On automorphic density results in higher rank

Habilitationsschrift

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Part 1. Introduction

1. Zusammenfassung

Meine Forschung beschäftigt sich mit Themen in der Schnittmenge von analytischer Zahlentheorie, harmonischer Analysis und Darstellungstheorie. Das Bindeglied zwischen diesen Gebieten ist die spektrale Theorie der automorphen Formen sowie das Langlands-Programm. Eine zentrale Vermutung in diesem Bereich ist die sogenannte allgemeine Ramanujan-Vermutung, welche viele weitreichende Implikationen hat. Ein vollständiger Beweis dieser Vermutung scheint aber mit heutigen Techniken noch aussichtslos zu sein. Nichtsdestotrotz haben viele interessante mathematische Entwicklungen ihren Ursprung in Untersuchungen, die mit Ramanujans-Vermutung zusammenhängen.

Die Dichtehypothese, welche das Thema der vorliegenden Arbeit ist, kann als praktische Approximation an die allgemeine Ramanujan-Vermutung gesehen werden. Diese Hypothese wurde formal in Sarnaks ICM Vortrag von 1990 formuliert und hat sich seitdem zu einem zentralen Forschungsthema in der Theorie der automorphen Formen entwickelt. Anfänglich gab es große Fortschritte im Zusammenhang mit Sarnaks Dichtehypothese für Rang Eins. Bevor Resultate in höherem Rang möglich waren, musste jedoch die nötige Theorie der automorphen Formen verfeinert werden. Dies ist nun geschehen, und es konnten Fälle der Dichtehypothese in beliebigem Rang bewiesen werden. Diese Entwicklungen haben schon zu neuen arithmetischen Anwendungen geführt, und es ist davon auszugehen, dass auch in Zukunft interessante Mathematik im Zusammenhang mit der Dichtehypothese entstehen wird.

Alle Arbeiten, die zu dieser Habilitation gehören, haben mit Sarnaks Dichtehypothese in höherem Rang zu tun. In den folgenden Abschnitten wollen wir die Philosophie hinter der Dichtehypothese erklären und den aktuellen Forschungsstand zusammenfassen. Diesem Prolog schließen sich die folgenden Arbeiten an, welche das Herz dieser kumulativen Habilitationschrift formen.

(1) E. Assing, V. Blomer: *The density conjecture for principal congruence subgroups*
Erschienen in: Duke Math. J. 173 (2024), no. 7, 1359–1426.
(2) E. Assing: *A density theorem for Borel-Type Congruence subgroups and arithmetic applications*
Arbeit erhältlich als: arXiv Preprint arXiv:2303.08925, 2023.
(3) E. Assing: *A note on Sarnak's density hypothesis for* Sp_4
Arbeit erhältlich als: arXiv Preprint arXiv:2305.04791, 2023.
(4) E. Assing, V. Blomer, P. D. Nelson: *Local analysis of the Kuznetsov formula and the density conjecture*
Arbeit erhältlich als: arXiv Preprint arXiv:2404.05561, 2024.

2. Summary

My research and in particular the part of it presented in this habilitation sits at the interface of analytic number theory, harmonic analysis and representation theory. The bridge between these topics is provided by the spectral theory of automorphic forms and parts of Langlands' program. An important cornerstone of the field is the generalized Ramanujan conjecture. This conjecture has many profound implications, but a complete proof seems to be far out of reach of current technology. Nonetheless many interesting mathematical developments originated in the study of it.

The density hypothesis is in some sense a convenient approximation to the generalized Ramanujan conjecture. It has been formally put forward in Sarnak's 1990 ICM address and has gotten much attention since. In particular, early on much progress has been made in rank one. Since then the theory of automorphic forms has matured significantly, so that very recently new instances of

Sarnak's density hypothesis have been established in higher rank. These developments have already led to interesting arithmetic applications. In the future we expect more exciting developments in connection with the density hypothesis and its applications.

All the articles contained in this habilitation are part of a series of recent breakthroughs in connection with Sarnak's density hypothesis in higher rank. In the following sections we will discuss the density hypothesis in detail and survey old and new results connected to it. This prelude is proceeded by the following works, which form the hearth of this cumulative habilitation thesis.

(1) E. Assing, V. Blomer: *The density conjecture for principal congruence subgroups*
Published in: Duke Math. J. 173 (2024), no. 7, 1359–1426.
(2) E. Assing: *A density theorem for Borel-Type Congruence subgroups and arithmetic applications*
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Preprint available at: arXiv Preprint arXiv:2404.05561, 2024.

3. The concept of a density theorem

Broadly speaking a density hypothesis is a precise quantification of the expectation that exceptional events occur rarely. The prototypical density hypothesis occurs in connection with the generalized Riemann hypothesis. Recall that it is conjectured that all non-trivial zeros of all Dirichlet L-functions lie on the line $\{\frac{1}{2}+it\colon t\in\mathbb{R}\}$. Proving this appears to be out of reach of current technology. However, one can often replace the full generalized Riemann hypothesis by suitable approximations, which are more approachable in practice. One of these approximations is the so-called zero density hypothesis and it arises as follows. We introduce the counting function

$$\mathfrak{N}(\sigma;T,q) := \sum_{\chi \bmod q} \sharp\{\rho=\beta+i\gamma\colon L(\rho,\chi)=0,\ |\gamma|\leq T \text{ and } \beta\geq\sigma\}.$$

If $\sigma=0$, then we count all non-trivial zeros up to height T of all Dirichlet characters modulo q. All these L-functions satisfy a functional equation relating $L(s,\chi)$ and $L(1-s,\overline{\chi})$. The zeros of these L-functions inherited this symmetry, so that it is sufficient to understand $\mathfrak{N}(\sigma;T,q)$ for $\frac{1}{2}\leq\sigma\leq 1$. The goal is to replace the generalized Riemann hypothesis by a suitable upper bound for $\mathfrak{N}(\sigma;T,q)$, which is small for σ close to 1 and captures the correct order of magnitude for $\sigma=\frac{1}{2}$.

Recall that for individual primitive Dirichlet character χ modulo q we have

$$\sharp\{\rho=\beta+i\gamma\colon L(\rho,\chi)=0,\ \beta\in[0,1] \text{ and } |\gamma|\leq T\} = \frac{T}{\pi}\log\left(\frac{qT}{2\pi e}\right)+O(\log(q(T+3))).$$

See [IwKo04, Theorem 5.24] for example. Summing this over all χ modulo q allows us to estimate

$$\mathfrak{N}(\frac{1}{2};T,q) \ll_\epsilon (qT)^{1+\epsilon}. \tag{1}$$

For convenience we have replaced $\log(qT)$ by $(qT)^\epsilon$ for arbitrary small $\epsilon>0$. This is common practice in many branches of analytic number theory and particularly convenient when keeping track of powers of logarithms is not relevant for the end result.

For $\sigma>\frac{1}{2}$ the generalized Riemann hypothesis would imply $\mathfrak{N}(\sigma;T,q)=0$. Such a result is of course not available at present, and the best we know is that $\mathfrak{N}(1;T,q)=0$ for all $T\in\mathbb{R}_{>0}$ and $q\in\mathbb{N}$. This is Dirichlet's theorem, see [IwKo04, Theorem 2.1]. Since we are ignoring factors like $(qT)^\epsilon$ for small $\epsilon>0$ in our current discussion, we can record this observation as

$$\mathfrak{N}(1;T,q) \ll_\epsilon (qT)^{\epsilon}. \tag{2}$$

The zero density hypothesis is the bound on $\mathfrak{N}(\sigma;T,q)$, that arises when linearly interpolating (1) and (2) in the exponent. More precisely, it is conjectured that the estimate

$$\mathfrak{N}(\sigma;T,q) \ll_\epsilon (qT)^{2(1-\sigma)+\epsilon} \tag{3}$$

holds for $\frac{1}{2}\leq\sigma<1$. The best bound available in practice is

$$\mathfrak{N}(\sigma;T,q) \ll (qT)^{\frac{12}{5}(1-\sigma)+\epsilon}. \tag{4}$$

We refer to [IwKo04, Chapter 10] for a more exhaustive discussion of zero density theorems.[1]

The big conjecture underpinning Sarnak's density hypothesis is the generalized Ramanujan conjecture. In this context exceptional events are triggered by non-tempered automorphic representations. In what follows we will make this more precise and explain how rareness can be quantified. This will lead naturally to a formulation of the automorphic density hypothesis.

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[1] In a recent breakthrough, see [GuMa24], Huxley's density theorem for the Riemann zeta function (i.e. $q=1$) was improved.

Ramanujan originally conjectured that the Fourier coefficients $\tau(n)$ of the Ramanujan Δ-function satisfy the bound

$$|\tau(n)| \leq d(n) \cdot n^{\frac{11}{2}}, \tag{5}$$

where $d(n)$ is the divisor function. What is nowadays known as Ramanujan conjecture goes far beyond this initial prediction. In order to fully appreciate the density hypothesis it is necessary to understand the full scope of the generalized Ramanujan conjecture. Therefore, we will start by giving a very brief introduction to this fascinating topic. For more details we refer to the surveys [Sar05, BlBr13] and the references within.

Let $\mathbb{A}_{\mathbb{Q}}$ denote the adele ring of $\mathbb{Q}$. It is a fact that Ramanujan's Δ-function gives rise to a representation π_Δ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character. This representation appears with multiplicity one in $L^2_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1)$. Flath's tensor product theorem allows us to write $\pi_\Delta = \pi_{\Delta,\infty} \otimes \bigotimes_p \pi_{\Delta,p}$. Here $\pi_{\Delta,\infty}$ is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ and $\pi_{\Delta,p}$ are spherical representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character. The bound (5) is equivalent to the statement that the representations $\pi_{\Delta,p}$ are tempered for all p. The following is a brave generalization.

Conjecture 3.1 (GL_2-Ramanujan conjecture)**.** *Let $\pi = \pi_\infty \otimes \bigotimes_p \pi_p$ be an irreducible representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with unitary central character. If π appears in $L^2_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1)$ and is infinite dimensional, then the representations $\pi_\infty, \pi_2, \pi_3, \dots$ are all tempered.*

As explained above this contains Ramanujan's original conjecture. It is a deep theorem of Deligne that the statement of Conjecture 3.1 holds for representations π that arise from classical holomorphic modular forms. On the other hand, when considering representations coming from Maaß forms, then the GL_2-Ramanujan conjecture implies Selberg's eigenvalue conjecture, which is still open.

We are tempted to further generalize Conjecture 3.1 as follows. Let G be a reductive linear algebraic group over a number field F. We write $\mathbb{A}_F$ for the adele ring of F and denote the places of F by v. The right regular representation of $G(\mathbb{A}_F)^1$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_F)^1)$ can be decomposed into a continuous part described by Eisenstein series and a discrete part

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_F)^1) \cong \sum_{\pi \in \widehat{G(\mathbb{A}_F)^1}} \mathrm{mult}_G(\pi) \cdot \pi.$$

Naively one might hope that the following is true.

Conjecture 3.2 (Naive Ramanujan conjecture)**.** *Let $\pi = \pi_\infty \otimes \bigotimes_v \pi_v$ be an irreducible infinite dimensional representation of $G(\mathbb{A}_F)$ with unitary central character. If $\mathrm{mult}_G(\pi|_{G(\mathbb{A}_F)^1}) > 0$, then $\pi = \bigotimes_v \pi_v$ for tempered representations $\pi_v \in \widehat{G(F_v)}$.*

This is false in general. Indeed, the discrete spectrum further decomposes into a cuspidal part and a residual part:

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_F)^1) = L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_F)^1) \oplus L^2_{\mathrm{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_F)^1). \tag{6}$$

An irreducible representation contributing to the cuspidal part will be called cuspidal automorphic.[2] For $G = \mathrm{GL}_n$ the residual spectrum is described in [MoWa89]. As a result of this description one finds that for many n there exist non-cuspidal (i.e. residual) infinite dimensional representations

[2]We refer to [BoJa79, Sections 4.4 and 4.6] for more precise definitions.

that are non-tempered at all places. We conclude that even for GL_n the naive conjecture needs to be modified.

Conjecture 3.3 (GL_n-Ramanujan conjecture). *Let $\pi = \bigotimes_v \pi_v$ be a a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with unitary central character. Then π_v is tempered for all places v of F.*

Remark 3.4. For $G = \mathrm{GL}_2$ the residual spectrum consists only of one dimensional representations and Conjecture 3.3 reduces to Conjecture 3.1 in this case.

For general groups G the situation is more subtle. Indeed, the construction of certain functorial lifts leads to examples of non-tempered cuspidal automorphic representations. These must be excluded in any reasonable formulation of the generalized Ramanujan conjecture. This can be done by introducing so-called CAP-representations. These arise in Arthur's classification of the discrete spectrum and encapsulate precisely the constituents thereof that are known to be non-tempered.

Conjecture 3.5 (Generalized Ramanujan conjecture). *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ with unitary central character. If π is non-CAP, then it is tempered at all places.*

Remark 3.6. If G is quasi-split, then one can define the notion of globally generic representations. A version of the generalized Ramanujan conjecture now states that all globally generic representations are everywhere tempered. If Arthur's conjectures concerning the discrete spectrum of reductive groups are true, then this formulation is essentially equivalent to the one given in Conjecture 3.5. Furthermore, the generalized Ramanujan conjecture would follow from the Ramanujan conjecture for GL_n and Arthur's conjectures. See [Shah11] for more details.

Recall that our goal is to find a suitable approximation to the generalized Ramanujan conjecture, which morally mimics the zero density hypothesis for L-functions. The first step is to introduce a measure for non-temperedness. Roughly speaking, given a place v such that $G(F_v)$ is non-compact, we are trying to assign a number to a (unitary) irreducible representation π_v which is minimal if π_v is tempered and maximal if π_v is trivial. There are two popular set-ups:

- One characterization of temperedness is through the regularity of matrix coefficients. To make this precise we fix a maximal compact subgroup $K_v \subseteq G(F_v)$ and let ω_v denote the central character of π_v. For a K_v-finite vector $w \in \pi_v$ we define[3]
$$p(\pi_v) = \inf\{2 \leq p \leq \infty\colon \langle w, \pi_v(\cdot)w\rangle \in L^p(G(F_v), \omega_v^{-1})\}. \tag{7}$$
By definition π_v is tempered if and only if $p(\pi_v) = 2$. Furthermore, for the trivial representation π_{triv} we have $p(\pi_{\mathrm{triv}}) = \infty$.
- For $G = \mathrm{GL}_n$ one can use the Langlands classification to characterize tempered representations. To do so we realize π_v as the unique subrepresentation of a representation arising through normalized parabolic induction $\mathrm{Ind}_{P(F_v)}^{G(F_v)}(\otimes_{i=1}^r |\cdot|^{\sigma_{\pi_v}(i)}\tau_i)$. Here P is a parabolic subgroup of G with Levi $M \cong \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$, the representations τ_i of $\mathrm{GL}_{n_i}(F_v)$ are tempered and $\sigma_{\pi_v}(i)$ are real numbers. We put
$$\sigma(\pi_v) = \max_i \sigma_{\pi_v}(i). \tag{8}$$
If π_v is tempered, then we must have $\sigma(\pi_v) = 0$. On the other hand, $\sigma(\pi_{\mathrm{triv}}) = \frac{n-1}{2}$.

[3]It can be shown that this definition is independent of the choice of w.

Remark 3.7. One way to approach the GL_n-Ramanujan conjecture is by showing upper bounds for $\sigma(\pi_v)$, where π_v is the vth-component of a non-trivial automorphic representation π. For archimedean v this is essentially a spectral gap. If we assume that π is cuspidal, then very strong bounds for $\sigma(\pi_v)$ are available. We refer to [LRS99, BlBr11] for a more thorough discussion.

