V}_\alpha = \mathbb{F}(-1)$ here.
(iii) The dimension of V is at least k and $v_1, \ldots, v_k$ are linearly independent. Then $\bar{V}_\alpha$ where $\alpha = v_1 \otimes \ldots \otimes v_k$ is isomorphic to $\mathbb{F}\mathscr{S}_k$ as a $\mathscr{S}_k$-module.

Next write $S_* X$ for the singular chains of a space X with a non-degenerate base point, and define

$$H_*(C^k(M);\bar{V}_\alpha)=H_*(S_*\,\mathbb{F}(M,k)\otimes_{\mathscr{S}_k}\bar{V}_\alpha).$$

If S_V is a bouquet of spheres with $\bar{H}_*(S_V)$ isomorphic to V as a graded vector space, then there are isomorphisms (over any field)

$$H_*(C^k(M);V^{\otimes k})\cong H_*\,D_k(M,S_V)$$

where $D_k(M, S_V)$ is the quotient space

$$\mathbb{F}(M,k)\times_{\mathscr{S}_k}X^{(k)}/\mathbb{F}(M,k)\times_{\mathscr{S}_k}*$$

with $*$ the base-point in X and $X^{(k)}$ is the k-fold smash product. Furthermore, there are isomorphisms

$$H_*(C^k(M);V^{\otimes k})\cong\bigoplus_\alpha H_*(C^k(M);V_\alpha)$$

where the right hand sum is over the obvious index set. For example if either (1) $M=\mathbb{R}^n$ or (2) if $M=M'\times\mathbb{R}^n$ and $\mathbb{F}=\mathbb{Q}$, the groups $H_*(C^k(M);V_\alpha)$ are implicit in [CT1].

We now determine $H_*(C^k(M);V^{\otimes k})$ as a direct summand of the homology of a much larger space $C(M, X)$. Namely, write

$$C(M,X)=\left\{[S,f]\,\middle|\,\begin{matrix}(1)\ S\text{ is a finite subset of }M\\(2)\ f\colon S\to X\end{matrix}\right\}/(\sim)$$

where $(\sim)$ is the equivalence relation generated by $[S,f]\sim[S-\{q\},f|_{S-\{q\}}]$ if and only if $f(q)=*$. The space $C(M,X)$ stably splits as $\bigvee_{k\geqq 1}D_k(M,X)$ for path-connected X, [B2], (the original reference here is an unpublished article of F. Cohen and L. Taylor but their proof is reproduced here). The space $C(M,X)$ is filtered with $F_j\,C(M,X)$ represented by $[S,f]$ where S is of cardinality at most j; similarly $H_*(C(M,X))$ has an induced filtration. Thus there are isomorphisms

$$\begin{aligned}\bar{H}_*\,C(M,X)&\cong\bigoplus_{k\geqq 1}\bar{H}_*\,D_k(M,X)\\&\cong\bigoplus_{k\geqq 1}H_*(C^k(M);\bar{H}_*\,X^{\otimes k}).\end{aligned}$$

We shall check the following two observations.

Lemma 8.1 *There are isomorphisms of vector spaces*

$$\bar{H}_q\,D_k(M,X)\to\bar{H}_{q+2nk}\,D_k(M,\Sigma^{2n}X).$$

Lemma 8.2 *Assume that X is $(q-1)$-connected, $H_j(X;\mathbb{F})=0$ if $j>N$ and the dimension of M is m. Then*

(1) *$D_t(M,X)$ is $(qt-1)$-connected,*

(2) *$H_j(D_t(M,X);\mathbb{F})=0$ if $j>t(m+N)$,*

(3) *If ϕ is any homology isomorphism of $H_* C(M, X)$, then ϕ also preserves filtration degrees in homological degree less than $t(m+N)$ if $t(m+N-q)<q$, and*
(4) *If $n \geqslant N-q$ there is an isomorphism*

$$H_s D_t(M, X) \cong H_{s+2nt} C(M, \Sigma^{2n} X)$$

provided $H_s D_t(M, X)$ is non-zero.

The point of 8.1 and 8.2 is that we know $H_* D_t(M, X)$ and hence we have

$$H_*(C^t(M); V^{\otimes t})$$

provided that we know $H_* D_t(M, X)$ and thus $H_* C(M, \Sigma^{2n} X)$ as a vector space for all n. In some cases such as [B-C-T] these answers follow directly.

