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Math. Ann. 251, 249–252 (1980)

Mathematische Annalen

Splitting the Künneth Sequence in K-Theory. II

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In [4] we proved that the complex K-theory Künneth sequence

$$0 \to \tilde{K}^{\#}(X) \otimes \tilde{K}^{\#}(Y) \to \tilde{K}^{\#}(X \wedge Y) \xrightarrow{\tau} \operatorname{Tor}(\tilde{K}^{\#}(X), \tilde{K}^{\#}(Y)) \to 0$$

splits if X and Y are compact metric spaces, extending results of Buhštaber and Miščenko [5], Mislin [8], Puppe [9], and Anderson [1]. As an addendum to [4] we give here a more direct proof of the general result.

Theorem A. *The Künneth sequence splits for all compact* X, Y.

The proof relies essentially on the observation that the universal coefficient sequences

$$\begin{array}{ccccccccc} 0 \to & \tilde{K}^{\#}(X) \otimes \mathbb{Z}_n & \xrightarrow{\varrho_n} & \tilde{K}^{\#}(X;n) & \xrightarrow{\beta_n} & \operatorname{Tor}(\tilde{K}^{\#}(X), \mathbb{Z}_n) & \to 0 \\ & \uparrow{\scriptstyle \kappa'_{n,m}} & & \uparrow{\scriptstyle \kappa_{n,m}} & & \uparrow{\scriptstyle \kappa''_{n,m}} & \\ 0 \to & \tilde{K}^{\#}(X) \otimes \mathbb{Z}_m & \xrightarrow{\varrho_m} & \tilde{K}^{\#}(X;m) & \xrightarrow{\beta_m} & \operatorname{Tor}(\tilde{K}^{\#}(X), \mathbb{Z}_m) & \to 0 \end{array}$$

admit splittings which commute with coefficient homomorphisms.

Theorem B. *The universal coefficient sequences split naturally in* n, *i.e. for each* n *there is an* $s_n : \tilde{K}^{\#}(X)[n] = \operatorname{Tor}(\tilde{K}^{\#}(X), \mathbb{Z}_n) \to \tilde{K}^{\#}(X;n)$ *such that*

(i) $\beta_n s_n = 1$, *for all* n;
(ii) $\kappa_{n,m} s_m = s_n \kappa''_{n,m}$, *for all* n, m.

This improves a result of [4] and holds in addition for all "good" cohomology theories in the sense of Deleanu and Hilton [6].

1. Proof of Theorem A, Assuming Theorem B

First we have to mention that several times we use results proved in [2, 3] and [9] only for finite CW complexes; but in [4, Sects. 2, 3] we always pointed out why

0025-5831/80/0251/0249/$01.00

these results hold for compact spaces with the same proofs. All definitions and notation are taken from [4].

Let T be the free abelian group generated by all triples (x, n, y) with $x \in \tilde{K}^{\#}(X)$, $y \in \tilde{K}(Y)$ and $n \geqq 2$ a natural number such that $nx = ny = 0$. The torsion product $\operatorname{Tor}(\tilde{K}^{\#}(X), \tilde{K}^{\#}(Y))$ is the quotient of T by the subgroup R generated by all elements of the following forms [7, Sect. 62, p. 264]:

$$\begin{array}{lll}
\text{(a)} & (x,n,y)+(x',n,y)-(x+x',n,y), & nx=nx'=ny=0, \\
\text{(b)} & (x,n,y)+(x,n,y')-(x,n,y+y'), & nx=ny=ny'=0, \\
\text{(c)} & (x,nm,y)-(nx,m,y), & nmx=my=0, \\
\text{(d)} & (x,nm,y)-(x,n,my), & nx=nmy=0.
\end{array}$$

Denote by $[x, n, y]$ the image of the generator (x, n, y) under the canonical quotient map $\Theta: T \to T/R = \operatorname{Tor}(\tilde{K}^{\#}(X), \tilde{K}^{\#}(Y))$. Let $K = \bigoplus_{n \geqq 2} \tilde{K}^{\#}(X; n) \otimes \tilde{K}^{\#}(Y; n)$ and consider the diagram

$$\begin{array}{ccc}
K & \xleftarrow{\quad \bar{S} \quad} & T \\
{\scriptstyle \Psi}\downarrow & \searrow^{-\Gamma} & \downarrow{\scriptstyle \Theta} \\
\tilde{K}^{\#}(X \wedge Y) & \xrightarrow[\tau]{} & \operatorname{Tor}(\tilde{K}^{\#}(X), \tilde{K}^{\#}(Y))
\end{array}$$

Ψ is given by $\beta_n \mu_n$ on each component, where μ_n is any admissible multiplication [2, 3], and β_n is the Bockstein. Likewise, Γ is determined by $\Gamma(x_n \otimes y_n) = [(-1)^i \beta_n(x_n), n, \beta_n(y_n)]$ for $x_n \in \tilde{K}^i(X; n)$ and $y_n \in \tilde{K}^{\#}(Y; n)$, $i = 0, 1$. The diagram commutes by [9, Lemma 2].

Let $\bar{S}: T \to K$ be defined by $\bar{S}(x, n, y) = (-1)^i s_n(x) \otimes s_n(y)$ (for x of degree i), where the s_n are the splittings of Theorem B for X and Y, respectively. Since β_n has degree 1 we have $\Gamma \bar{S} = -\Theta$. We claim that $\ker \Theta = R \subset \ker \Psi \bar{S}$ so that $\Psi \bar{S} = S \Theta$ for some $S: \operatorname{Tor}(\tilde{K}^{\#}(X), \tilde{K}^{\#}(Y)) \to \tilde{K}^{\#}(X \wedge Y)$. Such an S is clearly a splitting for τ.