Suppose we are given a finite family $\mathcal{F}$ of irreducible automorphic representations contributing to $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)^1)$. To each representation $\pi \in \mathcal{F}$ we associate a positive weight $m(\pi)$.[4] We write

$$\mathcal{S}(\mathcal{F}) = \sum_{\pi \in \mathcal{F}} m(\pi)$$

for the total mass (i.e. the weighted cardinality) of the family $\mathcal{F}$. In order to allow for a clean statement of the density hypothesis we assume that

$$\sum_{\substack{\pi \in \mathcal{F} \\ \dim(\pi)=1}} m(\pi) \ll_\epsilon \mathcal{S}(\mathcal{F})^\epsilon, \tag{9}$$

for any $\epsilon > 0$. This is the automorphic analogue of (2). Practical families $\mathcal{F}$ often contain only an absolutely bounded number of one dimensional representations and these will be weighted by one. Thus, (9) is a reasonable assumption to make.

Finally, fix a place v of F and let $r(\pi_v)$ denote a suitable measure for non-temperedness. For example, if $G = \mathrm{GL}_n$, then we can take $r(\pi_v) = p(\pi_v)$ or $r(\pi_v) = \sigma(\pi_v)$. Put $r_0 = r(\pi_v)$ for a tempered representation π_v and set $r_1 = r(\pi_{\mathrm{triv}})$. Linear interpolation in the exponent produces the *density hypothesis*

$$\sum_{\substack{\pi = \otimes_v \pi_v \in \mathcal{F} \\ r(\pi_v) \geq \sigma}} m(\pi) \ll_\epsilon \begin{cases} \mathcal{S}(\mathcal{F})^{1-\frac{\sigma - r_0}{r_1 - r_0}+\epsilon} & \text{if } r_1 < \infty, \\ \mathcal{S}(\mathcal{F})^{\frac{r_0}{\sigma}+\epsilon} & \text{if } r_1 = \infty. \end{cases} \tag{10}$$

This is a statistical statement concerning the weighted number of non-tempered representations of r-badness at least σ in the family $\mathcal{F}$. Note that this is only meaningful when $\mathcal{S}(\mathcal{F})$ tends to infinity.

Of course it is easy to construct families and weights for which (10) is false. However, for many families arising in practice this hypothesis is believed to hold.

Remark 3.8. In (10) we are interpolating between the full family (i.e. $\sigma = r_0$) and the sub-family of one dimensional representations (i.e. $\sigma = r_1$). Thus, we approximate the naive Ramanujan conjecture as stated in Conjecture 3.2. This is intentional. Indeed, the density hypothesis is believed to be robust enough to absorb all known cuspidal and residual counter-examples to the naive Ramanujan conjecture.

There are several ways to make the proceeding discussion precise. Here we will follow [GoKa23] to give a formal statement of Sarnak's density hypothesis. This is supposed to serve as a prototypical example for how such a statement may look in practice. For convenience we work over the real numbers.

Let $G(\mathbb{R})$ denote the $\mathbb{R}$-points of a semisimple algebraic group defined over $\mathbb{R}$ and fix a maximal compact subgroup $K_\infty \subseteq G(\mathbb{R})$. Given a lattice $\Gamma \subseteq G(\mathbb{R})$ we define the multiplicity of π_∞ in $L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))$ by

$$m(\pi_\infty, \Gamma) = \dim \mathrm{Hom}_{G(\mathbb{R})}(\pi_v, L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R}))).$$

[4] In practice $m(\pi)$ will be a certain multiplicity which is related to (but not equal in general) to the global multiplicity $\mathrm{mult}_G(\pi)$ of π in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)^1)$.

In particular,

$$L^2_{\text{disc}}(\Gamma\backslash G(\mathbb{R})) = \sum_{\pi\in\widehat{G(\mathbb{R})}} m(\pi_\infty,\Gamma)\cdot\pi_\infty.$$

In this setting Sarnak's density hypothesis is formulated in [GoKa23, Conjecture 1.1].

Conjecture 3.9 (Sarnak's density hypothesis)**.** *Let $G(\mathbb{R})$ be a real, semisimple, almost-simple and simply connected Lie group, let Γ_0 be an arithmetic lattice in $G(\mathbb{R})$ and let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of finite index congruence subgroups of Γ_0 such that $[\Gamma_0\colon\Gamma_n]\to\infty$ as $n\to\infty$. Then, for every pre-compact subset $\Omega\subseteq\widehat{G(\mathbb{R})}$ and every $\epsilon>0$ we have*

$$\sum_{\substack{\pi_\infty\in\Omega\\ p(\pi)\geq p}} m(\pi_\infty,\Gamma_n) \ll_{\Omega,\epsilon} [\Gamma_0\colon\Gamma_n]^{\frac{2}{p}+\epsilon}$$

uniformly in $p>2$ and $n\in\mathbb{N}$.

A weak form of this conjecture was proposed by Sarnak and Xue in [SaXu91, Conjecture 2.3] and is also known as *pointwise multiplicity hypothesis* or *L^p-conjecture*. Indeed, by applying Sarnak's density hypothesis with $p=p(\pi)$ to the singleton $\Omega=\{\pi\}$ we obtain:

Conjecture 3.10 (Sarnak-Xue conjecture)**.** *Let $G(\mathbb{R})$ be a real, semisimple, almost-simple and simply connected Lie group, let Γ_0 be an arithmetic lattice in $G(\mathbb{R})$ and let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of finite index congruence subgroups of Γ_0 such that $[\Gamma_0\colon\Gamma_n]\to\infty$ as $n\to\infty$. Then, for every $\pi_\infty\in\widehat{G(\mathbb{R})}$ and every $\epsilon>0$ we have*

$$m(\pi_\infty,\Gamma_n) \ll_{\pi_\infty,\epsilon} [\Gamma_0\colon\Gamma_n]^{\frac{2}{p(\pi_\infty)}+\epsilon}$$

uniformly in $p>2$ and $n\in\mathbb{N}$.

In practice it is often easier and sometimes sufficient to work with spherical representations. These are representations featuring a non-trivial K_∞-invariant vector. The isomorphism classes of spherical irreducible unitary representations form the spherical dual of $G(\mathbb{R})$, which we denote by $\widehat{G(\mathbb{R})}_{\text{sph}}$. It is well known that the Casimir operator acts on the subspace of K_∞-invariant vectors in π_∞ by scalar multiplication. For $\pi_\infty\in\widehat{G(\mathbb{R})}_{\text{sph}}$ this scalar is a well defined non-negative real number, which we denote by λ_{π_∞}. The spherical density hypothesis can now be stated as follows.

Conjecture 3.11 (Sarnak's spherical density hypothesis)**.** *Let $G(\mathbb{R})$ be a real, semisimple, almost-simple and simply connected Lie group, let Γ_0 be an arithmetic lattice in $G(\mathbb{R})$ and let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of finite index congruence subgroups of Γ_0 such that $[\Gamma_0\colon\Gamma_n]\to\infty$ as $n\to\infty$. If $G(\mathbb{R})$ has rank one, then*

$$\sum_{\substack{\pi_\infty\in\widehat{G(\mathbb{R})}_{sph}\\ p(\pi_\infty)\geq p}} m(\pi_\infty,\Gamma_n) \ll_{\epsilon} [\Gamma_0\colon\Gamma_n]^{\frac{2}{p}+\epsilon} \text{ for } p>2.$$

In the situation where $G(\mathbb{R})$ has rank at least two we let $M\geq 1$ be a parameter. Then there exists a constant K depending only on G such that

$$\sum_{\substack{\pi_\infty\in\widehat{G(\mathbb{R})}_{sph}\\ \lambda_{\pi_\infty}\leq M\\ p(\pi_\infty)\geq p}} m(\pi_\infty,\Gamma_n) \ll_{\epsilon} M^K[\Gamma_0\colon\Gamma_n]^{\frac{2}{p}+\epsilon} \text{ for } p>2.$$

Remark 3.12. The reason why we have to distinguish between rank one and higher rank is that the set

$$\Omega = \{\pi \in \widehat{G(\mathbb{R})}_{\mathrm{sph}} \colon p(\pi) > 2\} \tag{11}$$

is pre-compact as long as $G(\mathbb{R})$ has rank one. Otherwise, Ω is not pre-compact in general and we add the restriction on the Casimir eigenvalue to force pre-compactness. Note that our statement of the spherical density hypothesis includes the additional requirement that the dependence on (the bounds on) the Casimir eigenvalue are polynomial. This is often important in applications and was already proposed in [GoKa23, Conjecture 1.2]. Obvious variations of these conjectures are obtained by replacing

- $\mathbb{R}$ with some non-archimedean local field F_v;
- $p(\pi_\infty)$ with $\sigma(\pi_\infty)$.

Remark 3.13. We should explain how these conjectures relate to the general discussion above. Suppose that G, as in Conjecture 3.9, is defined over $\mathbb{Q}$ and has strong approximation. Then, for a congruence lattice $\Gamma \subseteq \Gamma_0$, there is some open compact subgroup K_Γ of $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ with

$$\Gamma\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_\Gamma.$$

Given a representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_{\mathbb{Q}})$ we let $\pi_{\mathrm{fin}}^{K_\Gamma}$ denote the space of K_Γ-invariant elements in $\pi_{\mathrm{fin}} = \otimes_{v<\infty}\pi_v$. We want to apply our heuristic to the family

$$\mathcal{F} = \{\pi = \otimes_v \pi_v \in \widehat{G(\mathbb{A}_{\mathbb{Q}})} \colon \pi_\infty \in \Omega,\, \pi_{\mathrm{fin}}^{K_\Gamma} \neq \{0\}\}$$

with weights

$$m(\pi) = \mathrm{mult}_G(\pi) \cdot \dim \pi_{\mathrm{fin}}^{K_\Gamma}.$$

It can be shown that (9) is satisfied for these families. Next we note that, as $[\Gamma_0 \colon \Gamma]$ grows, the compact subgroup K_Γ shrinks and the restriction $\pi_{\mathrm{fin}}^{K_\Gamma} \neq \{0\}$ gets weaker. As a result the family $\mathcal{F}$ grows as $[\Gamma_0 \colon \Gamma] \to \infty$. Even more, when Γ_0 and Ω are fixed, the index $[\Gamma_0 \colon \Gamma]$ is expected to be a good replacement for the total mass $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$. These observations allow us to write the density hypothesis from (10), with $v = \infty$ and $r(\pi_\infty) = p(\pi_\infty)$, as

$$\sum_{\substack{\pi = \otimes_v \pi_v \in \mathcal{F} \\ p(\pi_\infty) \geq \sigma}} m(\pi) \ll_{\Gamma_0, \Omega, \epsilon} [\Gamma_0 \colon \Gamma]^{\frac{2}{\sigma} + \epsilon}. \tag{12}$$

Finally, because

$$\sum_{\substack{\pi = \otimes_v \pi_v \in \mathcal{F} \\ p(\pi_\infty) \geq \sigma}} m(\pi) = \sum_{\substack{\pi_\infty \in \Omega \\ p(\pi_\infty) \geq \sigma}} m(\pi_\infty, \Gamma),$$

we see that (12) produces Conjecture 3.9.

In recent years there has been a lot of activity around Sarnak's density hypothesis. While Sarnak's density hypothesis as stated in Conjecture 3.9 is still open in general, much progress has been made in special cases. Best understood are the cases where $G(\mathbb{R})$ has (real) rank one or when $G = \mathrm{SL}_n$. These situations will be discussed in detail in Sections 4 and 5 below. In Section 6 we take a look at selected applications of the density hypothesis. The next section, namely Section 7, is devoted to a discussion of progress towards the (spherical) density hypothesis for $G = \mathrm{Sp}_4$. Finally, in Section 8 we reflect on the general conjecture and formulate some concrete open problems.

4. Automorphic density theorems in rank one

Here we will discuss results towards the density hypothesis in real rank one focussing mostly on the case $G = \mathrm{SL}_2$. We start with developments preceding the official formulation of Sarnak's density hypothesis. Indeed, to the best of our knowledge, the first spectral density theorems appear in the works of Iwaniec and Huxley.[5] More precisely, given a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ we consider the Laplace-Beltrami operator

$$\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

acting on $L^2(\Gamma\backslash\mathbb{H})$. It is known that $\Delta_{\mathbb{H}}$ has point spectrum

$$\sigma_{\Delta_{\mathbb{H}}}(\Gamma) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$$

and we write the eigenvalues as $\lambda_j = (\frac{1}{2} + t_j)(\frac{1}{2} - t_j) = \frac{1}{4} - t_j^2$. Note that the bottom eigenvalue λ_0 has $t_0 = \frac{1}{2}$ and appears with multiplicity one. Furthermore, by Weyl's law, we have

$$\sharp \left\{ \frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma) \colon |t| \leq M \right\} \sim \frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H})}{2\pi} M^2. \tag{13}$$

Selberg's eigenvalue conjecture predicts that $\lambda_1 \geq \frac{1}{4}$ or equivalently $t_j \in i\mathbb{R}$ for all $j \geq 1$. In view of the philosophy explained in Section 3 we expect

$$\sharp \left\{ \frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma) \colon t \in [\sigma, \infty) \right\} \ll_\epsilon \mathrm{Vol}(\Gamma\backslash\mathbb{H})^{1-2\sigma+\epsilon}. \tag{14}$$

Indeed, here we interpolate

$$\sharp \left\{ \frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma) \colon t \in [\frac{1}{2}, \infty) \right\} = 1 \text{ and } \sharp \left\{ \frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma) \colon t \in [0, \infty) \right\} \ll \mathrm{Vol}(\Gamma\backslash\mathbb{H}) \tag{15}$$

linearly in the exponent. Huxley, see [Hux86], establishes this for the three congruence subgroups

$$\Gamma_0(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z}), \tag{16}$$

$$\Gamma_1(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & 1 + N\mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z}) \text{ and} \tag{17}$$

$$\Gamma(N) = \begin{pmatrix} 1 + N\mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & 1 + N\mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z}) \tag{18}$$

of $\mathrm{SL}_2(\mathbb{R})$.

Theorem 4.1 (Section 4, [Hux86])**.** *Let $\epsilon > 0$, $N \in \mathbb{N}$ and let Γ be one of the congruence subgroups defined in* (16)-(18)*. Then, for $0 \leq \sigma \leq \frac{1}{2}$ and $\epsilon > 0$, we have*

$$\sharp \left\{ \frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma) \colon t \in [\sigma, \infty) \right\} \ll_\epsilon [\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma]^{1-2\sigma+\epsilon}.$$

[5] Related estimates appear already in [DeIw82], but a formal statement and the terminology *density theorem* seems to first appear in [IwSz85, Iwa85]. Note that [Iwa85] contains a reference to a preprint by Huxley, which is dated to 1983 and most likely contains results which later appeared in [Hux86].

Huxley's proof uses Selberg's trace formula. The basic idea is to reverse the strategy used when solving the hyperbolic circle problem. This approach works for a large variety of lattices and is an exercise with the trace formula. See [Iwa95, Section 11.4], where this is discussed for hyperbolic groups.

Remark 4.2. In the setting of compact orbifolds $X = \Gamma\backslash\mathbb{H}$ the analogy between the spectral density hypothesis (see (14)) and the density hypothesis for Dirichlet L-functions can be explained using the Selberg zeta function. Recall that the Selberg zeta function is defined by

$$Z_X(s) = \prod_{\gamma}(1-e^{-sl(\gamma)})^{-1} \text{ for } \Re(s) \gg 1. \tag{19}$$

Here γ runs over prime geodesics and $l(\gamma)$ denotes their length. This zeta function has a meromorphic continuation to $s \in \mathbb{C}$. Furthermore, if $\Re(\rho) \geq \frac{1}{2}$ and $Z_X(\rho) = 0$, then $\rho(1-\rho) \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma)$. In particular, we can rewrite (14) as

$$\sharp\{\rho \in \mathbb{C}\colon Z_X(\rho) = 0,\, \Re(\rho) > \sigma\} \ll_\epsilon \mathrm{Vol}(X)^{2-2\sigma+\epsilon},$$

for $\frac{1}{2} < \sigma \leq 1$. Since Weyl's law translates to counting zeros in a certain boxes, this estimate is a perfect analogue of the classical zero density estimate for Dirichlet L-functions. See (3) for comparison.

