Counterexamples to the local-global conjecture for Appolonian circle packings (after Haag/Kertzer/Rickards/Stange)

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Abstract

An ancient construction involving tangent circles leads to fractals called integral Appolonian circle packings. The curvatures of the circles appearing in such a packing give rise to a sequence of integers. The so called local-global conjecture predicts that up to at most finitely many exceptions every integer not ruled out by certain congruence obstructions appears in this list of curvatures. This conjecture has gotten much attention over the years and many interesting results in its favor have been obtained. In an surprising turn of events Haag, Kertzer, Rickards and Stange have found families of counterexamples. In this talk we plan to explain how these families are constructed using quadratic reciprocity.

1 From tangent circles to Apollonian gaskets

We assume that everyone is familiar with circles, usually denoted by C, in the euclidean plane. We write $r(C)$ for the radius of C. The intuitive notion of a circle in the plane is extended as follows:

- A point is a circle of radius 0;
- A straight line is a circle of infinite radius; and
- A circle is equipped with an orientation (i.e. a way of deciding where its interior is).

Given a circle C with $r(C) \in \mathbb{R}_{>0}$ and positive orientation (i.e. the interior is the intuitive inside of the circle), then we define the curvature by

$$\kappa(C) = \frac{1}{r(C)}.$$

This notion is extended to points and lines in the obvious way. Given an oriented circle C we write C^{op} for the same circle with flipped orientation and we define $\kappa(C^{\mathrm{op}}) = -\kappa(C)$. Two circles C_1 and C_2 are said to be tangent if they intersect at precisely one point. This unique intersection point is then called point of tangency.

The following theorem, due to Apollonius of Perga ($\approx$ 262-190 BC) and communicated by Pappus, is the starting point of the construction of Apollonian gaskets.

Theorem 1.1 (Apollonius). *Given three mutually tangent circles with distinct points of tangency there are exactly two circles that are tangent two all three.*

Proof. We only present the rough idea of proof following [20, Section 2]. Denote the three tangent circles by C_1, C_2 and C_3. After applying suitable motions of the plane that preserve tangencies and angles we can arrange that C_1 and C_3 are parallel lines both tangent to C_2. (Note that the point of tangency of C_1 and C_3 is infinity!) In this arrangement one can immediately write down the two solutions to the problem. Indeed the desired circles are simply shifted copies of the circle C_2. □

This theorem invites us to construct a collection of circles by applying it over and over. The idea for such an iterative process goes back to Leibniz (1646-1716). We consider the following special case. We start with a quadruple (C_0, C_1, C_2, C_3) of mutually tangent circles. We further assume that $\kappa(C_0) < 0$ and that C_1, C_2 and C_3 are external to C_0.[1] In the second step we apply Apollonius' theorem to construct three new circles that are mutually tangent to precisely three of our initial circles. Adding these circles to our set and iterating the procedure produces in total $4 \cdot 3^{n-1}$ circles after n steps. Continuing ad infinitum produces an infinite set $\mathscr{G}$ of circles called the Appolonian gasket.

The subset of the plane obtained by removing the interiors of the circles in $\mathscr{G}$ is called the residual set of $\mathscr{G}$. It is of fractal nature. Even though a Apollonian gaskets depends on the initial choice of the circles C_0, C_1, C_2 and C_3, they are rigid in the sense that any two gaskets can be mapped to each other using Möbius transformations of the plane. As a consequence the Hausdorff dimension δ of a Apollonian gasket is a universal constant. One can compute that $\delta \approx 1.30\ldots$.