Now generators of forms (a) and (b) already lie in the kernel of $\bar{S}$. For a generator of form (c) we have by Theorem B

$$\begin{aligned}
\Psi \bar{S}(x, nm, y) &= \beta_{nm} \mu_{nm}((-1)^i s_{nm}(x) \otimes s_{nm}(y)) \\
&= \beta_{nm} \mu_{nm}((-1)^i s_{nm}(x) \otimes s_{nm} \kappa''_{nm,m}(y)) \\
&= \beta_{nm} \mu_{nm}((-1)^i s_{nm}(x) \otimes \kappa_{nm,m} s_m(y)).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\Psi \bar{S}(nx, m, y) &= \beta_m \mu_m((-1)^i s_m(nx) \otimes s_m(y)) \\
&= \beta_m \mu_m((-1)^i s_m \kappa''_{m,nm}(x) \otimes s_m(y)) \\
&= \beta_m \mu_m(\kappa_{m,nm}((-1)^i s_{nm}(x)) \otimes s_m(y)),
\end{aligned}$$

and by [9, Lemma 3(c)] these are equal. A similiar computation for generators of form (d) completes the proof.

2. Splitting the Universal Coefficient Sequences

We recall the relations [4, Proposition 1.9] for the coefficient homomorphisms $\kappa_{n,m}$ and the fact that $\tilde{K}^{\#}(X;n)$ is a $\mathbb{Z}_n$-module. Therefore $\tilde{K}^{\#}(X;n)$ is isomorphic to the direct sum of all the $\tilde{K}^{\#}(X;p^k)$ with p^k the greatest power of a prime p dividing n. Hence it suffices to prove Theorem B for prime powers:

Lemma. *Let A and C be abelian groups and B_n p^n-bounded groups fitting in a simultaneously direct and inverse system of exact sequences*

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
0\to & A/p^n & \xrightarrow{\iota_n} & B_n & \xrightarrow{\pi_n} & C[p^n] & \to 0 \\
 & \alpha_{n,n+1}\uparrow\downarrow\alpha_{n+1,n} & & \beta_{n,n+1}\uparrow\downarrow\beta_{n+1,n} & & \gamma_{n,n+1}\uparrow\downarrow\gamma_{n+1,n} & \\
0\to & A/p^{n+1} & \xrightarrow{\iota_{n+1}} & B_{n+1} & \xrightarrow{\pi_{n+1}} & C[p^{n+1}] & \to 0 \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

(where $\alpha_{n,n+1}$, $\alpha_{n+1,n}$ and $\gamma_{n,n+1}$, $\gamma_{n+1,n}$ are the obvious maps). Assume that $\beta_{n,n+1}\beta_{n+1,n}$ and $\beta_{n+1,n}\beta_{n,n+1}$ are multiplication by p. Then there are splittings $s_n: C[p^n]\to B_n$ such that

(i) $\pi_n s_n = 1$,
(ii) $\beta_{n,n+1}s_{n+1} = s_n\gamma_{n,n+1}$,
(iii) $\beta_{n+1,n}s_n = s_{n+1}\gamma_{n+1,n}$.

Proof. We only indicate how to continue the proof in [4, Theorem 2.8]; note the change of notation.

Having achieved so far homomorphisms $s'_m: C[p^m]\to B_m$ such that

(a) $\pi_m s'_m = 1$,
(b) $\beta_{m,m+1}s'_{m+1} = s'_m\gamma_{m,m+1}$

for all m, we now consider commutativity in the opposite direction; assume therefore we have already constructed $s_1, \ldots, s_n$ such that (i), (ii), (iii), and

$$\beta_{n,n+1}s'_{n+1} = s_n\gamma_{n,n+1} \tag{1}$$

hold. Write

$$s'_{n+1}\gamma_{n+1,n} - \beta_{n+1,n}s_n = \iota_{n+1}\Delta \tag{2}$$

for some $\Delta: C[p^n]\to A/p^{n+1}$. It follows from our assumption $\beta_{n,n+1}\beta_{n+1,n} = p$ that $\iota_n\alpha_{n,n+1}\Delta = 0$, thus

$$\alpha_{n,n+1}\Delta = 0. \tag{3}$$

Given a basis [7, p. 78] $\mathscr{C}_{n+1}$ of $C[p^{n+1}]$, a direct sum of cyclic groups, we set for a $c\in\mathscr{C}_{n+1}$ $c' = c$ if c has order $p^k \leqq p^n$, and $c' = pc$ if c has order p^{n+1}; all these c' form a basis $\mathscr{C}'_n$ of $C[p^n]\subset C[p^{n+1}]$. In case $c' = pc$ we have $\beta_{n+1,n}s_n(c') = s'_{n+1}\gamma_{n+1,n}(c')$ by our assumption $\beta_{n+1,n}\beta_{n,n+1}=p$. Hence Δ vanishes on the direct summand of $C[p^n]$ generated by all $c'\in\mathscr{C}'_n$ with $c'=pc$.

Finally, we define $s_{n+1}(c)=s'_{n+1}(c)-\iota_{n+1}\Delta(c')$ for $c\in\mathscr{C}_{n+1}$.

Acknowledgement. The author thanks the referee for some comments on a first draft of this paper.

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Received February 22, 1980