In the series of papers [IwSz85, Iwa85, Iwa90] the Kuznetsov formula is used to count exceptional eigenvalues. These efforts result in a very strong estimate, which Iwaniec calls his *favorite estimate for the cardinality of the empty set!* See also [Iwa95, Theorem 11.7].

Theorem 4.3 (Theorem 1, [Iwa90])**.** *Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then we have*

$$\sharp\left\{\frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma_0(N))\colon t \in [\sigma, \frac{1}{2})\right\} \ll_\epsilon [\mathrm{SL}_2(\mathbb{Z})\colon \Gamma_0(N)]^{1-4\sigma+\epsilon}.$$

Remark 4.4. From our point of view (14) is the correct density hypothesis (or density conjecture). It is therefore interesting to note that in [IwSz85, Iwa85] the much stronger estimate given in Theorem 4.3 is referred to as *density conjecture.* This estimate is special for the Hecke congruence subgroups $\Gamma_0(N)$, where the exponent $1-4\sigma$ interpolates between Selberg's spectral gap $\lambda_1 \geq \frac{3}{16}$ on the one hand and Weyl's law on the other hand.

Remark 4.5. In this remark we will try to illustrate the strength of Iwaniec's density estimate for $\Gamma_0(N)$ by showing that it implies a strengthening of the density hypothesis for the principal congruence subgroup $\Gamma(N)$. This requires a slight extension of Theorem 4.3. Given a Dirichlet character χ modulo N we can define a character of $\Gamma_0(N)$, also denoted by χ, via

$$\chi(\gamma) = \chi(d) \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

One can consider the space $L^2(\Gamma_0(N)\backslash\mathbb{H}, \chi)$ consisting of square integrable automorphic functions satisfying $f(\gamma z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma_0(N)$. We write $\sigma_{\Delta_{\mathbb{H}}}(\Gamma_0(N), \chi)$ for the point spectrum of $\Delta_{\mathbb{H}}$ acting on $L^2(\Gamma_0(N)\backslash\mathbb{H}, \chi)$. A modification of the argument from [Iwa90] shows that

$$\sharp\left\{\frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma_0(N), \chi)\colon t \in [\sigma, \frac{1}{2})\right\} \ll_\epsilon [\mathrm{SL}_2(\mathbb{Z})\colon \Gamma_0(N)]^{1-4\sigma+\epsilon},$$

for all N and all Dirichlet character χ modulo M with $M^2 \mid N$.[6] Put

$$\Gamma(N)^\natural = \begin{pmatrix} N^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Gamma(N) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+N\mathbb{Z} & \mathbb{Z} \\ N^2\mathbb{Z} & 1+N\mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z})$$

and observe that

$$\sigma_{\Delta_{\mathbb{H}}}(\Gamma(N)) = \sigma_{\Delta_{\mathbb{H}}}(\Gamma(N)^\natural) = \bigsqcup_{\chi \bmod N} \sigma_{\Delta_{\mathbb{H}}}(\Gamma_0(N^2), \chi).$$

In particular, we can count exceptional eigenvalues for $\Gamma(N)$ as follows:

$$\begin{aligned} &\sharp\left\{\frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma(N)) \colon t \in [\sigma, \frac{1}{2})\right\} \\ &\quad = \sum_{\chi \bmod N} \sharp\left\{\frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma_0(N^2), \chi) \colon t \in [\sigma\frac{1}{2})\right\} \\ &\quad \ll_\epsilon \sum_{\chi \bmod N} [\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma_0(N^2)]^{1-4\sigma+\epsilon} \ll_\epsilon N^{3-8\sigma+\epsilon}. \end{aligned}$$

In summary, we have seen that

$$\sharp\left\{\frac{1}{4} - t^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma(N)) \colon t \in [\sigma, \frac{1}{2})\right\} \ll_\epsilon [\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma(N)]^{1-\frac{8}{3}\sigma+\epsilon},$$

which is still an improvement upon Theorem 4.1.

These results can be translated into the language of Section 3. To do so we recall that $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$ and that the eigenvalues $\lambda \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma)$ correspond to representations $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}_{\mathrm{sph}}$ with $m(\pi,\Gamma) > 0$. Furthermore, if π corresponds to $\lambda_\pi = \frac{1}{4} - t_\pi^2 \in \sigma_{\Delta_{\mathbb{H}}}(\Gamma)$, then we have

$$p(\pi) = \frac{2}{1 - 2\Re(t_\pi)}$$

and the multiplicity of λ_π in $\sigma_{\Delta_{\mathbb{H}}}(\Gamma)$ is precisely $m(\pi,\Gamma)$. These observations allow us to write Theorem 4.1 as

$$\sum_{\substack{\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}_{\mathrm{sph}} \\ p(\pi) > p}} m(\pi,\Gamma) \ll_\epsilon [\mathrm{SL}_2(\mathbb{Z}) \colon \Gamma]^{\frac{2}{p}+\epsilon},$$

for $2 < p \leq \infty$, Γ as in (16)-(18) and $\epsilon > 0$. This is precisely Conjecture 3.11 for $G = \mathrm{SL}_2$.

More generally, we can take $G(\mathbb{R})$ to be a semisimple Lie group of non-compact type with real rank one. Fix an Iwasawa decomposition $G(\mathbb{R}) = NAK$. Let ρ be the half sum of positive roots, W the Weyl group and let $\mathfrak{a}_{\mathbb{C}}^*$ be the complexified dual of the Lie algebra $\mathfrak{a}$ of A. We parametrize $\widehat{G(\mathbb{R})}_{\mathrm{sph}}$ by $\mathfrak{a}_{\mathbb{C}}^*/W$ and write this parametrization as $\pi \leftrightarrow \mu_\pi$. This is set up such that the trivial representation corresponds to ρ and tempered representations have parameters in $i\mathfrak{a}_{\mathbb{R}}^*$. All non-tempered unitary representations occur with parameter in $(0,\rho]$. It turns out that we can translate between $p(\pi)$ and μ_π using the equality

$$p(\pi) = \frac{2\rho}{\rho - \mathrm{Re}(\mu_\pi)}.$$

[6]More can be shown, but this is not necessary for our current discussion. We refer to [Hum18, (1.8)] or Theorem 4.10 below for more details.

Remark 4.6. If $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, then we have $\rho = \frac{1}{2}$. Selberg's eigenvalue conjecture can be phrased as

$$\bigcup_{\Gamma} \{\mu_\pi \colon m(\pi, \Gamma) > 0\} \subseteq i\mathfrak{a}_{\mathbb{R}}^* \cup \{\rho\},$$

where the union runs over all congruence subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

In order to introduce suitable families of lattices we further assume that G is defined over $\mathbb{Q}$ and that we have an embedding $\tau\colon G(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$. We say that $\Gamma \subseteq G(\mathbb{R})$ is arithmetic if $\tau(\Gamma)$ is commensurable with $G(\mathbb{Z}) = \tau(G(\mathbb{R})) \cap \mathrm{GL}_n(\mathbb{Z})$. This allows us to define (principal) congruence subgroups of Γ by

$$\Gamma(N) = \tau^{-1}(\tau(\Gamma) \cap \{\gamma \in G(\mathbb{Z}) \colon \gamma \equiv I_n \bmod N\}).$$

This is the context in which the original density hypothesis is formulated.

Conjecture 4.7 (Density Hypothesis 2.2, [Sar90])**.** *Assume $G(\mathbb{R})$ is as above and let $\Gamma \subseteq G(\mathbb{R})$ be an arithmetic lattice. Then we have*

$$\sum_{\substack{\pi \in \widehat{G(\mathbb{R})}_{sph} \\ \mu_\pi \geq \sigma}} m(\pi, \Gamma(N)) \ll_{\Gamma,\epsilon} [\Gamma \colon \Gamma(N)]^{1-\frac{\sigma}{\rho}+\epsilon}. \tag{20}$$

This is a special case of the general spherical density hypothesis formulated in Conjecture 3.11. It was observed by [SaXu91], see also [Sar90, Proposition 2.3], for co-compact Γ and by [HuKa93, Theorem 1.1] in general that Conjecture 4.7 follows from a certain uniform counting result. See Section 6 and in particular Conjecture 6.1 below for more details. This opens the door to a direct verification of Huxley's result Theorem 4.1 by counting matrices and also to some new instances of the density hypothesis.

Theorem 4.8 (Corollary 1, [SaXu91])**.** *The density hypothesis as stated in Conjecture 4.7 holds for co-compact arithmetic lattices Γ in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$.*

Remark 4.9. To be precise, it should be noted that [SaXu91, Corollary 1] only contains the slightly weaker statement

$$m(\pi, \Gamma(N)) \ll_{\Gamma,\epsilon} [\Gamma \colon \Gamma(N)]^{\frac{2}{p(\pi)}+\epsilon}.$$

This is the pointwise multiplicity hypothesis of Sarnak and Xue proposed in [SaXu91, Conjecture 1] see also Conjecture 3.10. However, the stronger statement given here is immediate. See also [Sar90, Remarks 2.3].

We conclude that our understanding of Sarnak's (spherical) density hypothesis for arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})$ is quite advanced. The same can be said about versions of the density hypothesis for SL_2 over non-archimedean local fields F. In this case the theory has a more combinatorial flavour. We refer to [GoKa22] and the references within for further discussion.

For more general groups $G(\mathbb{R})$ of real rank one the situation is more complicated. For arithmetic lattices Γ in $\mathrm{SU}_{2,1}(\mathbb{R})$ (arising from a Hermitian form in three variables) a result towards Conjecture 4.7 was obtained in [SaXu91]. Indeed the counting result [SaXu91, Theorem 2] together with a version of [HuKa93, Theorem 1.1] (see also [SaXu91, Theorem 3]) produces a density estimate with an exponent weaker than the one predicted by the density hypothesis. More precisely, Sarnak and Xue show that

$$m(\pi_\infty, \Gamma(N)) \ll_{\pi_\infty,\epsilon} [\Gamma \colon \Gamma(N)]^{\frac{7/3}{p(\pi_\infty)}+\epsilon},$$

which is slightly worse than the bound predicted by Conjecture 3.10. Using different tools, namely endoscopy, stronger bounds than predicted by Conjecture 3.10 were established for cohomological representations of $\mathrm{U}_{2,1}$ in [Mar14]. This was later generalized to cohomological representations of $\mathrm{U}_{n,1}$, see [MaSh19].

To the best of our knowledge a general density theorem along the lines of Conjecture 4.7 is still open for groups like $\mathrm{SU}_{n,1}$ and $\mathrm{SO}_{n,1}$ of real rank one. In [EGM87, CLPSS91] the theory of Poincaré series is developed in order to produce a spectral gap. However, the tools seem not yet sufficient to produce a density theorem. It should be noted that certain weighted density results have been established using Kuznetsov-type formulae in quite some generality. See for example [Rez93, BrMia98, BMP03, BrMia10].

We conclude this section by stating two very impressive results. Both go in some sense beyond the density hypothesis as stated in Conjecture 3.11, Conjecture 3.9 or Conjecture 4.7.

The methods developed in [Iwa90] extend to density theorem taking non-temperedness at finite places into account. This was worked out in [BBR14, Proposition 1] improving earlier results from [Sar84]. The method was pushed to its limits in [Hum18]. To state the result we take Γ as in (16)-(18) and fix an orthonormal basis $\mathcal{B}(\Gamma)$ of $L^2_{\mathrm{disc}}(\Gamma\backslash\mathbb{H})$. Given $f\in\mathcal{B}(\Gamma)$ we write $\lambda_f=\frac{1}{4}-t_f^2$ for the corresponding Laplace eigenvalue (i.e. $\lambda_f\in\sigma_{\Delta_{\mathbb{H}}}(\Gamma)$). We further assume that $\mathcal{B}(\Gamma)$ consists of Hecke eigenfunctions and write $\lambda_f(p)$ for the pth Hecke eigenvalue of $f\in\mathcal{B}(\Gamma)$.

Theorem 4.10 (Theorem 1.1 and 1.5, [Hum18])**.** *Let $\mathcal{P}$ be a finite set of primes. Let $N\in\mathbb{N}$ and assume that N is co-prime to all $p\in\mathcal{P}$. Further, for $p\in\mathcal{P}$ let $\alpha_p\in(2,p^{\frac{1}{2}}+p^{-\frac{1}{2}})$ be arbitrary and fix $\mu_p\in[0,1]$ with $\sum_{p\in\mathcal{P}}\mu_p=1$. For $T\geq 0$ and Γ as in* (16)-(18) *(with level N) we have*

$$\sharp\{f\in\mathcal{B}(\Gamma)\colon |\lambda_f(p)|\geq\alpha_p \text{ for all } p\in\mathcal{P} \text{ and } t_f\in i[0,T]\}$$
$$\ll_\epsilon [\mathrm{SL}_2(\mathbb{Z})\colon\Gamma]^{1-\alpha_\Gamma\sum_{p\in\mathcal{P}}\mu_p\frac{\log(\alpha_p/2)}{\log(p)}+\epsilon}T^{2-8\sum_{p\in\mathcal{P}}\mu_p\frac{\log(\alpha_p/2)}{\log(p)}+\epsilon}$$

with

$$\alpha_\Gamma=\begin{cases}4 & \text{if } \Gamma=\Gamma_0(N),\\ 3 & \text{if } \Gamma=\Gamma_1(N) \text{ and}\\ \frac{8}{3} & \text{if } \Gamma=\Gamma(N).\end{cases}$$

Furthermore, we can take $\alpha_\infty\in(0,\frac{1}{2})$ and pick $\mu_\infty\in[0,1]$ such that $\mu_\infty+\sum_{p\in\mathcal{P}}\mu_p=1$. Then we have

$$\sharp\{f\in\mathcal{B}(\Gamma)\colon |\lambda_f(p)|\geq\alpha_p \text{ for all } p\in\mathcal{P} \text{ and } \Re(t_f)\geq\alpha_\infty\}$$
$$\ll_\epsilon [\mathrm{SL}_2(\mathbb{Z})\colon\Gamma]^{1-\alpha_\Gamma\left(\alpha_\infty\mu_\infty+\sum_{p\in\mathcal{P}}\mu_p\frac{\log(\alpha_p/2)}{\log(p)}\right)+\epsilon}.$$

Remark 4.11. We make the following comments:

- For $\Gamma=\Gamma_0(N)$ one can extend the result to non-trivial nebentypi as indicated in Remark 4.5. However, for general nebentypus the Kloosterman sums can degenerate, so that the upper bound comes with certain correction factors and we omit a precise statement. Furthermore, the results in [Hum18] also include weight 1 Maaß forms.
- At a finite place $v=p$ we measure non-temperedness in terms of the size of the pth Hecke eigenvalue $\lambda_f(p)$. However, it is also natural to use the invariant defined in (8) instead. To be precise, we can write the Hecke eigenvalues as $\lambda_f(p)=p^{\mu_f(p,1)}+p^{\mu_f(p,2)}$ and define $\sigma_p(f)=\max_{i=1,2}\Re(\mu_f(p,i))$. As long as the implicit constants in the result above are allowed to depend on $\mathcal{P}$ one can modify the arguments from [Hum18] and obtain a version of Theorem 4.10 where the condition $|\lambda_f(p)|\geq\alpha_p$ is replaced by $\sigma_f(p)\geq\sigma_p$ for all $p\in\mathcal{P}$.