Given a Gasket $\mathscr{G}$ we obtain a (multi)-set of curvatures

$$\kappa(\mathscr{G}) = \{\kappa(C) \colon C \in \mathscr{G}\}$$

and (from the perspective of number theorists) it is natural to inquire about properties of these numbers. The most basic question is concerned with the counting function

$$N_{\mathscr{G}}(T) = \sharp\{C \in \mathscr{G} \colon \kappa(C) \leq T\}.$$

It turns out that this is well understood and we have the following theorem:

Theorem 1.2 (Kelmer-Kontorovich-Lutsko 2023)**.** *There is a positive constant $\mathfrak{C}_{\mathscr{G}}$ depending on the gasket $\mathscr{G}$ such that*

$$N_{\mathscr{G}}(T) = \mathfrak{C}_{\mathscr{G}} \cdot T^{\delta} + O(T^{\frac{3}{5}\delta+\frac{2}{5}} \log(T)^{\frac{2}{5}}).$$

This quite recent result is the finial piece of a long series of historical developments. Without claiming completeness we want to point towards some of the highlights:

1. To understand $N_{\mathscr{G}}(T)$ it is natural to study the generating function

$$\mathscr{L}_{\mathscr{G}}(s) = \sum_{C \in \mathscr{G}} \kappa(C)^{-s},$$

which is absolutely convergent for $\Re(s) \geq 2$. It was shown in [2] that δ is precisely the abscissa of convergence of this function. Using analytic properties of the generating function it was then shown in [3] that $N_{\mathscr{G}}(T) = T^{\delta+o(1)}$.

2. The constant δ has been studied numerically in many works. We refer to example to [17].

3. In 2011 Kontorovich and Oh established the asymptotic formula $N_{\mathscr{G}}(T) \sim \mathfrak{C}_{\mathscr{G}} \cdot T^{\delta}$ without specifying the error term. They proceed by giving a spectral interpretation of the counting problem and using dynamical properties of certain horocycle flows. See [15].

[1] Note that since C_0 has negative curvature it is negatively oriented, so that C_1, C_2 and C_3 are actually positioned in what we would intuitively call the inside of the circle C_0.

4. In 2013 the asymptotic formula was refined by Lee and Oh in [16]. Indeed, they show $N_{\mathscr{G}}(T) = \mathfrak{C}_{\mathscr{G}} \cdot T^{\delta} + O(T^{\delta-\beta_{LO}})$ for $\beta_{LO} = \frac{2}{63}(\delta - s_1) > 0$, where s_1 is related to a certain spectral gap. They also give an interpretation of $\mathfrak{C}_{\mathscr{G}}$ in terms of the δ-dimensional Hausdorff measure of the residual set of $\mathscr{G}$. Similar results were independently established in Vinogradov's thesis.
5. A refined analysis performed in [13] improves the power saving in the error term from β_{LO} to $\beta_{KL} = \frac{2}{5}(\delta - s_1)$. (Their approach also applies to Kleinian packings in higher dimension.)
6. The theorem stated above is finally obtained in [11, Corollary 2] after resolving Sarnak's spectral gap question. More precisely, in loc. cit. it is shown that one can take $s_1 = 1$ in β_{KL} defined in the previous bullet point. This is to say that the relevant spectral gap is maximal.

2 Integrality and the rise of the group action

While the above argument is purely geometrically, it is natural to ask for an analytic verification. In this direction we define the so called Descartes quadratic form[2]

$$Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2.$$

With this at hand Descartes (1596-1650) was able to deduce an algebraic relation between the curvatures of four mutually tangent circles. This beautiful result was re-discovered several times over the years and we refer to [6] for more discussion.

Theorem 2.1 (The Descartes Circle Theorem 1643)**.** *Let C_1, C_2, C_3 and C_4 be four mutually tangent circles. Then we have*

$$Q(\kappa(C_1), \kappa(C_2), \kappa(C_3), \kappa(C_4)) = 0.$$

Proof. Instead of giving a full proof of this let us just say the following. One first checks that the theorem is true if C_1 and C_2 are parallel lines. Indeed, in this case $\kappa(C_1) = \kappa(C_2) = 0$ and the equality reduces to

$$(\kappa(C_3) - \kappa(C_4))^2 = 0,$$

which happens exactly when $\kappa(C_3) = \kappa(C_4)$. The latter is obviously the case if the four circles are mutually tangent.