In particular if $D^m = [0, 1]^m$ with (1) $\bar{M} = M \cup D^m$ and (2) $M \cap D^n = [0, 1]^{n-q} \times \partial [0, 1]^q$ then we have by following work of Segal, Gromov, Dold and Thom

Proposition 8.3 [MD, B1] *If X is a connected* CW-*complex then up to homotopy equivalence there are fibrations*

$$C(M, X) \to C(\bar{M}, X) \to \Omega^{m-q} \Sigma^m X.$$

As a corollary we get the following where $\beta_q = \dim_{\mathbb{F}} H_q(M; \mathbb{F})$.

Corollary 8.4 [B-C-T] (i) *If $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ or m is odd, then $H_* C^k(M)$ depends only on $H_* M$ and m. Furthermore there is an isomorphism of vector spaces for $n >> 0$ and $0 < q < mk$ given by*

$$\begin{aligned} H_q C^k(M) &\cong H_{q+2nk} C(M, S^{2n}) \\ &\cong H_{q+2nk}\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n}\right). \end{aligned}$$

(ii) *if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ or m is even, then $H_*(C^k(M); \mathbb{F}(-1))$ depends only on $H_* M$ and m. Furthermore, there is an isomorphism of vector spaces for $n >> 0$ and $0 < q < mk$ given by*

$$\begin{aligned} H_q(C^k(M); \mathbb{F}(-1)) &\cong H_{q+(2n+1)k} C(M, S^{2n+1}) \\ &\cong H_{q+(2n+1)k}\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right). \end{aligned}$$

(Compare 7.3 where we must assume M is oriented.)

Next we restrict attention to a closed orientable Riemann surface of genus g, M_g. Write $\hat{M}_g = M_g - \{\text{point}\}$ and $\text{map}_*(,)$ for the pointed mapping space. If $\text{map}_*(X, Y)$ is not connected, write $\text{map}_*^0(X, Y)$ for the component of the constant map. The next theorem gives the homotopy type of the mapping space $\text{map}_*(M_g, S^{2n})$.

Theorem 8.5 *If $n > 1$, there exist principal $\Omega^2 S^{2n}$ fibrations*

$$\Omega^2 S^{2n} \to Y_{g,2n} \to (S^{2n-1})^{2g}$$

and a principal $\Omega^2 S^3$-fibration

$$\Omega^2 S^3 \to Y_{g,2} \to (S^1)^{2g}$$

such that

(1) *there is a homotopy equivalence* $\mathrm{map}_*^0(M_g, S^2) \xrightarrow{\simeq} (\Omega S^3)^{2g} \times Y_{g,2}$ *and*
(2) *there is a homotopy equivalence after inverting* 6

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$$

for $n > 1$,
(3) *there is a fibration for* $n > 1$

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}$$

which has a section after inverting 2.

Theorem 8.6 *After localizing at p, $p > 3$, there are stable decompositions*

$$Y_{g,2n} \xrightarrow[\text{stable}]{\simeq} \bigvee_{k \geqq 1} D_k(Y_{g,2n}) \text{ and}$$

$$C^k(\hat{M}_g) \xrightarrow[\text{stable}]{\simeq} \bigvee_{i=0}^{k} D_i(Y_{g,2n}) \wedge D_{k-i}(\Omega S^3)^{2g}$$

where $(\Omega S^3)^{2g}$ is given the product filtration obtained from the James filtration [J]. *Thus there are isomorphisms*

$$\bar{H}_N C^k(\hat{M}_g) \xrightarrow{\simeq} \bigoplus_{I_j} \bigoplus_{i} \bar{H}_{N-2j} D_i(Y_{g,2})$$

where I_j is the number of ordered partitions of j having length $2g$, $(\alpha_1, \ldots, a_{2g})$.

Thus it suffices to understand the homology of the spaces $Y_{g,2n}$ in order to know the homology of $C^k(\hat{M}_g)$. As in (5.2) in studying $H_* Y_{g,2n}$ the exterior algebra

$$\Delta = E[x_1, \ldots, x_{2g}, z]$$

with differential given by

$$d(z) = 2\left(\sum_{i=1}^{g} x_{2i-1} x_{2i}\right),$$

is needed but here the degrees are specified by $|x_i| = 2n-1$, $|z| = 4n-3$.