Another remarkable theorem is obtained in [FHMM24]. As we will see the result goes beyond rank one, but we nonetheless state it at the end of this section because it is closely related to the (algebraic) group SL_2. Fix $a,b,c\in\mathbb{N}$ and let $\mathcal{F}_{a,b,c}$ denote the family of all congruence lattices $\Gamma\subseteq \mathrm{SL}_2(\mathbb{R})^a\times\mathrm{SL}_2(\mathbb{C})^b$ that arise from certain (congruence) orders in quaternion division algebras of signature (a,b,c) over a number field k of degree $a+c+2b$. See [FHMM24, Definition 1.1] for a precise definition. We let $G(\mathbb{R})=\mathrm{SL}_2(\mathbb{R})^a\times\mathrm{SL}_2(\mathbb{C})^b$ and parametrize representations $\pi\in\widehat{G(\mathbb{R})}_{\mathrm{sph}}$ by a tuple $(s_\pi(j))_{j\in S_\infty}$, where $S_\infty=\{1,\ldots,a+b\}$ and $s_\pi(j)\in(0,\frac{1}{2}]\cup i[0,\infty)$.

Theorem 4.12 (Theorem 1, [FHMM24])**.** *Let $a,b,c\in\mathbb{N}$, $\emptyset\neq S\subseteq S_\infty$ and take tuples $(\sigma_j)_{j\in S}\in[0,\frac{1}{2}]^{\sharp S}$ and $(T_j)_{j\in S_\infty\setminus S}\in\mathbb{R}_{\geq 0}^{\sharp S_\infty\setminus S}$. Then, for $\epsilon>0$, we have*

$$\sum_{\substack{\pi\in\widehat{G(\mathbb{R})}_{\mathrm{sph}}\\ s_\pi(j)\in[\sigma_j,\frac{1}{2}]\text{ for } j\in S\\ s_\pi(j)\in i[T_j-1,T_j+1]\text{ for } j\in S_\infty\setminus S}} m(\pi,\Gamma)\ll_{a,b,c,\epsilon}\left[\mathrm{Vol}(\Gamma\backslash G(\mathbb{R}))\prod_{j\in S_\infty\setminus S}(1+|T_j|)^{1+\delta_{j>a}}\right]^{\min\limits_{j\in S}(1-2\sigma_j)+\epsilon}$$

uniformly in $\Gamma\in\mathcal{F}_{a,b,c}$.

The key feature here is the uniformity within the family of lattices $\mathcal{F}$. Also note that if $S=S_\infty$, then

$$p(\pi)=\frac{2}{\min_j(1-2\Re(s_j))}.$$

Thus, Theorem 4.12 covers (among other things) Conjecture 3.11 for a very large family of congruence lattices of $G(\mathbb{R})$. This result is established using the trace formula and can be seen as a vast extension of the work of Huxley mentioned above. In particular, the high uniformity in the lattices requires the introduction of many interesting new ideas.

5. A density hypothesis for SL_n

We will now change gears and work over SL_n for fixed $n\geq 3$.[7] Our goal is to discuss recent progress towards the spherical density hypothesis as formulated in Conjecture 3.11.

Given a congruence lattice $\Gamma\subseteq\mathrm{SL}_n(\mathbb{R})$ we will begin classically and consider automorphic forms $\varpi\in L^2_{\mathrm{disc}}(\Gamma\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$. Throughout we will assume that these forms ϖ are eigenfunctions of the algebra of invariant differential operators. We denote the corresponding spectral parameter by $\mu_\varpi(\infty)=(\mu_\varpi(\infty,1),\ldots,\mu_\varpi(\infty,n))$. Similarly, if $v=p$ is a finite place of $\mathbb{Q}$ and ϖ is an eigenfunction of all (spherical) Hecke operators at v, then we write $\mu_\varpi(v)$ for the Satake parameter attached to ϖ at v. In this case the entries of $\mu_\varpi(v)$ are defined modulo $\frac{2\pi i}{\log(p)}\mathbb{Z}$. We normalize our parameters such that the following statements hold:

- The constant function $\mathbf{1}$ has $\mu_{\mathbf{1}}(v)=((1-n)/2,\ldots,(n-1)/2)$ for all places v.
- The form ϖ is tempered at v if and only if $\mu_\varpi(v)\in(i\mathbb{R})^n$.

To measure non-temperedness we define

$$\sigma_\varpi(v)=\max_{j=1,\ldots,n}|\Re(\mu_\varpi(v,j))|. \tag{21}$$

[7] Let us stress that we view n as fixed and we claim no uniformity in this parameter. In particular, all the hidden constants in our estimates will depend on n.

Remark 5.1. Note that $\sigma_\varpi(v)$ is the spherical version of the invariant $\sigma(\pi_v)$ defined in (8). Indeed, an automorphic form ϖ gives rise to an automorphic representation $\pi_\varpi = \otimes_v \pi_{\varpi,v}$ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. If ϖ is an eigenfunction of all spherical Hecke operators at v (with non-trivial eigenvalues), then π_v is spherical. Thus, after changing the ordering of the tuple $\mu_\varpi(v)$ if necessary, we can realize π_v as the unique subrepresentation of $\mathrm{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_n(\mathbb{Q}_v)}(|\cdot|^{\mu_\varpi(v,1)} \otimes \ldots \otimes |\cdot|^{\mu_\varpi(v,n)})$. In particular, we have $\sigma_\varpi(v) = \sigma(\pi_{\varpi,v})$.

The same discussion is valid for $n = 2$ and we can compare our current notation to the one from Section 4. Indeed, given an automorphic form ϖ with Laplace-Beltrami eigenvalue $\lambda_\varpi = \frac{1}{4} - t_\varpi^2 \in \sigma_{\Delta_\mathbb{H}}(\Gamma)$ the corresponding spectral parameter is nothing but $\mu_\varpi(\infty) = (-t_\varpi, t_\varpi)$. In particular, $\sigma_\varpi(\infty) = |\Re(t_\varpi)|$.

Remark 5.2. In view of the assignment $\varpi \rightsquigarrow \pi_\varpi$ mentioned in the previous remark we can easily define another measure of non-temperedness. Recall $p(\cdot)$ as defined in (7) and set

$$p_\varpi(v) = p(\pi_{\varpi,v}). \tag{22}$$

For $n = 3$ we have

$$p_\varpi(v) = \frac{2}{1 - \sigma_\varpi(v)}.$$

In general, for $n \geq 4$ the two numbers $\sigma_\varpi(v)$ and $p_\varpi(v)$ do not contain the same information. They are however related by the inequality:

$$p_\varpi(v) \geq \frac{2(n-1)}{(n-1) - 2\sigma(\mu)}.$$

See [JaKa24, Remark 3.4] for details.

Let $\mathcal{F}$ be a finite family of (orthogonal) automorphic forms ϖ and $\sigma \geq 0$. We define

$$N_v(\sigma, \mathcal{F}) := \#\{\varpi \in \mathcal{F} \mid \sigma_{\varpi,v} \geq \sigma\}. \tag{23}$$

In the case at hand the GL_n-Ramanujan conjecture predicts that, if $\mathcal{F}$ consists of cusp forms, then $N_v(\sigma, \mathcal{F}) = 0$ for all $\sigma > 0$. In general we allow $\mathcal{F}$ to contain residual forms such as the constant function $\varpi = \mathbf{1}$. Thus, $N_v(\sigma, \mathcal{F})$ might be non-zero for $0 \leq \sigma \leq \frac{n-1}{2}$. In this context Sarnak's philosophy leads to the robust estimate

$$N_v(\sigma, \mathcal{F}) \ll_\epsilon (\#\mathcal{F})^{1 - \frac{2\sigma}{n-1} + \epsilon}. \tag{24}$$

As soon as $\mathcal{F}$ contains the constant function this estimate is sharp up to the tolerable factor $(\#\mathcal{F})^\epsilon$. However, for certain special families one can hope to find $c_n > \frac{2}{n-1}$ such that the stronger estimate

$$N_v(\sigma, \mathcal{F}) \ll_\epsilon (\#\mathcal{F})^{1 - c_n \cdot \sigma + \epsilon} \tag{25}$$

holds.

In practice we will encounter certain spectral families, which we will describe now. Let Γ be a lattice and fix a symmetric set $\Omega \subseteq \mathbb{C}^n$ such that $\Omega \cap (i\mathbb{R})^n$ is compact. We define the family $\mathcal{F}(\Omega, \Gamma)$ to consist of a maximal linearly independent set of automorphic forms $\varpi \in L^2_{\mathrm{disc}}(\Gamma \backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}))$ with $\mu_\varpi(\infty) \in \Omega$. We write $\mathcal{F}_{\mathrm{cusp}}(\Omega, \Gamma)$ for the subfamily consisting of cusp forms.

These families grow if the (Plancherel) volume of Ω or the co-volume of Γ gets large. We distinguish these two aspects:

- **(Spectral Aspect)** One can consider Γ as (essentially) fixed and let Ω grow. In this situation the size of the family $\mathcal{F}(\Omega,\Gamma)$ (resp. $\mathcal{F}_{\text{cusp}}(\Omega,\Gamma)$) is asymptotically captured by the volume of Ω with respect to the Plancherel measure. In practice we often encounter balls $\Omega_M=\{\mu\in\mathbb{C}^n\colon \|\mu\|\leq M\}$ of radius $M>n$. In this case the Weyl law predicts
$$\sharp\mathcal{F}(\Omega_M,\Gamma)\sim_\Gamma M^{d_n} \text{ and } \sharp\mathcal{F}_{\text{cusp}}(\Omega_M,\Gamma)\sim_\Gamma M^{d_n}$$
for $d_n=\frac{(n+2)(n-1)}{2}$. See [Mue23, Corollary 1.3] for suitable Weyl laws.
- **(Level Aspect)** On the other hand one can fix Ω and vary the lattice Γ. In practice one often fixes an ambient lattice Γ_0 and considers certain families $\{\Gamma_m\}_{m\in\mathbb{N}}$ of sublattices with growing index. In this case we have
$$\sharp\mathcal{F}_{\text{cusp}}(\Omega_M,\Gamma_m)\leq\sharp\mathcal{F}(\Omega_M,\Gamma_m)\ll_{\Gamma_0,\Omega}[\Gamma_0\colon\Gamma_m].$$
For generic not to small Ω this upper bound is a good replacement for the size of the family $\mathcal{F}(\Omega,\Gamma_n)$ (resp. $\mathcal{F}_{\text{cusp}}(\Omega,\Gamma_m)$) as $m\to\infty$.

Remark 5.3. Sarnak's spherical density hypothesis is essentially the estimate (24) for the family $\mathcal{F}(\Omega,\Gamma)$ in the level aspect. Note that in Conjecture 3.11 it is formulated for
$$\Omega=\Omega'_M=\{\lambda_{\pi_\infty}\leq M^2\}$$
and it contains the additional requirement that the implied constant depends at most polynomially on M. If $\pi_\infty=\pi_{\varpi,\infty}$ for an automorphic form ϖ with spectral parameters $\mu_\varpi(\infty)$, then one can compute λ_{π_∞} in terms of $\mu_\varpi(\infty)$ and compare Ω'_M and Ω_M. For example, if $n=3$, then
$$\Omega'_M=\{\mu\in\mathbb{C}^3\colon 1-\frac{1}{2}(\mu(1)^2+\mu(2)^2+\mu(3)^2)\leq M^2\}\subseteq\mathbb{C}^3.$$
The general case is similar and in practice we can replace Ω'_M by Ω_M without loosing any information.

We will now turn towards explaining recent progress towards the density hypothesis (24) (or even its strengthening (25)) for families of the form $\mathcal{F}(\Omega_M,\Gamma)$. The first breakthroughs were made for $n=3$ in [Blo13, BBR14, BBM17] using the Kuznetsov formula.

Theorem 5.4 (Theorem 2, [Blo13])**.** *For $\sigma>0$ and $\epsilon>0$ we have*
$$N_\infty(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z})))\ll_\epsilon M^{3-4\sigma+\epsilon}. \tag{26}$$

Remark 5.5. The numerical values in the exponent of this density theorem require some explanation. First, we note that the spectral parameter of an exceptional Maaß form ϖ is very restricted. Indeed it can be written as
$$\mu_\varpi(\infty)=(\sigma_\varpi(\infty)+i\gamma,-\sigma_\varpi(\infty),-2i\gamma), \tag{27}$$
for $\gamma\geq0$ and $0<\sigma_\varpi(\infty)\leq\frac{5}{14}$. The upper bound on $\sigma_\varpi(\infty)$ is given in [BlBr11, Theorem 1]. Due to the constraint on the imaginary part of the spectral parameter the spectral density of exceptional Maaß forms drops. Indeed, by [LaMue09, Corollary 1.2] we have
$$\sharp\{\varpi\in\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))\colon \sigma_\varpi(\infty)>0\}\ll M^3. \tag{28}$$
In view of this, the result fits within our usual philosophy. Indeed, for $\sigma=\epsilon/4>0$ close to zero we count all exceptional Maaß forms and as σ increases fewer forms contribute to the count. Note

that the statement of [Blo13, Theorem 3] actually reads

$$\sum_{\substack{\varpi\in\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))\\ \|\mu_\varpi(\infty)\|_\infty=T+O(1)\\ \sigma_\varpi(\infty)>0}} T^{4\sigma_\varpi(\infty)} \ll T^{2+\epsilon}. \tag{29}$$

The result as stated is deduced from this by cutting Ω_M into $O(1)$-pieces and taking (27) into account.

Theorem 5.6 (Theorem 2, [BBR14])**.** *Let $v=p$ be a finite place of $\mathbb{Q}$. Then, for $\epsilon>0$, we have*

$$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_{v,\epsilon} (M^5)^{1-\sigma+\epsilon}. \tag{30}$$

Remark 5.7. We should note that the statement given in [BBR14, Theorem 2] is slightly weaker than what we are claiming here. However, the argument given in loc. cit. is easily adapted to produce the bound above. This is also discussed in the final remark of [MaTe21, Section 1.D]. For completeness let us provide a brief sketch of the argument using notation and results from [BBR14].

Fix $\epsilon>0$. If $\sigma<\epsilon$, then $(M^5)^{1-\sigma+\epsilon}\leq M^5$ and Weyl's law gives the desired bound. Thus we can assume that $\sigma\geq\epsilon$. In this case one applies [BBR14, (18)] to obtain

$$p^{l\sigma_\varpi(v)} \ll_\epsilon |A_\varpi(p^l,1)|.$$

At this point we estimate

$$\begin{aligned} N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) &\leq p^{-2l\sigma}\sum_{\varpi\in\mathcal{F}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))} p^{2l\sigma_\varpi(v)} \\ &\ll_\epsilon p^{-2l\sigma}\sum_{\varpi\in\mathcal{F}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))} |A_\varpi(p^l,1)|^2. \end{aligned} \tag{31}$$

Estimating this as in the proof of [Blo13, Theorem 3] gives

$$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_\epsilon (Mp^l)^\epsilon(M^5+M^2p^l)p^{-2l\sigma}. \tag{32}$$

To establish the density hypothesis we need to pick l so that $p^l \asymp_p M^{\frac{5}{2}}$. Since l should be an integer, we choose $l=\left\lfloor\frac{5\log(M)}{2\log(p)}\right\rfloor$. In view of our bound above this leads to

$$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_\epsilon p^{2\sigma+\epsilon}(M^5)^{1-\sigma+\epsilon}$$

as desired. We conclude this remark by discussing the strength of the exponent:

- Using [BBR14, (23)] with $k=1$ instead of the argument from [Blo13] to estimate (31) gives the weaker bound

  $$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_\epsilon (Mp^l)^\epsilon(M^5+M^2p^{2l}+M^3p^{\frac{7}{16}l}+p^{\frac{20}{3}l})p^{-2l\sigma}.$$

  In particular, this estimate only allows us to take $p^l\asymp_p M^{\frac{3}{4}}$. This way one obtains the density result

  $$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_{v,\epsilon} (M^5)^{1-\frac{3}{10}\sigma+\epsilon},$$

  which is weaker than Sarnak's density hypothesis.
- The optimal choice in (32) is $p^l\asymp M^3$ and yields an improved density theorem. Even more, [BuZh20, Theorem 3.3] can be used to estimate (31) even better. This way one can prove

  $$N_v(\sigma,\mathcal{F}_{\text{cusp}}(\Omega_M,\mathrm{SL}_3(\mathbb{Z}))) \ll_{v,\epsilon} (M^5)^{1-\frac{8}{5}\sigma+\epsilon}.$$

  This improves upon the density hypothesis and was already recorded in [JaKa24, Proposition 4.15].