To handle the general case one has to derive formulae for the radii (or curvatures) of the circles under planar motions. Doing so one can reduce to the easy case described above and verify directly that the desired equality still holds. We omit the details. $\square$

We can now take a look at our Apollonian gasket $\mathscr{G}$ or more precisely the set $\kappa(\mathscr{G})$ from a new perspective. First, recall that C_0, C_1 and C_2 determine the gasket $\mathscr{G}$ completely. The curvatures of the two circles tangent to the first three circles solve the quadratic equation

$$\begin{aligned} 0 &= Q(\kappa(C_0), \kappa(C_1), \kappa(C_2), x) \\ &= x^2 - 2(\kappa(C_0) + \kappa(C_1) + \kappa(C_2))x + 2(\kappa(C_0)^2 + \kappa(C_1)^2 + \kappa(C_2)^2) - (\kappa(C_0) + \kappa(C_1) + \kappa(C_2)). \end{aligned}$$

Suppose the circles in our gasket are numbered such that the two solutions are precisely $x_+ = \kappa(C_3)$ and $x_- = \kappa(C_4)$. We can solve this as

$$x_{\pm} = \kappa(C_0) + \kappa(C_1) + \kappa(C_2) \pm 2\sqrt{\Delta},$$

[2]This quadratic form can be diagonalized to $x^2 + y^2 + z^2 - w^2$ and thus has signature $(3, 1)$.

where

$$\Delta = \kappa(C_0)\kappa(C_1) + \kappa(C_0)\kappa(C_2) + \kappa(C_1)\kappa(C_2).$$

In particular we obtain the nice formula

$$\kappa(C_3) + \kappa(C_4) = x_+ + x_- = 2(\kappa(C_0) + \kappa(C_1) + \kappa(C_2)).$$

Thus, in the numbering above C_3 and C_4 are the two circles tangent to C_0, C_1 and C_2. If $\kappa(C_3)$ is given, then we can compute $\kappa(C_4)$ by

$$\kappa(C_4) = -\kappa(C_3) + (\kappa(C_3) + \kappa(C_4)) = 2(\kappa(C_0) + \kappa(C_1) + \kappa(C_2)) - \kappa(C_3).$$

We can write this as

$$\begin{pmatrix}\kappa(C_0) & \kappa(C_1) & \kappa(C_2) & \kappa(C_4)\end{pmatrix} = \begin{pmatrix}\kappa(C_0) & \kappa(C_1) & \kappa(C_2) & \kappa(C_3)\end{pmatrix} \cdot S_4,$$

where

$$S_4 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Similarly we can obtain new curvatures using the matrices

$$S_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \; S_2 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \text{ and } S_3 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

Observe that $S_i^2 = 1_4$. We are therefore led to defining

$$A = \langle S_1, S_2, S_3, S_4 \rangle \subseteq \mathrm{GL}_4(\mathbb{Z}).$$

This is the so called Apollonian group. This group governs the (curvatures of) a packing. Indeed, refining the above discussion, shows that the orbit

$$\mathscr{O}_{\mathbf{a}} = \mathbf{a} \cdot A,$$

for any quadruple of curvatures of mutually tangent circles in $\mathscr{G}$, consists precisely of all quadruples of curvatures of mutually tangent circles.

Let us consider an example. We denote the gasket generated by three circles with curvatures 5, 8 and 8 by $\mathscr{G}_0$. The equation

$$Q(X, 5, 8, 8) = 0$$

leads to the quadratic equation $X^2 - 42X - 135 = 0$. We have $\Delta = 242 = (24)^2$, so that we find the two solutions $x_- = -3$ and $x_+ = 45$. This leads gives us the root quadruple $\mathbf{a}_0 = (-3, 5, 8, 8)$ and we obtain the structure pictured in Figure 1. In particular we see that

$$\{-3, 5, 8, 12, 20, 21, 29, 44, 45, 53, 77, 108, 117\} \subseteq \kappa(\mathscr{G}_0).$$

This leads to the following observation, which goes back to F. Soddy (1877-1956) in [21].[3] If an Apollonian gasket $\mathscr{G}$ contains four mutually tangent circles with integral curvatures, then

$$\kappa(\mathscr{G}) \subseteq \mathbb{Z}.$$

[3] Note that F. Soddy was actually a chemist, who won the Nobel price for chemistry in 1921.