Theorem 8.7 *Assume that n is greater than one.*

(1) *In characteristic zero there are isomorphisms*

$$H^* C(\hat{M}_g, S^{2n-2}) \xrightarrow{\simeq} H_*(\Delta \otimes \mathbb{Q}; d) \otimes H^*(\Omega S^{4n-1})^{2g} \otimes H^* \Omega S^{2n-1}.$$

(2) *In characteristic p there are isomorphisms*

$$H^* C(\hat{M}_g, S^{2n-2}) \xrightarrow{\simeq} H^* Y_{g,2n} \otimes H^*(\Omega S^{3n-1})^{2g},$$

and
(3) *In characteristic n* with $p \geqq g$, *there are isomorphisms*

$$H^* Y_{g,2n} \xrightarrow{\simeq} H_*(\Delta \otimes \mathbb{F}; d) \otimes H^* \Omega S^{2n-1} \otimes H^* \Omega^2 S^{4n-1} // \Lambda[z].$$

Remark 8.8 8.7(1) is, of course, dual to 5.3, (2) is immediate, and (3) is dual to 5.12.

Remark 8.9 We conjecture that the stable decompositions above for $\mathrm{map}_*(M_g, S^{2n})$ and $Y_{g,2n}$ are also satisfied at $p=3$.

9 Proof of 8.1, 8.2 and 8.4

We first prove Lemma 8.1. Notice that if X_t is the fat wedge in X^t, then there is a cofibration [May]

$$\mathbf{F}(M,t) \times_{\mathscr{S}_t} X_t \to \mathbf{F}(M,t) \times_{\mathscr{S}_t} X^t \to D_t(M,X).$$

Next recall that $H_*(\mathbf{F}(M,t) \times \mathscr{S}_t X^t)$ is the homology of the chain complex

$$S_* \mathbf{F}(M,t) \otimes_{\mathscr{S}_t} S_* X^{\otimes t}$$

and that over $\mathbb{F}$, there is a homology isomorphism $g\colon H_* X \to S_* X$ given by sending a class (in a choice of basis) to a representing cycle. Thus $H_*(\mathbf{F}(M,t) \times_{\mathscr{S}_t} X^t)$ is the homology of the complex $S_* \mathbf{F}(M,t) \otimes_{\mathscr{S}_t} H_* X^{\otimes t}$ and 8.1 follows as there is an isomorphism of $\mathscr{S}_t$-modules

$$(H_* X)^{\otimes t} \to (H_{*+2} \Sigma^2 X)^{\otimes t}.$$

Next we give the proof of 8.2: Notice that 8.2 (i) and 8.2 (ii) follow immediately from the cofibration given above. To check 8.2 (iii), let a_j be a non-zero element in $\bar{H}_*(D_j(M,X); \mathbb{F})$ and $|a_j|$ it's degree. Then $j(q) \leqq |a_j| \leqq j(N+m)$ if $j<t$. But since $t(m+N-q)<q$ by assumption, we have $j(m+N)<(j+1)q$ and the results in 8.2 (iii) and 8.2 (iv) follow.

Finally in this section we prove 8.4: By the above remarks it suffices to check that these are isomorphisms

$$H_* C(\bar{M}, S^{2n}) \cong H_*\left(\prod_{q=0}^{n} \prod_{\beta(q)} \Omega^{n-q} S^{m+2n}\right)$$

for m odd and

$$H_*(C(\bar{M}, S^{2n+1})) \cong H_*\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right)$$

for m even.