Theorem 5.8 (Theorem 4 and 5, [BBM17])**.** *Fix $\Omega_M \subseteq \mathbb{C}^3$, a place v of $\mathbb{Q}$ and let $\Gamma_0(N) \subseteq \mathrm{SL}_3(\mathbb{Z})$ denote the Hecke congruence subgroup*

$$\Gamma_0(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_3(\mathbb{Z}).$$

If $v = p$ is finite then we assume $(N,p) = 1$. For N prime we have

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{SL}_3(\mathbb{Z})\colon \Gamma_0(N)]^{1-2\sigma+\epsilon}. \tag{33}$$

Remark 5.9. Some remarks are in order:

(1) The statement for finite v appearing in [BBM17, Theorem 5] is slightly weaker than what is claimed in the statement above. However, what we state is easily deduced from [BBM17, Theorem 4]. See Remark 5.7 for a similar discussion.

(2) For general N the proof given in [BBM17] reveals the weaker bound

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{SL}_3(\mathbb{Z})\colon \Gamma_0(N)]^{1-\sigma+\epsilon}, \tag{34}$$

which matches Sarnak's density hypothesis (see (24)).

In view of Remark 5.2 we can rephrase (24) as

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{SL}_3(\mathbb{Z})\colon \Gamma_0(N)]^{\frac{2}{p_\varpi(v)}+\epsilon}.$$

All the results mentioned so far are for $n = 3$ only and crucially rely on explicit versions of the Kuznetsov formula. Another natural approach is to use the Arthur-Selberg trace formula. This gives rise to the following general result.

Theorem 5.10 (Corollary 1.8, [MaTe21])**.** *Let $v = p$ be a finite place of $\mathbb{Q}$. Then we have*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \mathrm{SL}_n(\mathbb{Z}))) \ll p^{2\sigma}(M^d)^{1-c_n\sigma}.$$

Here $c_n > 0$ is a small unspecific constant depending (at least quadratically) on n^{-1}.

Remark 5.11. To the best of our knowledge this is the first result which gives quantitative control on the density of exceptional Maaß cusp forms for SL_n with unbounded n. Note that [MaTe21] appeared as an arXiv preprint already in 2015. Unfortunately the estimate is much weaker than what is predicted by Sarnak's density hypothesis.

Another general density estimate is contained in [BrMil24, Proposition 9.1]. This result is crucial to the argument in loc. cit., but takes a quite peculiar shape in the sense that it intertwines level aspect and spectral aspect. Also [BrMil24] was published as a preprint already in 2018.

A major breakthrough was obtained in [Blo23], where the density hypothesis (and more) was established for SL_n, with no assumptions on n, in the level aspect.

Theorem 5.12 (Theorem 1, [Blo23])**.** *Let $M, \epsilon > 0$, $\sigma \geq 0$ and let $N \in \mathbb{N}$. Then, for a place v of $\mathbb{Q}$ with $p \nmid N$ if $v = p$ is finite, we have*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_0(N)]^{1-\frac{2}{n-1}\sigma+\epsilon}.$$

Even more, if N is prime, then

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_0(N)]^{1-\frac{4}{n-1}\sigma+\epsilon}.$$

Remark 5.13. The statement of [Blo23, Theorem 1] only covers the case when N is prime. However, the bound in the general case is a by-product of the argument. This is the content of the discussion below [Blo23, (1.4)].

The family of Hecke congruence subgroups $\Gamma_0(N)$ is very nice in several aspects. Most notably it admits an elegant theory of oldforms and newforms, which was developed in [JPS81, Ree91, Jac12]. Given a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ this theory allows us to estimate

$$\dim_\mathbb{C} \pi^{K_0(N)} \ll_\epsilon N^\epsilon, \tag{35}$$

where $K_0(N)$ is the open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ such that $K_0(N) \cap \mathrm{GL}_n(\mathbb{Q}) = \Gamma_0(N)$. Given a cusp form $\varpi \in L^2_{\mathrm{cusp}}(\Gamma_0(N)\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n)$ the standard procedure of adelization allows us to construct a corresponding cuspidal automorphic representation π_ϖ with $\pi_\varpi^{K_0(N)} \neq \{0\}$. In view of global multiplicity one (see [Shal74]) and (35) we can replace $N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N)))$ by the counting function

$$\widetilde{N}_v(\sigma, M, K_0(N)) = \sharp\{\pi = \otimes_v \pi_v \text{ cuspidal automorphic}\colon \sigma(\pi_v) \geq \sigma,\ \pi^{K_0(N)} \neq \{0\} \text{ and } \lambda_{\pi_\infty} \leq M^2\}$$

without loosing to much information. Reformulating Theorem 5.12 accordingly yields

$$\widetilde{N}_v(\sigma, M, K_0(N)) \ll_{M,v,\epsilon} \begin{cases} [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_0(N)]^{1-\frac{2}{n-1}\sigma+\epsilon} & \text{for arbitrary } N, \\ [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_0(N)]^{1-\frac{4}{n-1}\sigma+\epsilon} & \text{for } N \text{ prime.} \end{cases}$$

A different family of automorphic forms was studied in [Jan21]. Indeed given a representation π_∞ of $\mathrm{PGL}_n(\mathbb{R})$ we associate the analytic conductor $C(\pi_\infty)$ as in [IwSa00]. Note that we are not making any assumption on the minimal K-type of π_∞. We define $\mathcal{F}^{\mathrm{co}}(X, \Gamma)$ to be a maximal orthogonal set of automorphic forms ϖ for Γ such that $C(\pi_{\varpi,\infty}) \leq X$. From Weyl's law it can be deduced that

$$\sharp \mathcal{F}^{\mathrm{co}}_{\mathrm{cusp}}(X, \Gamma) \asymp_\Gamma X^{n-1}.$$

Theorem 5.14 (Theorem 3, [Jan21])**.** *Let v be a finite place of $\mathbb{Q}$. Then we have*

$$N_v(\sigma, \mathcal{F}^{\mathrm{co}}_{\mathrm{cusp}}(X, \mathrm{SL}_n(\mathbb{Z}))) \ll_{v,\epsilon} (X^{n-1})^{1-\frac{2}{n-1}\sigma+\epsilon}.$$

So far we only have discussed families of automorphic forms which are closely linked to families of (automorphic) L-functions. See [SST16] for a detailed account on the notion of families.[8] However, there are many other reasonable families of automorphic forms. The level structure underlying one of the original formulations of Sarnak's density hypothesis, see for example Conjecture 4.7, arises from so-called principal congruence subgroup. For $\mathrm{SL}_n(\mathbb{Z})$ these are defined as

$$\Gamma(N) = \begin{pmatrix} 1+N\mathbb{Z} & N\mathbb{Z} & \cdots & N\mathbb{Z} \\ N\mathbb{Z} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & N\mathbb{Z} \\ N\mathbb{Z} & \cdots & N\mathbb{Z} & 1+N\mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_n(\mathbb{Z}) \text{ for } N \in \mathbb{N}. \tag{36}$$

As we will see in Section 6 below the arithmetic applications arising from this family of congruence lattices are particularly beautiful.

Establishing the density hypothesis for the lattices $\Gamma(N)$ is the content of the two papers [AsBl24, ABN24], which are both part of this habilitation. The first theorem in this direction is the following:

[8] The attentive reader will note that we slightly abuse the term family when using it for $\mathcal{F}(\Omega, \Gamma)$.

Theorem 5.15 (Theorem 1.1, [AsBl24])**.** *Let $M, \epsilon > 0$, $N \in \mathbb{N}$ squarefree and fix a place v of $\mathbb{Q}$. If $v = p$ is finite, then we assume $p \nmid N$. There exists $K > 0$ depending only on n such that*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{v,\epsilon} M^K \cdot [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{2}{n-1}\sigma+\epsilon}.$$

Remark 5.16. Some remarks are in order:

(1) The dependence on M (i.e. the spectral aspect) is not very impressive. However, for applications it turns out to be important that it is polynomial. Achieving this uniformity requires some delicate tweaks of the archimedean argument in [Blo13]. A convenient by-product is that also the density estimate stated in Theorem 5.12 above can be made polynomial in M.

(2) As a corollary, see [AsBl24, Corollary 1.2], we get a Sarnak-Xue type multiplicity bound

$$m(\pi_\infty, \Gamma(N)) \ll_{\lambda_{\pi_\infty}, v, \epsilon} [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{2}{n-1}\sigma_{\pi_v}+\epsilon}.$$

As discussed in Remark 5.2 this is different from the Sarnak-Xue L^p-conjecture as soon as $n > 3$.

(3) Using Langlands' theory of Eisenstein series and the classification of the discrete spectrum by Moeglin and Waldspurger given in [MoWa89] one can upgrade the cuspidal density theorem to the full spectrum, see [AsBl24, Theorem 7.1]. We will come back to this in Corollary 5.21 below.

Remark 5.17. In Remark 4.5 we have seen that for $n = 2$ the density hypothesis for the principal congruence group follows from a strong $\Gamma_0(N)$ density theorem. Recall that the key to this deduction was that $\Gamma(N)$ is conjugate to a nice subgroup of $\Gamma_0(N^2)$. This argument does not generalize to $n \geq 3$. In the following discussion, which is kept rather informal, we will illustrate the issue. Write $K(N)$ for the open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ corresponding to the principal congruence subgroup $\Gamma(N)$. Strong approximation fails for this subgroup, so that the assignment $\varpi \rightsquigarrow \pi_\varpi$ looses uniqueness. However, one still obtains

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \approx \frac{1}{N} \sum_{\substack{\pi=\otimes_v \pi_v, \text{ cuspidal} \\ \lambda_{\pi_\infty} \ll M^2 \\ \sigma(\pi_v) \geq \sigma}} \dim_{\mathbb{C}} \pi^{K(N)}.$$

It can be shown that for cuspidal automorphic representations one has $\dim_{\mathbb{C}} \pi^{K(N)} \asymp N^{\frac{n(n-1)}{2}}$. Furthermore, if $\pi^{K(N)} \neq \{0\}$, then π has level $\leq N^n$. In view of (35) we conclude that

$$\sum_{\substack{\pi=\otimes_v \pi_v \\ \text{cuspidal} \\ \lambda_{\pi_\infty} \ll M^2}} \dim_{\mathbb{C}} \pi^{K(N)} \approx N^{\frac{n(n-1)}{2}} \sum_{\chi \bmod N} \sum_{\substack{\pi=\otimes_v \pi_v \text{ cuspidal} \\ \omega_\pi=\chi,\, \sigma(\pi_v)\geq\sigma \\ \lambda_{\pi_\infty} \ll M^2}} \dim_{\mathbb{C}} \pi^{K_0(N^n)},$$

where ω_π denotes the central character of π. A minor modification of Theorem 5.12 shows that

$$\sum_{\chi \bmod N} \sum_{\substack{\pi=\otimes_v \pi_v \text{ cuspidal} \\ \omega_\pi=\chi,\, \sigma(\pi_v)\geq\sigma \\ \lambda_{\pi_\infty} \ll M^2}} \dim_{\mathbb{C}} \pi^{K_0(N^n)} \ll_{M,v,\epsilon} N^{n(n-1)-2n\sigma+\epsilon}.$$

Putting every together yields

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{M,v,\epsilon} N^{\frac{3n(n-1)}{2}-2n\sigma+\epsilon}. \tag{37}$$

For $n \geq 3$ this estimate falls short of anything resembling the density hypothesis. Indeed, even for $\sigma = 0$ we overestimate $N_v(0, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \asymp N^{n^2-1}$. Thus, not even an improvement in the exponent of the density theorem for $\Gamma_0(N)$ can save this argument.

However, for $n = 2$ we have a numerical coincidence and (37) reads

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{M,v,\epsilon} (N^3)^{1-\frac{4}{3}\sigma+\epsilon},$$

which is slightly weaker than the density hypothesis. However, an improved density hypothesis for $\Gamma_0(N)$, as given in Theorem 4.3 for example, is strong enough to recover the density hypothesis for $\Gamma(N)$. This is a reincarnation of the argument from Remark 4.5.

In view of the strengthening of the density hypothesis for Hecke congruence subgroups, given in Theorem 5.12, it is a natural challenge to obtain similar improvements for the principal congruence subgroup. This is addressed in [BlMa24].

Theorem 5.18 (Proposition 4, [BlMa24]). *Let $M, \epsilon > 0$, $N \in \mathbb{N}$ prime and fix a place v of $\mathbb{Q}$. If $v = p$ is finite, then we assume $p \nmid N$. There exist $K, \delta > 0$ depending only on n such that*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{v,\epsilon} M^K \cdot [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{2+\delta}{n-1}\sigma+\epsilon}.$$

Theorems 5.15 and 5.18 come with severe restrictions on the level N of the principal congruence subgroup. These restrictions are due to several hard technical problems that arise in the general case. Removing them requires new ideas and was achieved in [ABN24].

Theorem 5.19 (Theorem 1.1, [ABN24]). *Let $M, \epsilon > 0$, $N \in \mathbb{N}$ arbitrary and fix a place v of $\mathbb{Q}$. If $v = p$ is finite, then we assume $p \nmid N$. There exists K depending only on n such that*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{v,\epsilon} M^K \cdot [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{2}{n-1}(1+\frac{1}{n+1})\sigma+\epsilon}.$$

Note that this is an explicit improvement on Sarnak's density hypothesis. The result goes beyond Theorem 5.15 and 5.18 in strength and in generality simultaneously. It is reasonable to believe that for general n the factor $\frac{2}{n-1}(1 + \frac{1}{n+1})$ in the exponent is the limit of current technology. However, for $n = 3$ one can do slightly better.

Theorem 5.20 (Theorem 1.8, [ABN24]). *Let $M, \epsilon > 0$, $N \in \mathbb{N}$ arbitrary and fix a place v of $\mathbb{Q}$. If $v = p$ is finite, then we assume $p \nmid N$. There exists K such that*

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma(N))) \ll_{v,\epsilon} M^K \cdot [\mathrm{SL}_3(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{3}{2}\sigma+\epsilon} = M^K N^{8-12\sigma+\epsilon}.$$

Finally, we can extend the density theorem to the full discrete spectrum. Note that the resulting family contains the trivial representation, so that one can not hope to improve upon Sarnak's density hypothesis.

Corollary 5.21 (Corollary 1.3, [ABN24]). *Let $M, \epsilon > 0$, $N \in \mathbb{N}$ arbitrary and fix a place v of $\mathbb{Q}$. If $v = p$ is finite, then we assume $p \nmid N$. There exists K depending only on n such that*

$$N_v(\sigma, \mathcal{F}(\Omega_M, \Gamma(N))) \ll_{v,\epsilon} M^K \cdot [\mathrm{SL}_n(\mathbb{Z})\colon \Gamma(N)]^{1-\frac{2}{n-1}\sigma+\epsilon}.$$

One can go a step further and include the full spectrum. This means adding the contribution of Eisenstein series as well. A statement in this direction is given in [ABN24, Corollary 1.5].

A straight forward application of the Jacquet-Langlands correspondence allows one to transfer Theorem 5.19 to certain co-compact quotients of $\mathrm{SL}_n(\mathbb{R})$.