$\mathbf{a}_0 = (-3, 5, 8, 8)$

- $\mathbf{a}_0 S_1 = (45, 5, 8, 8)$
  - $\mathbf{a}_0 S_1 S_2 = (45, 117, 8, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_1 S_3 = (45, 5, 108, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_1 S_4 = (45, 5, 8, 108) \longrightarrow \cdots$
- $\mathbf{a}_0 S_2 = (-3, 21, 8, 8)$
  - $\mathbf{a}_0 S_3 S_1 = (77, 21, 8, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_2 S_3 = (-3, 21, 44, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_2 S_4 = (-3, 21, 8, 44) \longrightarrow \cdots$
- $\mathbf{a}_0 S_3 = (-3, 5, 12, 8)$
  - $\mathbf{a}_0 S_3 S_1 = (53, 5, 12, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_3 S_2 = (-3, 29, 12, 8) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_3 S_4 = (-3, 5, 12, 20) \longrightarrow \cdots$
- $\mathbf{a}_0 S_4 = (-3, 5, 8, 12)$
  - $\mathbf{a}_0 S_4 S_1 = (53, 5, 8, 12) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_4 S_2 = (-3, 29, 8, 12) \longrightarrow \cdots$
  - $\mathbf{a}_0 S_4 S_3 = (-3, 5, 20, 12) \longrightarrow \cdots$

Figure 1: This figure contains the first layers of the orbit $\mathbf{a}_0 \cdot A$ containing the curvatures of the Apollonian gasket $\mathscr{G}_0$.

This is when things become very interesting for number theorists and we will discuss some natural questions in the following section.

Note that we now also have a natural dynamical interpretation of (quadruples of) curvatures. Indeed, if $\mathbf{a} = (\kappa(C_0), \kappa(C_1), \kappa(C_2), \kappa(C_3))$, then we call the orbit $\mathscr{O}_{\mathbf{a}} = \mathbf{a} \cdot A$ integral if $\mathscr{O}_{\mathbf{a}} \subseteq \mathbb{Z}^4$. (As observed above this is equivalent to saying that $\mathbf{a}$ is integral.) Similarly, we say that an integral orbit $\mathscr{O}_{\mathbf{a}}$ is primitive if the elements of it have co-prime coordinates. Again this is equivalent to saying that $\mathbf{a}$ is primitive. Analogously, we call an Apollonian gasket $\mathscr{G}$ integral (resp. primitive) if the corresponding orbit of curvature quadruples is integral (resp. primitive).

Since the Apollonian group A plays an important role in the study of Apollonian gaskets let us summarize some of its properties:

1. Let O_Q be the orthogonal group of Q. This is the group[4]

$$O_Q = \{g \in \mathrm{GL}_4 \colon Q(\mathbf{x} \cdot g) = Q(\mathbf{x})\}.$$

It is easy to check that the matrices S_1, S_2, S_3 and S_4 preserve Q, so that we obtain

$$A \subseteq O_Q(\mathbb{Z}).$$

[4]It is actually an algebraic group defined by quadratic equations in 16 variables.

Here $O_Q(\mathbb{Z})$ is nothing but the matrices in O_Q with entries in the integers, which is an arithmetic lattice.

2. The group A has infinite index in $O_Q(\mathbb{Z})$ but is Zariski dense in O_Q. See [7, Lemma 1.6] Thus it is a thin group. We refer to [12] for a nice introductory discussion to thin groups.

So far we have discovered that, if we arrange the curvatures in quadruples, then we can describe all curvature quadruples that occur as the orbit of the Apollonian group. Note that the gasket $\mathscr{G}$ is classified by the curvature quadruple $(\kappa(C_0), \kappa(C_1), \kappa(C_2), \kappa(C_3))$. We call this the root quadruple.[5] Of course a quadruple of curvatures stores more information then a single curvature and we immediately face the question: How much information is lost if we forget all but the first entry in a curvature quadruple? This is answered by the following two results.