This follows directly from naturality of the Serre spectral sequence, induction on a handle decomposition for $\bar{M}$ and that the stabilization map $\Omega^i S^N \to \Omega^\infty \Sigma^\infty S^{N-i}$ is a monomorphism in homology when $\mathbb{F} = \mathbb{F}_2$ or $N-i$ is odd: Namely consider the map of fibrations

$$\begin{array}{ccc} C(\bar{M}, S^{2n}) & \longrightarrow & \Omega^\infty \Sigma^\infty (\bar{M} \times S^q / \bar{M} \times *) \\ \downarrow & & \downarrow \\ \Omega^{m-q} S^{m+n} & \longrightarrow & \Omega^\infty \Sigma^\infty (S^{n+q}). \end{array}$$

If $m+n$ is odd, the homology of $C(\bar{M}, S^n)$ injects in the homology of $\Omega^\infty \Sigma^\infty (\bar{M} \times S^n / \bar{M} \times *)$ by induction where $\bar{M}$ is as in Proposition 8.3. The result follows.

10 Examples related to surfaces

If M is a closed, stably parallelizeable manifold then one has $C(M - \{\text{point}\}, \Sigma^l X)$ is homotopy equivalent to the pointed mapping space $\text{map}_*(M, \Sigma^l X)$ for some l, [B2]. If M is $(2m)$-dimensional and connected up to the middle dimension then M is very close to being parallelizable. If m is even, its index must be zero, and there is one further condition: M cannot contain any (connected sum) summands which are fibrations $S^m \to E \to S^m$, the associated sphere bundles of vector bundles that are stably non-trivial. In particular all the Riemann surfaces M_g^2 are stably parallelizable, $g > 0$.

Hence, in these cases we have, first, a stable splitting of $\text{map}_*(M, \Sigma^m X)$ using the stable splitting of $C(M - \{\text{point}\}, X)$, and second, a good homotopy type description of $\text{map}_*(M, \Sigma^m X)$ since there is a fibration

$$\text{map}_*(M, \Sigma^m X) \to (\Omega^m \Sigma^m X)^r \xrightarrow{w^*} \Omega^{2m-1} \Sigma^m X$$

where w^* is the "hom-dual" of the attaching map $w: S^{2m-1} \to \bigvee_r S^m$ with cofiber M. Thus one can determine the homology of $C(M - \{\text{point}\}, X)$ by using this fibration (and either the Eilenberg-Moore spectral sequence or backing up and using the Serre spectral sequence).

In case $M = M_g$ is a closed orientable Riemann surface of genus g, there is a map

$$w_g: S^1 \to \bigvee_{2g} S^1$$

with cofiber M_g and w_g is given by sending a generator of $\pi_1(S^1, *)$ to

$$[x_1, x_2] \cdot [x_3, x_4] \dots [x_{2g-1}, x_{2g}]$$

where the x_i run over a choice of generators for $\pi_1(\bigvee_{2g} S^1, *)$. Write

$$ad: \Omega X \times \Omega X \to \Omega X$$

for the commutator $[f,g]=fgf^{-1}g^{-1}$. Let μ_g: $(\Omega X)^g \to \Omega X$ be given as loop multiplication taken in a given, fixed order. By definition we obtain the following lemma,

Lemma 10.1 *There is a homotopy commutative diagram*

$$\begin{array}{ccc}
\operatorname{map}_*(\bigvee_{2g} S^1, X) & \xrightarrow{w_g^*} & \Omega X \\
\Big\downarrow = & & \Big\downarrow \mu_g \\
[(\Omega X)^2]^g & \xrightarrow{(a\ d)^g} & (\Omega X)^g
\end{array}$$

where w_g^* *is the "hom-dual" of* w_g.

Thus a first step to computing the cohomology of $\operatorname{map}_*(M_g, X)$ is to understand w_g^* in cohomology. We shall work out a single example here where $X=S^{2n}$. Recall that there are isomorphisms of Hopf algebras

$$H^*(\Omega S^{2n})^{2g} \cong E[x_1, \dots, x_{2g}] \otimes \Gamma[y_1, \dots, y_{2g}]$$

and

$$H^* \Omega S^{2n} \cong E[u] \otimes \Gamma[v]$$

with $|x_i|=|u|=2n-1$ and $|y_i|=|v|=4n-2$. Applying Lemma 10.1 in cohomology we get

$$(w_g^*)^*(u)=0 \text{ and}$$

$$(w_g^*)^*(v)=2\sum_{i=1}^{q} x_{2i-1}x_{2i}.$$

Next we compare the morphism of fibrations in cohomology given by

$$\begin{array}{ccc}
\Omega^2 S^{2n} & \xrightarrow{1} & \Omega^2 S^{2n} \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{map}_*(M_g, S^{2n}) & \longrightarrow & * \\
\Big\downarrow & & \Big\downarrow \\
(\Omega S^{2n})^{2g} & \xrightarrow{w_g^*} & \Omega S^{2n}.
\end{array}$$