Corollary 5.22 (Theorem 1.9, [ABN24]). *Let $\Gamma_{\mathcal{O}} \subseteq \mathrm{SL}_n(\mathbb{R})$ be a co-compact lattice arising from an order in a quaternion division algebra over $\mathbb{Q}$. For $N \in \mathbb{N}$ co-prime to the discriminant we define the principal congruence subgroups $\Gamma_{\mathcal{O}}(N)$ and obtain*

$$N_v(\sigma, \mathcal{F}(\Omega_M, \Gamma_{\mathcal{O}}(N))) \ll_{\mathcal{O},v,\epsilon} M^K \cdot [\Gamma_{\mathcal{O}} \colon \Gamma_{\mathcal{O}}(N)]^{1-\frac{2}{n-1}\sigma+\epsilon}.$$

Different families of (congruence) lattices give rise to new applications and also to new challenges. To illustrate this we have studied level families arising from Borel type congruence subgroups:

$$\Gamma_2(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} \\ N\mathbb{Z} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{Z} \\ N\mathbb{Z} & \cdots & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_n(\mathbb{Z})$$

in [Ass23a]. The density theorem takes the usual shape.

Theorem 5.23 (Theorem 1.1, [Ass23a]). *For N prime and $\sigma, \epsilon > 0$ we have*

$$N_\infty(\sigma, \mathcal{F}(\Omega_M, \Gamma_2(N))) \ll_\epsilon M^K [\mathrm{SL}_n(\mathbb{Z}) \colon \Gamma_2(N)]^{1-\frac{2}{n-1}\sigma+\epsilon}, \tag{38}$$

for an absolute constant K depending only on n.

Remark 5.24. The assumption that N is prime seems to be difficult to remove. It can probably be relaxed to squarefree levels, but the general case would require new ideas going even beyond those developed in [ABN24].

We conclude this section by briefly sketching the basic method behind most of the density results presented in this section. Similar discussions have appeared in [Blo23, Section 1.1] and [AsBl24, Section 1.4]. Let $\Gamma \subseteq \mathrm{SL}_n(\mathbb{R})$ be a (non co-compact) congruence subgroup. In order to establish the density hypothesis we use (a version of) the Rankin-Trick and replace $N_\infty(\sigma, \mathcal{F}(\Omega_M, \Gamma))$ by

$$N_\infty(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)) \leq Z^{-2\sigma} \sum_{\varpi \in \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)} Z^{2\sigma_\varpi(\infty)}.$$

The density hypothesis follows if we can show

$$\sum_{\varpi \in \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)} Z^{2\sigma_\varpi(v)} \ll_\epsilon \mathrm{Vol}(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}))^{1+\epsilon}, \tag{39}$$

for $Z \asymp \mathrm{Vol}(\Gamma\backslash \mathrm{SL}_n(\mathbb{R}))^{\frac{1}{n-1}}$. Allowing larger Z will result in estimates that go beyond the density hypothesis. This strategy has appeared several times above, see for example Remark 5.7. The next step is to realise the right hand side of (39) as the spectral side of the Kuznetsov formula. With an appropriately chosen test function the $\mathrm{SL}_n(\mathbb{R})$-Kuznetsov formula is an identity that roughly reads

$$\frac{1}{\sharp \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)} \sum_{\varpi \in \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)} |A_\varpi(\mathbf{1})|^2 \cdot Z^{2\sigma_\varpi(\infty)} \approx 1 + \sum_{1 \neq w \in W} \sum_{\substack{\mathbf{c} \in \mathbb{N}^{n-1} \\ |c_1|, \ldots, |c_{n-1}| \ll Z}} \frac{S_{\Gamma,w}(\mathbf{1}, \mathbf{1}, \mathbf{c})}{|c_1 \cdots c_{n-1}|}. \tag{40}$$

Here $W \cong S_n$ is the Weyl group, $A_\varpi(\mathbf{1})$ is the *first* Fourier coefficient of ϖ and $S_{\Gamma,w}(\mathbf{1}, \mathbf{1}, \mathbf{c})$ is the w-Kloosterman sum for Γ.

The spectral side (i.e. the right hand side of (40)) is a version of the right hand side of (39) weighted by Fourier coefficients. In order to remove these weights one needs to show that

$$\sum_{\substack{\varpi\in\mathcal{F}_{\mathrm{cusp}}(\Omega_M,\Gamma)\\ \mu_\varpi(\infty)=\mu_0}} |A_\varpi(\mathbf{1})|^2 \approx \sharp\{\varpi\in\mathcal{F}_{\mathrm{cusp}}(\Omega_M,\Gamma)\colon \mu_\varpi(\infty)=\mu_0\}. \tag{41}$$

This yields

$$\frac{1}{\sharp\mathcal{F}_{\mathrm{cusp}}(\Omega_M,\Gamma)}\sum_{\varpi\in\mathcal{F}_{\mathrm{cusp}}(\Omega_M,\Gamma)} Z^{2\sigma_\varpi(v)} \approx 1+\sum_{1\neq w\in W}\sum_{\substack{\mathbf{c}\in\mathbb{N}^{n-1}\\ |c_1|,\ldots,|c_{n-1}|\ll Z}} \frac{S_{\Gamma,w}(\mathbf{1},\mathbf{1},\mathbf{c})}{c_1\cdots c_{n-1}}. \tag{42}$$

Establishing (41) is not an easy task in general. For $\Gamma=\Gamma_0(N)$ it follows from the Rankin-Selberg method and newform theory. On the other hand, for the principal congruence subgroup $\Gamma=\Gamma(N)$ and the Borel type congruence subgroup $\Gamma=\Gamma_2(N)$ more complicated local considerations are required. See [ABN24, Section 1.3] for more discussion.

To finish the proof of (39) it suffices to estimate the right hand side of (42). In particular, we must show that

$$\sum_{\substack{\mathbf{c}\in\mathbb{N}^{n-1}\\ |c_1|,\ldots,|c_{n-1}|\ll Z}} \frac{S_{\Gamma,w}(\mathbf{1},\mathbf{1},\mathbf{c})}{c_1\cdots c_{n-1}} \ll 1 \tag{43}$$

for all $q\neq w\in W$ and Z sufficiently large. The trivial bound for the Kloosterman sums was established in [DaRe98] and takes the form

$$\frac{S_{\Gamma,w}(\mathbf{1},\mathbf{1},\mathbf{c})}{|c_1\cdots c_{n-1}|} \ll_\Gamma 1. \tag{44}$$

Unfortunately this is insufficient for our purposes. In order to efficiently estimate the geometric side (i.e. (43)) we need to exploit the following two phenomena:

- The Kloosterman sums vanish by default unless certain compatibility relations between w and $\mathbf{c}$ are satisfied.
- The congruence subgroup Γ forces certain divisibility conditions on the moduli $\mathbf{c}$. More precisely $S_{\Gamma,w}(\mathbf{1},\mathbf{1},\mathbf{c})=0$ unless $c_1,\ldots,c_{n-1}$ satisfy certain congruence conditions that depend on Γ and w.

For $\Gamma=\Gamma_0(N)$ these observations are actually sufficient to establish the density hypothesis as stated in Theorem 5.12 for general N. For other Γ these observations vastly reduce the complexity of the geometric side, but are in general insufficient to yield the full density hypothesis. Establishing (43) for suitable Z still relies on good bounds for $S_{\Gamma,w}(\mathbf{1},\mathbf{1},\mathbf{c})$ that are uniform in Γ. Establishing these is an interesting and difficult problem in general. For the principal congruence subgroup the required bounds have been established in [ABN24, Section 3] by direct computation.

We see that the (spherical) density hypothesis in the level aspect follows from the Kuznetsov formula as long as the estimates (41) and (43) are available. Both of these estimates can be reduced to local problems. In all instances where these local problems have been solved their solution relies on structural features of the lattice family in question.

6. Arithmetic applications

The density hypothesis and its variations have many interesting applications. We collect a few of them in the following list.

- In the co-compact setting the density hypothesis yields bounds on Betti number. See [SaXu91, Theorem 4] for example. For an example of this phenomenon in higher rank we refer to the more recent work [EGG23, Theorem 1.5].
- In some situations the density hypothesis implies a (uniform) spectral gap. See for example [SaXu91, Corollary 2].
- The density hypothesis for spectral families implies optimal diophantine exponents. See for example [JaKa24, Theorem 3].
- On a combinatorial level the density hypothesis can be used to construct families of graphs (or complexes) with good expansion properties. One can also formulate the density hypothesis on the level of graphs. In this setting its applications include cut-off and optimal almost diameter. We refer to [GoKa22, GoKa23, EGG23] and the references within for a more thorough discussion of these matters.

So far we have not mentioned uniform counting and optimal lifting, which are both important arithmetic concepts closely connected to the density hypothesis. We will discuss these in more detail now.

Uniform counting was one of the main motivations behind the formulation of the density hypothesis in [Sar90]. To set things up we start with a semisimple group $G(\mathbb{R}) \subseteq \mathrm{GL}_m(\mathbb{R}) \subseteq \mathbb{R}^{m^2}$ and assume that $K = G(\mathbb{R}) \cap O_m(\mathbb{R})$ is a maximal compact subgroup. The euclidean norm $\|\cdot\|$ on $\mathbb{R}^{m^2}$ descends to $G(\mathbb{R})$ and we define

$$\alpha_{\|\cdot\|}(G) = \lim_{T\to\infty} \log(T)^{-1} \cdot \log\left(\mathrm{Vol}(\{g \in G(\mathbb{R})\colon \|g\| \leq T\}, dg)\right)$$

for a Haar measure dg on $G(\mathbb{R})$. We have suppressed the dependence of α on $\|\cdot\|$ and the embedding $G(\mathbb{R}) \subseteq \mathrm{GL}_m(\mathbb{R})$ in the notation. For example taking the standard inclusion $\mathrm{SL}_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ yields $\alpha(\mathrm{SL}_n) = n(n-1)$. Using the realization of G in GL_n we can define $\Gamma(1) = G(\mathbb{Z}) = G(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z})$ and

$$\Gamma(N) = \{\gamma \in G(\mathbb{Z})\colon \gamma \equiv I \bmod N\}.$$

One has the well known asymptotic expansion

$$\sharp\{\gamma \in \Gamma(N)\colon \|\gamma\| \leq T\} = \frac{c_G \cdot T^{\alpha}}{[G(\mathbb{Z})\colon \Gamma(N)]} + O_{N,G}(T^{\alpha-\delta}), \tag{45}$$

for some $\delta > 0$. See [DRS93]. This asymptotic formula alone is a very important result. However, its lacking uniformity in the level N can be problematic in practice. To remedy this Sarnak proposed the following uniform counting conjecture.

Conjecture 6.1 (Main Conjecture 2.1, [Sar90]). *For G as above and $T, N \geq 1$ we have*

$$\sharp\{\gamma \in \Gamma(N)\colon \|\gamma\| \leq T\} \ll_{G,\epsilon} (TN)^{\epsilon}\left(\frac{T^{\alpha}}{[G(\mathbb{Z})\colon \Gamma(N)]} + T^{\frac{\alpha}{2}}\right).$$

For $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ a beautiful elementary argument, going back to Sarnak and Xue, produces the following estimate.

Theorem 6.2 (Proposition 5.3, [Gam02][9]). *We have*

$$\sharp\{\gamma \in \mathrm{SL}_2(\mathbb{Z})\colon \gamma \equiv I \text{ mod } N \text{ and } \|\gamma\|_{\infty} \leq T\} \ll_{\epsilon} (NT)^{\epsilon}\left(\frac{T^2}{N^3} + \frac{T}{N} + 1\right). \tag{46}$$

In particular, Conjecture 6.1 holds for SL_2.

[9] This statement and its proof appears in many places. Choosing this reference is historically inaccurate but still justifiable. Note that we have fixed a small typo appearing in loc. cit..

Similarly, the analogous conjecture was established for principal congruence subgroups of arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. See [SaXu91, Theorem 1]. On the other hand, it is shown in [HuKa93, Theorem 1.1] that Conjecture 6.1 implies Sarnak's density hypothesis for G of real rank one.

In higher rank the relation between the density hypothesis and Conjecture 6.1 becomes more subtle. The main difficulty arises due to the existence of non-generic Eisenstein series, which are analytically hard to handle. A first step was taken in [AsBl24], where the following conditional result was obtained:

Theorem 6.3 (Theorem 1.4, [AsBl24])**.** *Let $\Gamma(N) \subseteq \mathrm{SL}_n(\mathbb{Z})$ be the principal congruence subgroup defined in* (36)*. Conditional on* [AsBl24, Hypothesis 1] *we have*

$$\sharp\{\gamma \in \Gamma(N) \colon \|\gamma\| \leq T\} \ll_\epsilon (TN)^\epsilon \left(\frac{T^{n(n-1)}}{[\mathrm{SL}_n(\mathbb{Z}) \colon \Gamma(N)]} + T^{\frac{n(n-1)}{2}} \right).$$

for squarefree N and $T \geq 1$.

Here [AsBl24, Hypothesis 1] essentially postulates that the norms of all relevant truncated Eisenstein series for SL_n are essentially bounded in the N-aspect and of at most polynomial growth in the spectral aspect. Establishing this hypothesis would require an intricate analysis of the Maaß-Selberg relations and the availability of strong zero free regions for Rankin-Selberg L-functions. See [AsBl24, Section 1.3] and [JaKa22, Section 1.4] for a more detailed discussion.

It turns out that [AsBl24, Hypothesis 1] is stronger than what is actually needed to deduce uniform counting results from density theorems. Indeed, in practice it suffices to control the L^2-growth of Eisenstein on average. This crucial idea was carefully worked out in [JaKa22].

Theorem 6.4 (Theorem 5, [JaKa22])**.** *For $G = \mathrm{SL}_n$ and N squarefree Conjecture 6.1 holds unconditionally.*

The squarefreeness assumption on N is a relict from the [AsBl24] density theorem as stated in Theorem 5.15, which is a key ingredient for the proof of [JaKa22]. Since this assumption was recently removed in [ABN24] we now have uniform counting for the principal congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ in full generality:

Theorem 6.5 (Corollary 1.6, [ABN24])**.** *Conjecture 6.1 is true for $G = \mathrm{SL}_n$.*

Remark 6.6. The second error term $T^{\alpha/2}$ in Conjecture 6.1 is (up to the index) exactly the square root of the main term. This numerology is no coincidence and should be no surprise to analytic number theorists. Indeed, a classical analogue is counting primes in arithmetic progressions modulo q. Here it is known that under the generalized Riemann hypothesis one has a main term of size x/q and an error of roughly size $x^{\frac{1}{2}}$.

The significance of the $T^{\alpha/2}$-term can be seen for $G(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$. Indeed, one clearly has

$$\sharp\{\gamma \in \Gamma(N) \colon \|\gamma\|_\infty \leq T \text{ and } \gamma \text{ upper triangular}\} \asymp \left(\frac{T}{N}\right)^{\frac{n(n-1)}{2}}.$$

This matches the error term up to the saving in N. In particular, the claimed bound

$$\sharp\{\gamma \in \Gamma(N) \colon \|\gamma\| \leq T\} \ll_{n,\epsilon} \frac{T^{n(n-1)}}{[\mathrm{SL}_n(\mathbb{Z}) \colon \Gamma(N)]} \log(T) + 1$$

from [Kat93, Theorem 4] is to good to be true. See also [GoKa23, Section 2.6] for a related discussion.

For general (congruence) lattices $\Gamma \subseteq G(\mathbb{Z})$ the direct generalization of uniform counting is more subtle. However, one still expects that the the following is true

$$\frac{1}{[G(\mathbb{Z})\colon \Gamma]} \sum_{x \in \Gamma\backslash G(\mathbb{Z})} \sharp\{\gamma \in x^{-1}\Gamma x \colon \|\gamma\| \leq T\} \ll_\epsilon T^\epsilon \left(\frac{T^\alpha}{[G(\mathbb{Z})\colon \Gamma(N)]} + T^{\frac{\alpha}{2}} \right) \tag{47}$$

Note that, if Γ is normal in $G(\mathbb{Z})$, then $x^{-1}\Gamma x = \Gamma$ for all relevant x and the average is superfluous. In some situations the quotient $\Gamma\backslash G(\mathbb{Z})$ has a nice geometric description endowing the average in (47) with interesting interpretations.