Proposition 2.1 (Graham-Lagarias-Mallows-Wilks-Yan 2003)**.** *Let $0 \neq \kappa \in \mathbb{Z}$. There is a bijection between primitive integral positive definite binary quadratic forms of discriminant $-4\kappa^2$ and primitive curvature quadruples $\mathbf{a} = (\kappa, a_2, a_3, a_4)$ with $\kappa + a_2 + a_3 + a_4 > 0$.*[6]

Proof. This is essentially [9, Theorem 4.2] and we can describe the correspondence explicitly as follows. Given a solution $\mathbf{a} = (\kappa, a_2, a_3, a_4)$ of $Q(\mathbf{a}) = 0$ with $\kappa + a_2 + a_3 + a_4 > 0$ we associate the form

$$q_{\mathbf{a}}(x, y) = (\kappa + a_2)x^2 + (\kappa + a_2 + a_3 - a_4)xy + (\kappa + a_3)y^2.$$

This is obviously a integral binary quadratic form. We compute its discriminant

$$\begin{aligned}\operatorname{disc}(q_{\mathbf{a}}) &= (\kappa + a_2 + a_3 - a_4)^2 - 4(\kappa + a_2)(\kappa + a_3)\\ &= -4\kappa^2 + \kappa^2 + a_2^2 + a_3^2 + a_4^2 - 2\kappa a_2 - 2\kappa a_3 - 2\kappa a_4 - 2a_2a_3 - 2a_2a_4 - 2a_3a_4\\ &= -4\kappa^2 + Q(\kappa, a_2, a_3, a_4) = -4\kappa^2.\end{aligned}$$

To see that $q_{\mathbf{a}}$ is positive definite it remains to show that $\kappa + a_2 > 0$. Suppose that $\kappa + a_2 \leq 0$. Then we write $Q(\mathbf{a}) = 0$ as

$$(\kappa - a_2)^2 + (a_3 - a_4)^2 = (\kappa + a_2)(a_3 + a_4).$$

We consider two cases:

- Suppose $\kappa + a_2 < 0$. Since the left hand side of the above equality is non-negative we get $a_3 + a_4 \leq 0$. This allows us to deduce $\kappa + a_2 + a_3 + a_4 < 0$, which is a contradiction.
- If $\kappa + a_2 < 0$, then the above equality reads $(a_3 - a_4)^2 = 0$, so that $a_3 = a_4$. But this can only happen if $\kappa = 0$, which is also a contradiction.

This proves positivity.

To establish primitivity we first observe that looking at the equality $Q(\mathbf{a}) = 0$ modulo 2 implies $\kappa + a_2 + a_3 + a_4 \equiv 0 \bmod 2$. Now suppose 2 divides all coefficients of $q_{\mathbf{a}}$. Then we easily deduce that 2 divides (κ, a_2, a_3, a_4), which is a contradiction to the primitivity of $\mathbf{a}$. Now take a

[5] There is a technically deffinition of a root quadruple, which ensure that it is the smallest or reduced element in a given orbit. We do not need this notion here and omit the details. See [9, Definition 3.2] for more information.

[6] A binary quadratic form is a homogenous polynomial of degree two in two variables: $q(x, y) = ax^2 + bxy + cy^2$. Its discriminant is given by $\operatorname{disc}(q) = b^2 - 4ac$. We call q integral if $a, b, c \in \mathbb{Z}$ and we call it primitive if $(a, b, c) = 1$. Finally, we say that q is positive definite if $a, c > 0$ and $\operatorname{disc}(q) < 0$. There is a natural action of $\mathrm{GL}_2(\mathbb{Z})$ on integral binary quadratic forms given by $\gamma.q(x, y) = q(ax + cy, bx + dy)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This action preserves primitivity, definiteness and discriminant. We write $[q]$ for the equivalence class of (positive definite primitive) integral binary quadratic forms containing q.

prime $p > 2$ and suppose it divides the coefficients of $q_{\mathbf{a}}$. Then we note that $p^2 \mid \mathrm{disc}(q_{\mathbf{a}})$ so that $p \mid \kappa$. Using this one concludes that p must also divide a_2, a_3 and a_4 giving a contradiction again.