The Serre spectral sequence in mod-2 cohomology collapses while the image of the stabilization map

$$H^*(QS^{2n-2}; \mathbb{F}) \to H^*(\Omega^2 S^{2n}; \mathbb{F})$$

consists of infinite cycles for any $\mathbb{F}$ by inspection of the map

$$\mathrm{map}_*(M_g, S^{2n}) \to \mathrm{map}_*(M_g, QS^{2n})$$

as $\mathrm{map}_*(M_g, QS^{2n})$ splits as $(QS^{2n-1})^{2g} \times QS^{2n-2}$. As a corollary, one gets the rational homology of $C(\hat{M}_g, S^{2n-2})$ and, so, again, that of $C^k(\hat{M}_g)$.

Theorem 10.2 *If $n>1$ and $\mathbb{F}=\mathbb{Q}$, there is an isomorphism*

$$H^* C(\hat{M}_g, S^{2n}) \cong H^*(\Lambda(x_1, \ldots, x_{2g}, z); d) \otimes H^*(\Omega S^{4n-1})^{2g} \otimes H^*(\Omega S^{2n-1})^{2g},$$

where $d(z)=2\sum x_{2i-1}x_{2i}$.

Proof. Recall that there is a homotopy equivalence $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$ after rationalization. Thus there is an equivalence $\Omega^2 S^{2n} \simeq \Omega S^{2n-1} \times \Omega^2 S^{4n-4} \cong \Omega S^{2n-1} \times S^{4n-3}$. In the Serre spectral sequence for

$$\Omega S^{2n-1} \times S^{4n-3} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{2n})^{2g},$$

the E_2-term is

$$E[x_1, \ldots, x_{2g}] \otimes \mathbb{Q}[y_1, \ldots, y_{2g}] \otimes \mathbb{Q}[w] \otimes \Lambda[z]$$

with $|w|=2n-2$ and $|z|=4n-3$. The differential $d=d_{4n-2}$ is given by $d(z)=2\sum x_{2i-1}x_{2i}$. By inspection, the algebra generators for E_{4n-1} are all infinite cycles.

11 Surfaces and product decompositions

In this section we again consider the pointed function spaces $\mathrm{map}_*(M_g, S^{2n})$, $n>1$, and $\mathrm{map}^0_*(M_g, S^2)$. Notice that there are fibrations

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{j} (\Omega S^{2n})^{2g} \xrightarrow{w_g^*} \Omega S^{2n}, n>1, \text{and}$$

$$\mathrm{map}^0_*(M_g, S^2) \xrightarrow{j} (\Omega S^2)^{2g} \xrightarrow{w_g^*} \Omega S^3.$$

Recall that there is a fibration (over $\mathbb{Z}$) given by James [J]

$$S^{2n-1} \longrightarrow \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$$

and thus a commutative diagram

$$\begin{array}{ccc}
\mathrm{map}_*(M_g, S^{2n}) & \xrightarrow{H^{2g}\cdot j} & (\Omega S^{4n-1})^{2g} \\
\downarrow j & & \downarrow 1 \\
(\Omega S^{2n})^{2g} & \xrightarrow{H^{2g}} & (\Omega S^{4n-1})^{2g}.
\end{array}$$

Enlarging the above to a commutative diagram giving a morphism of fibrations, we obtain spaces $Y_{g,2n}$ as follows:

$$
\begin{array}{ccccc}
\Omega^2 S^{2n} & \xrightarrow{1} & \Omega^2 S^{2n} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
Y_{g,2n} & \longrightarrow & \operatorname{map}_*(M_g, S^{2n}) & \longrightarrow & (\Omega S^{4n-1})^{2g} \\
\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1} \\
(S^{2n-1})^g & \longrightarrow & (\Omega S^{2n})^{2g} & \xrightarrow{H^{2g}} & (\Omega S^{4n-1})^{2g} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w_g^*} & & \downarrow \\
\Omega S^{2n} & \xrightarrow{1} & \Omega S^{2n} & \longrightarrow & *
\end{array}
$$

(where ΩS^{2n} is replaced by ΩS^3 if $n=1$ in the bottom row).