We now turn back to $G = \mathrm{SL}_n$ and discuss some instances of (47) in this setting. We start by considering the the case of $\Gamma_2(N)$ assuming that $N = l$ is prime. We denote the set of complete flags in $\mathbb{F}_l^n$ by

$$B_l = \{(V_1, \dots, V_{n-1})\colon 0 < V_1 < \ldots < V_{n-1} < \mathbb{F}_l^n\}.$$

Reduction modulo l defines an action

$$\Phi_l \colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Sym}(B_l).$$

We write

$$\mathbf{1} = \{\langle e_n \rangle, \langle e_n, e_{n-1} \rangle, \dots\}$$

for the standard flag and observe that[10]

$$\Gamma_2(l)^\top = \{\gamma \in \mathrm{SL}_n(\mathbb{Z})\colon \Phi_l(\gamma)\mathbf{1} = \mathbf{1}\}. \tag{48}$$

The following is an adaption of Conjecture 6.1 to the setting at hand.

Conjecture 6.7. *For every prime l and $\epsilon > 0$ we have*

$$\sharp\{(\gamma, x) \in \mathrm{SL}_n(\mathbb{Z}) \times B_l \colon \|\gamma\|_\infty \leq T \text{ and } \Phi_l(\gamma)(x) = x\} \ll_\epsilon (Tl)^\epsilon \left(T^{n(n-1)} + (Tl)^{\frac{n(n-1)}{2}} \right).$$

This set-up was already considered in [KaLa23], where the case $n = 3$ was treated. In loc. cit. they prove a slightly modified version of Conjecture 6.7. To state it we need to define the following gauge:

$$\|g\|_* = \|g\|_\infty \cdot \|g^{-1}\|_\infty \tag{49}$$

on $\mathrm{SL}_n(\mathbb{R})$.

Remark 6.8. Note that for $n = 2$ we have $\|g\|_\infty = \|g^{-1}\|_\infty$ for $g \in \mathrm{SL}_2(\mathbb{R})$. Thus in this case we would simply have $\|g\|_* = \|g\|_\infty^2$. However, for higher rank (i.e. $n \geq 3$) the two gauges $\|\cdot\|_\infty$ and $\|\cdot\|_*$ can differ drastically.

Theorem 6.9 (Theorem 5.2, [KaLa23])**.** *There is a constant $C > 0$ such for every prime l and $T \leq Cl^3$ we have*

$$\sharp\{(\gamma, x) \in \mathrm{SL}_3(\mathbb{Z}) \times B_l \colon \|\gamma\|_* \leq T \text{ and } \Phi_l(\gamma)(x) = x\} \ll_\epsilon (l^3T)^{1+\epsilon}.$$

Note that the exponents do not exactly match the ones in Conjecture 6.7. The reason for this is that the balls $\{\|g\|_\infty \leq T\}$ and $\{\|g\|_* \leq T\}$ have different volumes. However, qualitatively the two statement essentially agree.

The proof of Theorem 6.9 proceeds without making any reference to the density hypothesis. Instead it is obtained by a direct counting argument. This counting argument is very involved and

[10]For technical reasons we have chosen the standard flag so that its stabilizer is the transpose of the Borel type congruence subgroup. This is however inessential.

is unlikely to generalize to $n > 3$. However, by using ideas from [JaKa22] together with the density theorem for $\Gamma_2(N)$ we can establish the following:

Theorem 6.10 (Theorem 10.1, [Ass23a])**.** *Conjecture 6.7 holds for all $n \geq 3$.*

Since it seems fitting we can formulate a similar statement for the Hecke congruence subgroup $\Gamma_0(N)$. Here we have to replace the space of flags by projective space. Let

$$P_l = P^{n-1}(\mathbb{F}_l) = \{\mathbf{x} \in \mathbb{F}_l^n \setminus \{0\}\}/\sim,$$

where the equivalence relation $\sim$ on $\mathbb{F}_l^n \setminus \{0\}$ is given by $\mathbf{x} \sim \mathbf{y}$ if there is $\alpha \in \mathbb{F}_l^\times$ such that $\mathbf{y} = \alpha\mathbf{x}$. Again reduction modulo l (coupled with the canonical action of $\mathrm{SL}_n(\mathbb{F}_l)$ on P_l) yields an action

$$\Psi_l \colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Sym}(P_l).$$

The Hecke congruence subgroup arises as

$$\Gamma_0(l)^\top = \{\gamma \in \mathrm{SL}_n(\mathbb{Z}) \colon \Psi_l(\gamma)\overline{e_n} = \overline{e_n}\}, \tag{50}$$

where $\overline{e_n}$ denotes the image of e_n in P_l.

Theorem 6.11 (Theorem 1.4, [KaLa23])**.** *Then there exists a constant $C > 0$ such that for every prime l, $T \leq Cl^2$ and $\epsilon > 0$ it holds that*

$$\sharp\{(\gamma, x) \in \mathrm{SL}_3(\mathbb{Z}) \times P_l \colon \|\gamma\|_* \leq T \text{ and } \Psi_l(\gamma)x = x\} \ll_\epsilon (l^2T)^{1+\epsilon}.$$

Again this result is obtained by a direct counting argument, which is hard to generalize directly to higher rank. However, using the density hypothesis for $\Gamma_0(l)$ established in [Blo23] one can prove the following:

Theorem 6.12. *For every prime l and $\epsilon > 0$ we have*

$$\sharp\{(\gamma, x) \in \mathrm{SL}_n(\mathbb{Z}) \times P_l \colon \|\gamma\|_\infty \leq T \text{ and } \Psi_l(\gamma)(x) = x\} \ll_\epsilon (Tl)^\epsilon \left(T^{n(n-1)} + T^{\frac{n(n-1)}{2}} l^{n-1}\right).$$

Remark 6.13. The proof of this result combines the density estimate Theorem 5.12 with the methods from [JaKa22]. The proof is analogous to the one given in [Ass23a, Section 10] and was anticipated in [GoKa23, Section 2.5] as well as in the introduction of [KaLa23].

We turn towards optimal lifting. Optimal lifting was introduced in [Sar15] as a close analogue of studying *the least prime in arithmetic progressions.* For $\mathrm{SL}_2(\mathbb{Z})$ the following statement is taken from [KaLa23, Theorem 1.1].

Theorem 6.14 ([Sar15])**.** *For all $\epsilon > 0$ as $N \to \infty$ there exists a set $Y \subseteq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ with*

$$\sharp Y \geq \sharp\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cdot (1 + o_\epsilon(1))$$

such that for every $y \in Y$ there is $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\|\gamma\|_\infty \leq N^{\frac{3}{2}+\epsilon}$ and $\gamma \equiv y$ mod N.

Remark 6.15. In [Sar15] this is proved using the spectral theory of automorphic forms. As a key input a density theorem for the principal congruence subgroup is used. On the other hand, in [KaLa23] a different proof is given. Indeed, in loc. cit. the elementary counting result (46) is combined with Selberg's spectral gap to establish the optimal lifting property.

It is now very natural to conjecture the following for $\mathrm{SL}_n(\mathbb{Z})$.

Conjecture 6.16 (Conjecture 1.2, [KaLa23][11])**.** *For all $\epsilon > 0$ as $N \to \infty$ there exists a set $Y \subseteq \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$ with*

$$\sharp Y \geq \sharp \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z}) \cdot (1 + o_\epsilon(1))$$

such that for every $y \in Y$ there is $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ with $\|\gamma\|_\infty \leq N^{1+\frac{1}{n}+\epsilon}$ and $\gamma \equiv y \bmod N$.

Remark 6.17. The bound $\|\gamma\|_\infty \leq N^{1+\frac{1}{n}+\epsilon}$ for the least pre-image of an element in $\mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$ is suggested by the pigeon hole principle. Thus, Conjecture 6.16 is based on the expectation that the pigeon hole principle gives the correct upper bound for generic elements in $\mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$. On the other hand, in [Sar15] Sarnak constructed examples of matrices with exceptionally large smallest pre-image. Thus, one can not expect the statement to hold with $Y = \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$, making the conjecture in some sense optimal. We refer to [KaVa23] for more interesting results concerning the size of lifts of matrices.

The first step towards the resolution of Conjecture 6.16 was taken in [AsBl24] using the density theorem for the principal congruence subgroup.

Theorem 6.18 (Theorem 1.5, [AsBl24])**.** *For N squarefree Conjecture 6.16 holds conditionally on* [AsBl24, Hypothesis 1].

As in the case of the uniform counting conjecture this was made unconditional in [JaKa22].

Theorem 6.19 (Theorem 6, [JaKa22])**.** *For N squarefree Conjecture 6.16 holds unconditionally.*

Finally, optimal lifting was established in full generally in [ABN24].

Theorem 6.20 (Corollary 1.7, [ABN24])**.** *Conjecture 6.16 holds unconditionally for arbitrary N.*

Let us briefly discuss optimal lifting in the context of different families of congruence lattices. The prototypical result in this direction were formulated in [KaLa23] for $\Gamma_0(N) \subseteq \mathrm{SL}_3(\mathbb{Z})$ and $\Gamma_2(N) \subseteq \mathrm{SL}_3(\mathbb{Z})$.

Theorem 6.21 (Theorem 1.3, [KaLa23])**.** *For every $\epsilon > 0$, as $l \to \infty$ through primes, there exists a set $Y \subseteq P_l$ of size*

$$\sharp Y \geq \sharp P_l \cdot (1 + o_\epsilon(1)),$$

such that for every $x \in Y$, there exists a set $Z_x \subseteq P_l$ of size

$$\sharp Z_x \geq \sharp P_l \cdot (1 + o_\epsilon(1)),$$

such that for every $y \in Z_x$, there exists an element $\gamma \in \mathrm{SL}_3(\mathbb{Z})$ satisfying $\|\gamma\|_\infty \leq l^{\frac{1}{3}+\epsilon}$, such that $\Psi_l(\gamma).x = y$.

Theorem 6.22 (Theorem 5.1, [KaLa23])**.** *For every $\epsilon > 0$, as $l \to \infty$ through primes, there exists a set $Y \subseteq B_l$ of size*

$$\sharp Y \geq \sharp B_l \cdot (1 + o_\epsilon(1)),$$

such that for every $x \in Y$, there exists a set $Z_x \subseteq B_l$ of size

$$\sharp Z_x \geq \sharp B_l \cdot (1 + o_\epsilon(1)),$$

such that for every $y \in Z_x$, there exists an element $\gamma \in \mathrm{SL}_3(\mathbb{Z})$ satisfying $\|\gamma\|_\infty \leq l^{\frac{1}{2}+\epsilon}$, such that $\Phi_l(\gamma).x = y$.

[11] We refer to [KaLa23, Conjecture 1.2], because it offers a clean statement of the optimal lifting conjecture in our context. Note however that this is not the place where the conjecture was first formulated. We refer also to [Sar15, GoKa23, JaKa22] for more discussion.

In [KaLa23] these two lifting theorems are obtained from the counting results Theorem 6.11 and Theorem 6.9 respectively together with a spectral gap. In view of the generalizations Theorem 6.12 and Theorem 6.10 one can hope to adapt this argument to higher rank (i.e. for $n > 3$). Another approach, which was taken in [Ass23a], is to use appropriate density theorems and the spectral theory of automorphic forms directly. Either way one obtains the following theorems.

Theorem 6.23. *For every $\epsilon > 0$, as $l \to \infty$ through primes, there exists a set $Y \subseteq P_l$ of size*

$$\sharp Y \geq \sharp P_l \cdot (1 + o_\epsilon(1)),$$

such that for every $x \in Y$, there exists a set $Z_x \subseteq P_l$ of size

$$\sharp Z_x \geq \sharp P_l \cdot (1 + o_\epsilon(1)),$$

such that for every $y \in Z_x$, there exists an element $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ satisfying $\|\gamma\|_\infty \leq l^{\frac{1}{n}+\epsilon}$, such that $\Psi_l(\gamma).x = y$.

Theorem 6.24 (Theorem 10.2, [Ass23a])**.** *For every $\epsilon > 0$, as $l \to \infty$ through primes, there exists a set $Y \subseteq B_l$ of size*

$$\sharp Y \geq \sharp B_l \cdot (1 + o_\epsilon(1)),$$

such that for every $x \in Y$, there exists a set $Z_x \subseteq B_l$ of size

$$\sharp Z_x \geq \sharp B_l \cdot (1 + o_\epsilon(1)),$$

such that for every $y \in Z_x$, there exists an element $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ satisfying $\|\gamma\|_\infty \leq l^{\frac{1}{2}+\epsilon}$, such that $\Phi_l(\gamma).x = y$.

7. A density hypothesis for Sp_4

The local and global theory of automorphic forms for groups beyond GL_n is more intricate and comes with several new features. This can already be seen when comparing the Ramanujan conjecture for GL_n, see Conjecture 3.3, with the generalized Ramanujan conjecture stated in Conjecture 3.5. The group $G = \mathrm{GSp}_4$ (or PGSp_4) is one of the easiest groups where (essentially) all the complications arising for general classical groups are visible. On the other hand, the theory of automorphic forms is developed well enough for analytic purposes. Therefore, studying the density hypothesis for this setting promises to shed some light on the general case.

The endoscopic classification of automorphic representations for GSp_4 and PGSp_4 has been addressed in [Art04, Art13, GeTa19].[12] This classification gives a detailed description of the discrete spectrum of GSp_4 in terms of automorphic forms on GL_4 and partitions it in so-called Arthur packets. These packets have been explicitly computed in [Sch18, Sch20]. As a consequence a precise description of the part of the spectrum that is known to violate the naive Ramanujan conjecture (i.e. the residual spectrum and the CAP-representations) is available. Note that the residual spectrum was already described in [Kim95]. Another by-product of the classification is a proof of the local Langlands conjecture, which was also established in [GaTa11]. Also the analytic theory of automorphic forms on GSp_4 is quite well developed. We refer to [KWS24] for an explicit version of the trace formula with interesting applications. More generally, but less explicit, the methods from [FiMa21] apply to the symplectic group. More precisely, [FiMa21, Theorem 1.2] shows that representations that are non-tempered at infinity are of lower order. On the other hand, an explicit

[12] Strictly speaking these works and the results within seem to be conditional on the resolution of certain issues in connection with some not yet published items in the reference list of [Art13]. See also [LLS24] for recent developments. Throughout this section we will assume in good fate that these will be sorted out sooner or later. In particular, we take the endoscopic classification for GSp_4 and its variants for granted.

Kuznetsov formula has been developed in [Com21] using the relative trace formula approach. See also [Man22a, Man22b] for relevant results concerning Poincaré series and Kloosterman sums.

We see all this as good evidence that the automorphic machinery for GSp_4 is sufficiently mature to allow for good progress towards Sarnak's density hypothesis in this setting. The first density estimate was established by Man in [Man22b]. In loc. cit. the ideas from [Blo23] are transferred to Sp_4. To state the theorem we introduce some notation.

We closely follow the set-up from Section 5 and work with automorphic forms

$$\varpi \in L^2_{\mathrm{disc}}(\Gamma\backslash \mathrm{Sp}_4(\mathbb{R})/K_\infty).$$

We assume that ϖ is an eigenfunctions of the algebra of invariant differential operators. The spectral parameter is denoted by $\mu_\varpi(\infty) = (\mu_\varpi(\infty,1), \mu_\varpi(\infty,2)) \in \mathbb{C}^2$. At a finite place $v = p$, where ϖ is unramified, we can further assume that ϖ is an eigenfunction of all (spherical) Hecke operators. This leads to the definition of the Satake parameter $\mu_\varpi(v) \in \mathbb{C}^2$. For comparison we note that the constant function $\varpi = \mathbf{1}$ satisfies

$$\mu_\varpi(v) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

in our normalization. As before ϖ is tempered at v if and only if $\mu_\varpi(v) \in (i\mathbb{R})^2$. We define

$$\sigma_\varpi(v) = \max_{j=1,\ldots,n} |\Re(\mu_\varpi(v,j))|.$$

This is a direct adaption of (8) to the current setting. Translating (23) from the SL_n-setting to Sp_4 leads us to the following definition. For any finite orthogonal family $\mathcal{F} \subseteq L^2_{\mathrm{disc}}(\Gamma\backslash \mathrm{Sp}_4(\mathbb{R})/K_\infty)$ we define the counting function

$$N_v(\sigma, \mathcal{F}) = \sharp\{\varpi \in \mathcal{F} \colon \sigma_\varpi(v) \geq \sigma\}. \tag{51}$$

Put $\Omega_M = \{\mu \in \mathbb{C}^2 \colon \|\mu\| \leq M\}$. We introduce the family $\mathcal{F}(\Omega_M, \Gamma)$ as a maximal orthogonal set of automorphic forms in $L^2_{\mathrm{disc}}(\Gamma\backslash \mathrm{Sp}_4(\mathbb{R})/K_\infty)$ with $\mu_\varpi(\infty) \in \Omega_M$. As usual we can form the sub-family $\mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)$ of cusp forms. Going even further we write $\mathcal{F}_{\mathrm{cusp,gen}}(\Omega_M, \Gamma)$ for the sub-family of all $\varpi \in \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma)$ that are (globally) generic.