Finally, given a primitive positive definite binary quadratic form $q(x, y) = Ax^2 + Bxy + Cy^2$ of discriminant $-4\kappa^2$ we associate the quadruple

$$\mathbf{a}_q = (\kappa, A - \kappa, C - \kappa, A + C - B - \kappa).$$

we directly see that this is the inverse to the mapping defined above and one verifies directly that $Q(\mathbf{a}_q) = 0$ and that $\mathbf{a}_q$ is primitive. Note that

$$\kappa + (A - \kappa) + (C - \kappa) + (A + C - B - \kappa) = 2A + 2C - 2\kappa - B.$$

It can be checked that this must be positive. □

Given a primitive integral Apollonian gasket $\mathscr{G}$ and a circle $C \in \mathscr{G}$ we can use the proposition above to associate a $\mathrm{GL}_2(\mathbb{Z})$ equivalence class of quadratic forms, called $[f_C]$, as follows. First we note that one can find four mutually tangent circles C, C_1, C_2 and C_3 in $\mathscr{G}$ satisfying $\kappa(C) + \kappa(C_1) + \kappa(C_2) + \kappa(C_3) > 0$. Now we use the proposition above and put $f_C = q_{\mathbf{a}}$ for $\mathbf{a} = (\kappa(C), \kappa(C_1), \kappa(C_2), \kappa(C_3))$. It needs to be verified that the class $[f_C]$ is independent of our initial choice of mutually tangent circles. We omit this.

Proposition 2.2 (Sarnak 2007)**.** *Let $\mathscr{G}$ be a primitive integral Apollonian gasket and let $C \in \mathscr{G}$ with corresponding equivalence class $[f_C]$. The multiset of curvatures of circles tangent to C in $\mathscr{G}$ is $\{f_C(x, y) - \kappa(C) \colon (x, y) = 1\}$.*[7]

Proof. This is established in [19]. We start by writing $\mathbf{a} = (a, \mathbf{x})$ for $a \in \mathbb{Z}_{\neq 0}$ and $\mathbf{x} = (x_1, x_2, x_3)$ and decompose

$$Q(\mathbf{a}) = g(\mathbf{x} + (a, a, a)) + 4a^2,$$

where

$$g(\mathbf{y}) = y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_1y_3 - 2y_2y_3 = \mathbf{y} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \mathbf{y}^\top.$$

Next we observe that curvature quadruples $\mathbf{a}$ with fixed first coordinate are fixed by the subgroup

$$A_1 = \langle S_2, S_3, S_4 \rangle \subseteq A.$$

We can study how this group acts on g. Indeed, we find that A_1 induces a linear action Γ on triplets satisfying $\Gamma \subseteq O_g(\mathbb{Z})$. By direct computation one verifies that $\Gamma = \langle \tilde{S_2}, \tilde{S_3}, \tilde{S_4} \rangle$ for

$$\tilde{S_2} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \tilde{S_3} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } \tilde{S_4} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

We note that the point $\mathbf{x} + (a, a, a)$ is a primitive point on the quadric $g(\mathbf{y}) = -4a^2$ and the orbit $\mathbf{a} \cdot A_1$ is given by $(a, (\mathbf{x} + (a, a, a)) \cdot \Gamma - (a, a, a))$. We make the change of variables

$$A = a + x_1,\ B = (a + x_1 + x_2 - x_3)/2 \text{ and } C = a + x_2.$$

[7]Note that we can replace f_C by any representative in the same equivalence class. This is because equivalent quadratic forms have the same value sets.