Theorem 11.1 *There are homotopy equivalences*
(1) $\operatorname{map}_*^0(M_g, S^2) \simeq (\Omega S^3)^{2g} \times Y_{g,2}$, *and*
(2) $\operatorname{map}_*(M_g, S^{2n}) \simeq (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$ *after inverting* 6.

Next, recall the bundle

$$v\colon \mathbf{F}(\hat{M}_g, k) \times_{\mathscr{S}_k} \mathbb{R}^k \to \mathbf{F}(\hat{M}_g, k)/\mathscr{S}_k$$

and

Proposition 11.2 [C²M²] *The bundle v has order* 4 *and hence there are homotopy equivalences* $D_k(\hat{M}_g, S^{n+4}) \simeq \Sigma^{4k} D_k(\hat{M}_g, S^n)$ *and* $\hat{D}_k(\hat{M}_g, S^4) \simeq \Sigma^{4k}(C^k(\hat{M}_g)_+)$.

Filter $(\Omega S^{2n+1})^{2g}$ by the product filtration obtained from the James filtration and write $D_k(\Omega S^{2n+1})^{2g}$ for the filtration quotients.

Theorem 11.3 *After localizing at p, $p>3$, $Y_{g,2n}$ stably splits as $\bigvee_{k\geq 1} D_k(Y_{g,2n})$ and there is a stable decomposition*

$$C^k(\hat{M}_g) \overset{\simeq}{\text{stably}} \bigvee_{i=0}^{k} D_i(Y_{g,2n}) \wedge D_{k-i}(\Omega S^3)^{2g}.$$

Thus there is an isomorphism

$$\bar{H}_N(C^k(\hat{M}_g); \mathbb{F}) \cong \bigoplus_{I_j} \bar{H}_{N-2j}(D_i(Y_{g,2}); \mathbb{F})$$

where I_j is the number of ordered partitions of j having length $2g$.

Theorem 11.4 *If $p \geqq g$ and $\mathbb{F}=\mathbb{Z}/p\mathbb{Z}$ for $p \geqq 3$, there are isomorphisms*

$$H^*(Y_{g,2n}) \xrightarrow{\simeq} H(\Lambda[x_1, \dots, x_{2g}, z]; d) \otimes H^*\Omega S^{2n-1} \otimes H^*\Omega^2 S^{4n-1}//\Lambda[z],$$

and

$$H^* \mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} H^* Y_{g,2n} \otimes H^*(\Omega S^{4n-1})^{2g}.$$

Proof of 11.1 Consider the unit sphere bundle in the tangent bundle of S^{2n}, $\tau(S^{2n})$. There is a bundle

$$S^{2n-1} \longrightarrow \tau(S^{2n}) \xrightarrow{\pi} S^{2n}$$

and $\tau(S^{2n})$, after localization at an odd prime is S^{4n-1}. Thus there is a p-local fibration

$$S^{2n-1} \longrightarrow S^{4n-1} \xrightarrow{\pi} S^{2n}$$

where π is the Whitehead square $[\iota_{2n}, \iota_{2n}]$. Since the triple Whitehead product

$$[\iota_{2n}, [\iota_{2n}, \iota_{2n}]] = 0$$

after localizing at primes at least 5, there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{4n-1} \vee S^{2n} & \xrightarrow{w_{2n} \vee 1} & S^{2n} \vee S^{2n} & \xrightarrow{\text{fold}} & S^{2n} \\ \downarrow & & & & \downarrow \\ S^{4n-1} \times S^{2n} & & \xrightarrow{\mu} & & S^{2n}_{(p)}. \end{array}$$

Using this action we illustrate the proof of 11.1 by proving the case (1) for the space $\mathrm{map}^0_*(M_g, S^2)$.