Remark 7.1. In contrast to the SL_n case we have a strict inclusion

$$\mathcal{F}_{\mathrm{cusp,gen}}(\Omega_M, \Gamma) \subset \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma).$$

The reason for this is that Sp_4 (or PGSp_4 when working on the level of representations) features certain functorial lifts which are known to violate the naive Ramanujan conjecture and which are non generic.

Note that generic forms will never belong to CAP-representations, and are therefore believed to be tempered at all places, see Conjecture 3.5. Thus, we expect that all elements in $\mathcal{F}_{\mathrm{cusp,gen}}(\Omega_M, \Gamma)$ are tempered everywhere.

In [Sar05] the generalized Ramanujan conjecture is precisely formulated for (globally) generic representations. However, even for Sp_4 there are non-generic forms that are non-CAP. According to the formulation given in Conjecture 3.5 these are also believed to be tempered. (Using structural results concerning the GSp_4 Arthur packets one can actually show that Conjecture 3.5 follows from the formulation in [Sar05], which in turn follows from the Ramanujan conjecture for GL_n with $n = 2, 4$. See also Remark 3.6 above.)

The first density estimate concerns families of generic forms.

Theorem 7.2 (Theorem 1.1, [Man22b]). *Take $M \geq 1$, a place v of $\mathbb{Q}$ and a prime N. Define the Siegel congruence subgroup*

$$\Gamma_0(N) = \begin{pmatrix} \mathrm{Mat}_{2\times 2}(\mathbb{Z}) & \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \\ N \cdot \mathrm{Mat}_{2\times 2}(\mathbb{Z}) & \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}).$$

If $v = p$ is finite, then we assume that $(N, p) = 1$. We have

$$N_v(\sigma, \mathcal{F}_{\mathrm{cusp},gen}(\Omega_M, \Gamma_0(N))) \ll_{M,v,\epsilon} [\mathrm{Sp}_4(\mathbb{Z})\colon \Gamma_0(N)]^{1-\frac{4}{3}\sigma+\epsilon}.$$

Remark 7.3. Unfortunately the proof of Theorem 7.2 contains a gap. See [Ass23b, Remark 5.1] for a detailed explanation. With some additional local computations this gap can be filled conditional on a conjecture by Lapid-Mao formulated in [LaMao15]. However, the machinery from [Man22b] applies with minor modifications to other congruence subgroups. It turns out that for the paramodular group

$$\Gamma_{\mathrm{pa}}(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q})$$

the result can even be made unconditional. The details were carried out in [Ass23b, Theorem 5.6] and lead to the bound:

$$N_\infty(\sigma, \mathcal{F}_{\mathrm{cusp},gen}(\Omega_M, \Gamma_{\mathrm{pa}}(N))) \ll_{M,\epsilon} \mathrm{Vol}(\Gamma_{\mathrm{pa}}(N)\backslash \mathrm{Sp}_4(\mathbb{R}))^{1-\sigma+\epsilon} \tag{52}$$

for squarefree N and

$$N_\infty(\sigma, \mathcal{F}_{\mathrm{cusp},gen}(\Omega_M, \Gamma_{\mathrm{pa}}(N))) \ll_{M,\epsilon} \mathrm{Vol}(\Gamma_{\mathrm{pa}}(N)\backslash \mathrm{Sp}_4(\mathbb{R}))^{1-\frac{9}{8}\sigma+\epsilon}$$

for N prime.

One caveat of Theorem 7.2 and its variation (52) is that it only concerns the generic part of the discrete spectrum. However, Sarnak's density hypothesis should apply to the full discrete spectrum. This is natural on a philosophical level but it is also important for potential applications.

The main tool behind the proof of Theorem 7.2 is the Kuznetsov formula, which by default only detects generic representations. Thus, in order to upgrade the density result to the full spectrum, one has to manually account for all the missing representations.[13] Carrying out this argument is the main achievement of [Ass23b].

Theorem 7.4 (Theorem 1.1, [Ass23b]). *For N squarefree we have*

$$N_\infty(\sigma, \mathcal{F}(\Omega_M, \Gamma_{\mathrm{pa}}(N))) \ll_{M,\epsilon} \mathrm{Vol}(\Gamma_{\mathrm{pa}}(N)\backslash \mathrm{Sp}_4(\mathbb{R}))^{1-\sigma+\epsilon} + 1.$$

Remark 7.5. We make some comments:

- The bound presented in Theorem 7.4 is slightly sharper than the density hypothesis. Indeed, because $\boldsymbol{\sigma_1}(\infty) = \frac{3}{2}$, we would expect

  $$N_\infty(\sigma, \mathcal{F}(\Omega_M, \Gamma_{\mathrm{pa}}(N))) \ll_{M,\epsilon} \mathrm{Vol}(\Gamma_{\mathrm{pa}}(N)\backslash \mathrm{Sp}_4(\mathbb{R}))^{1-\frac{2}{3}\sigma+\epsilon}.$$

  The improved version stated above has its root in the estimate (52), which is also subconvex. A more detailed statement containing the densities of all types of forms for Sp_4 separately is given in [Ass23b, Theorem 5.9].

[13] Of course, one can also start with a different strategy. For instance, if one can proof the density hypothesis directly using the Arthur-Selberg trace formula, then it would include the full discrete spectrum by default.

- The method developed in [Ass23b] is robust enough to carry over to other families of congruence lattices. Treating the Siegel congruence subgroup $\Gamma_0(N)$ (for squarefree N) or the principal congruence subgroups in $\mathrm{Sp}_4(\mathbb{Z})$ would require some additional local computations and some other minor modifications. However, the results would most likely be conditional on the conjecture of Lapid and Mao as formulated in [Ass23b, Conjecture 4.1]. See [Ass23b, Remark 3.3, 4.4, 4.6, 4.8 and 5.7] for more explanation in this direction.
- Note that the density theorem above uses $\sigma_{\varpi}(\infty)$ as a measure of non-temperedness. Formulations of the density hypothesis using this invariant are called the *Satake variant* in [EGG23]. In loc. cit. it is also stressed that the original formulation of Sarnak's density hypothesis uses the formulation involving $p_{\varpi}(\infty)$ (corresponding to $p(\pi_{\infty})$ as defined in 7 for appropriate π_{∞} depending on ϖ). However, using the classification of unitary unramified representations one observes that
$$1-\sigma_{\varpi}(\infty) \leq \frac{2}{p_{\varpi}(\infty)}.$$
Thus, Theorem 7.4 directly implies
$$\sharp\{\varpi \in \mathcal{F}(\Omega_M, \Gamma_{\mathrm{pa}}(N)) \colon p_{\varpi}(\infty) \geq p\} \ll_{M,\epsilon} \mathrm{Vol}(\Gamma_{\mathrm{pa}}(N)\backslash \mathrm{Sp}_4(\mathbb{R}))^{\frac{2}{p}+\epsilon}.$$
This is essentially Conjecture 3.11 for the paramodular group $\Gamma_{\mathrm{pa}}(N)$ of squarefree level. With slightly more work in the archimedean analysis of the Sp_4-Kuznetsov formula one can also show the desired uniformity in M obtaining Conjecture 3.11 in full.
- Artificially adding the non-generic part to the estimate (52) heavily uses Arthur's endoscopic classification of representations for $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$ and the explicit computations of Arthur packets given in [Sch18, Sch20]. At the time of writing this is all conditional on the validity of the main results from [Art13]. Thus, Theorem 7.4 inherits this conditionality.

Shortly after [Ass23b] first appeared closely related ideas were independently developed in [EGG23]. More precisely, in [EGG23, Theorem 1.4] a cohomological version of the pointwise multiplicity hypothesis (see Conjecture 3.10) is shown for principal congruence subgroups of $\mathrm{PGSp}_4(\mathbb{R}) \cong \mathrm{SO}_5(\mathbb{R})$. The article also contains results towards p-adic versions of the Sarnak-Xue conjecture for so-called Gross inner forms of SO_5. While the strategy suggested in [Ass23b] is to take a density theorem for the generic part of (spherical) spectrum and upgrade it to a full (spherical) density theorem using the endoscopic classification, in [EGG23] the seed is the Ramanujan conjecture for cohomological self dual cuspidal representations of GL_n (see [EGG23, Theorem 6.2]). This is then used together with Arthur's endoscopic classification, more precisely with the resulting multiplicity formula, to show that the pointwise multiplicity hypothesis holds for cohomological, (i.e. algebraic) representations. The two approaches have in common that they require a detailed analysis of local and global Arthur packets as computed by Schmidt in [Sch18, Sch20].

8. Open questions

Of course all the conjectures formulated in Section 3 are still open in general. While most of them, in particular the Ramanujan conjecture and its generalizations, are out of reach of current technology, there is hope of establishing Sarnak's (spherical) density hypothesis in quite some generality. Indeed, we expect that it is feasible to establish Conjecture 3.11 for (quasi)-split classical groups. Modelled on the strategy pioneered in [Mar14, MaSh19, Ass23b, EGG23] this is a two step process:

(1) Establish a suitable density estimate for the generic part of the spectrum using a Kuznetsov-type relative trace formula. Handling the geometric side might require assuming a conjecture of Lapid and Mao concerning the Whittaker-Fourier coefficients put forward in [LaMao15].
(2) Bootstrapping the generic estimate to a full estimate using Arthur's multiplicity formula for the discrete spectrum. This requires a careful analysis of local and global Arthur packets and is by default conditional on the endoscopic classification for the groups in question.

Turning this two step sketch into a complete argument is an ambitious undertaking and will require many new technical insights. However, at least for certain families of congruence lattices, we believe that the obstacles can be overcome. A slight caveat is that the resulting density theorem would be conditional on a conjecture of Lapid and Mao as well as on the endoscopic classification of representations. Fortunately there has been much progress towards these two conjecture, so that we believe these are reasonable assumptions to be made.

We end with a list of problems and question that we find worth thinking about. These are of varying difficulty, but all promise to provide some new stepping stones on the road to a fuller understanding of Sarnak's density hypothesis and applications.

- For $n = 3$ versions of a non-spherical Kuznetsov formula have been developed in [But20, But21]. Using these one can prove density theorems involving non-spherical representations. Ultimately it seems possible to establish Sarnak's full density hypothesis as stated in Conjecture 3.9 for certain families of congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$.
- Theorems 5.12, 5.15 and 5.19 establish strong density estimates for the number of representations that are non-tempered at a fixed place v. If $v = p$ is finite the implicit constant is allowed to depend on p. Making the dependence explicit is not to hard, but the result is quite large. More precisely it can be shown that

  $$N_p(\sigma, \mathcal{F}_{\mathrm{cusp}}(\Omega_M, \Gamma_0(N))) \ll_{M,\epsilon} p^{2(n-1)\sigma+1}[\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_0(n)]^{1-\frac{2}{n-1}\sigma+\epsilon}$$

  and similar estimates are true for the principal congruence subgroup. It is an interesting problem to find possible combinations of Hecke-eigenvalues that effectively amplify forms that are highly non-tempered and improve the dependency on p. Remarks in this direction have been made by Matz and Templier, see [MaTe21, Example 3.2].
- Playing with different test functions (archimedean and non-archimedean) on the spectral side of the Kuznetsov formula for SL_n allows one to replace $\sigma_\varpi(v)$ in the density estimate by different measures of non-temperedness. Note that making such changes will necessarily effect the lengths (and shapes) of the sums on the geometric side. Working out the required combinatorics is an interesting exercise and can be useful in other applications of the Kuznetsov formula to high rank analytic number theory.
- A challenging problem is to prove a general density theorem for SL_n in the spectral aspect. More precisely, we are asking for an extension of Theorem 5.6 from SL_3 to SL_n. This requires some hard analysis of the archimedean orbital integrals. Note that, as discussed in [JaKa24, Theorem 3], such a result has immediate applications to optimal diophantine exponent.
- Theorem 5.14 establishes the density hypothesis for cuspidal automorphic forms of level 1 sorted by archimedean conductor. Going beyond the density hypothesis is an interesting problem already mentioned in [Jan21]. Furthermore, if such an improvement can be made while also allowing different levels, there is hope of establishing the density hypothesis for

the *universal family* of $\mathrm{PGL}_n(\mathbb{Q})$. Note that the corresponding Weyl-Schanuel-law for this family established in [BrMil24].

- For SL_n density theorems are available for the principal congruence subgroup $\Gamma(N)$ (resp. the Hecke congruence subgroup $\Gamma_0(N)$), see Theorem 5.19 (resp. Theorem 5.12). It is an interesting problem to consider more general families of lattices. Good candidates to start are lattices in the Moy-Prasad Filtration or parahoric families. One example of a parahoric lattice is the Borel type congruence subgroup, which has been studied in [Ass23a] for prime level. See also Theorem 5.23.
- In Corollary 5.22 a density theorem for co-compact lattices in $\mathrm{SL}_n(\mathbb{R})$ is given. More precisely we fix a lattice $\Gamma_{\mathcal{O}}$ arising from an order $\mathcal{O}$ in a quaternion division algebra over $\mathbb{Q}$ and then move through a family of principal congruence subgroups $\Gamma_{\mathcal{O}}(N)$ thereof. Doing so we ignore any dependence on the original lattice. It would be interesting to achieve uniformity in say the discriminant of $\mathcal{O}$. One way of doing so would be to careful analysis how local representations with $\mathcal{O}$-invariant vectors are transformed under the Jacquet-Langlands correspondence. Even for SL_2, where this can be done very concretely, this seems be interesting.
- In [GoKa22] the density hypothesis is discussed in the context of graphs. An particular interesting feature is the so-called *density amplification* feature discussed in [GoKa22, Theorem 1.5]. Roughly speaking it is established there that if a graph X_t satisfies the density hypothesis, then certain quotients of X_t also satisfy a density hypothesis with slightly worse exponent. It would be interesting to find instances of this phenomenon also in the setting of automorphic forms.
- In Section 6 we have discussed how the density hypothesis implies certain uniform counting results. Here we have usually used the Frobenius norm $\|\gamma\|$. Often it is however useful to count with respect to other gauges such as for example $\|\gamma\|_\sharp = \max(\|\gamma\|, \|\gamma^{-1}\|)$. This corresponds to a different choice of α in (45). The asymptotic formula given in (45) is for a large variety of gauges, see [Mau07]. It is reasonable to expect that the density hypothesis should imply uniform counting results for other gauges (i.e. it should imply Conjecture 6.1 for different choices of α.) Even though this is certainly folklore, working out the details can shed some light on the mechanics of the spectral theory of automorphic forms in higher rank.
- With some more work the uniform counting results stated in Theorem 6.5 can be upgraded to an asymptotic formula with good error term in the level (i.e. volume) aspect. On the other hand, an impressive asymptotic formula with power saving in the radius was established recently in [BlLu24]. Making this result uniform in the level can then lead to an asymptotic formula for the counting problem with hybrid power saving in the error term.

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