Since $\mathrm{map}_*(M_g, S^3)$ splits as $(\Omega S^3)^{2g} \times \Omega^2 S^3$, the action of S^3 on S^2 induces a homotopy commutative diagram

$$\begin{array}{ccccc} Y_{g,2} \times (\Omega S^3)^{2g} & \longrightarrow & Y_{g,2} \times \mathrm{map}_*(M_g, S^3) & \longrightarrow & \mathrm{map}^0_*(M_g, S^2) \\ \downarrow{\scriptstyle \text{project}} & & & & \downarrow{\scriptstyle H^{2g} \cdot j} \\ (\Omega S^3)^{2g} & & \xrightarrow{1} & & (\Omega S^3)^{2g}. \end{array}$$

But the map of $Y_{g,2} \times \{\text{point}\}$ induced by the action is the inclusion of a fibre in the space $\mathrm{map}^0_*(M_g, S^2)$ as $Y_{g,2}$ is also the fibre in

$$Y_{g,2} \to \mathrm{map}^0_*(M_g, S^2) \to (\Omega S^3)^{2g}$$

obtained from the diagram

$$\begin{array}{ccccc}
\Omega^2 S^3 & \xrightarrow{1} & \Omega^2 S^3 & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
Y_{g,2} & \longrightarrow & \mathrm{map}^0_*(M_g, S^2) & \longrightarrow & (\Omega S^3)^{2g} \\
\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1} \\
(S^1)^{2g} & \longrightarrow & (\Omega S^2)^{2g} & \xrightarrow{H^{2g}} & (\Omega S^3)^{2g} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega S^3 & \longrightarrow & \Omega S^3 & \longrightarrow & *.
\end{array}$$

The theorem follows.

Remark 11.5 The fibration $Y_{g,2} \to (S^1)^{2g}$ is classified by a map

$$(S^1)^{2g} \to \Omega S^3$$

which sends the generator of $H^2(\Omega S^3, \mathbb{Z})$ to $2(\sum x_{2i-1} x_{2i})$.

Remark 11.6 When localized at $p=3$, there is a p-local fibration

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}, n > 1.$$

This fibration has a cross-section although we don't know that it is a trivial fibration. Nevertheless notice that there is a diagram of fibrations

$$\begin{array}{ccccc}
\mathrm{map}_*(M_g, S^{2n-1}) & \xrightarrow{1} & \mathrm{map}_*(M_g, S^{2n-1}) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 S^{4n-1} & \longrightarrow & \mathrm{map}_*(M_g, \tau S^{2n}) & \longrightarrow & (\Omega S^{4n-1})^{2g} \\
\downarrow & & \downarrow & & \downarrow \\
Y_{g,2n} & \longrightarrow & \mathrm{map}_*(M_g, S^{2n}) & \longrightarrow & (\Omega S^{4n-1})^{2g}
\end{array}$$

Furthermore, since τS^{2n} is a 3-local H-space, S^{4n-1}, the function space $\mathrm{map}_*(M_g, \tau S^{2n})$ is homotopy equivalent to a product $(\Omega S^{4n-1})^{2g} \times \Omega^2 S^{4n-1}$.

Proof of 11.3 If $p > 3$, this theorem follows from the action $S^{4n-1} \times S^{2n} \to S^{4n-1}$ given in the proof of 11.1 together with the periodicity in 8.1, 8.2, and 11.2. The proof is analogous to that given in [$\mathrm{C}^2\mathrm{M}^2$] for the stable decomposition of $D_k(\hat{M}_g, S^{2n-1})$ and we omit the details.

Proof of 11.4 Consider the morphisms of fibrations

$$\begin{array}{ccc} \Omega^2 S^{2n} & \xrightarrow{1} & \Omega^2 S^{2n} \\ \downarrow & & \downarrow \\ Y_{g,2n} & \longrightarrow & * \\ \downarrow & & \downarrow \\ (S^{2n-1})^{2g} & \longrightarrow & \Omega S^{2n} \end{array}$$

and the p-local decomposition $\Omega^2 S^{2n} \simeq \Omega S^{2n-1} \times \Omega^2 S^{4n-1}$. Since a direct degree check gives that the A-module indecomposables of $H^* \Omega^2 S^{4n-1}$: degrees greater than $4n-3$, must be infinite cycles by naturality. The theorem follows for $Y_{g,2n}$.

To check the case of $\mathrm{map}_*(M_g, S^{2n})$, appeal to the product decomposition for the space $\mathrm{map}_*^0(M_g, S^2)$ and use periodicity in 11.2.

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