For parity reasons one finds that $B \in \mathbb{Z}$. Note that $q_{\mathbf{a}}(x,y) = Ax^2 + 2Bxy + Cy^2$. Write $\mathbf{y} = \mathbf{x} + (a,a,a)$. Then we have

$$\begin{pmatrix} A & B & C \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & -1/2 & 0 \end{pmatrix}.$$

We compute that

$$\begin{aligned} g(\mathbf{y}) &= \begin{pmatrix} A & B & C \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \\ &= \begin{pmatrix} A & B & C \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 4B^2 - 4AC = \Delta(A,B,C). \end{aligned}$$

One directly computes that the image of Γ in $O_\Delta(\mathbb{Z})$, lets call it Γ', is generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In summary, we have translated the set-up as follows. We have a primitive point (A,B,C) (i.e. the image of $\mathbf{x} + (a,a,a) = \mathbf{y}$ under our change of variables) solving

$$\Delta(A,B,C) = -4a^2$$

and we want to understand the orbit $(A,B,C)\Gamma'$. Here it gets a bit technical and we refer to [1, pp.953-954] for details. We write $\tilde{\Gamma} = \Gamma' \cap \mathrm{SO}_\Delta$. The spin double cover of SO_Δ is identified with the image of $\mathrm{GL}_2(\mathbb{Z})$ under the homomorphism

$$\rho\colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \delta^2 \end{pmatrix}.$$

We have

$$\rho^{-1}(\tilde{\Gamma}) = \{\gamma \in \mathrm{GL}_2(\mathbb{Z})\colon \gamma - I_2 \equiv 0 \bmod 2\},$$

where the right hand side is the principal congruence subgroup of level 2. That $\rho^{-1}(\tilde{\Gamma})$ contains the principal congruence subgroup can be easily seen by playing with the generators. That actually equality holds can be verified using a volume computation.

Now we can take an element

$$\tilde{\gamma} = \begin{pmatrix} 2k+1 & 2n \\ 2m & 2l+1 \end{pmatrix} \in \rho^{-1}(\tilde{\Gamma}).$$

Computing $\rho(\tilde{\gamma})$ as well as its action on (A,B,C) and changing variables back to $\mathbf{y}$ yields

$$(A,B,C)\rho^{-1}(\tilde{\Gamma}) \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = (q_{\mathbf{a}}(2k+1, 2m), q_{\mathbf{a}}(2k+1-2l, 2m-2n-1), q_{\mathbf{a}}(2l, 2n+1)).$$

When ranging over all allowed k,n,m,l the right hand side ranges over all values of $q_{\mathbf{a}}$ at primitive points. We recover $\mathbf{x} = \mathbf{y} - (a,a,a)$ and the result follows. $\square$

3 The local to global conjecture

Suppose we have an integral Apollonian gasket $\mathscr{G}$. It is very natural to wonder which curvatures appear in this setting. In other words we are asking about the structure of the set $\kappa(\mathscr{G})$.

Given $k \in \mathbb{Z}$ we put

$$m_{\mathscr{G}}(k) = \sharp\{C \in \mathscr{G} \colon \kappa(C) = k\}.$$

Note that we can phrase Theorem 1.2 as

$$\sum_{|k| \leq T} m_{\mathscr{G}}(k) \sim T^{\delta}.$$

Since $\delta > 1$, we must have large multiplicities. However, if the multiplicities are not to badly behaved, then one might (very naively) expect that every integer appears as a curvature in an integral primitive packing. (Primitivity is important because otherwise all curvatures that appear have a common multiple!) The following lemma shows that there is an immediate stumbling block.

Lemma 3.1. *Let $\mathscr{G}_0$ be the Apollonian gasket given by the root quadruple $(-3, 5, 8, 8)$. All circles in $\mathscr{G}_0$ have curvatures congruent 0 or 1 modulo 4.*

It can even be seen that

$$\{\kappa(C) \bmod 24 \colon C \in \mathscr{G}_0\} = \{0, 5, 8, 12, 20, 21\} \subseteq \mathbb{Z}/24\mathbb{Z}.$$

This shows that, in the notation of [10, Proposition 2.1], $\mathscr{G}$ is of type $(6, 5)$.

Proof. We claim that any curvature quadruple $\mathbf{a} = (a_1, a_2, a_3, a_4)$ contains exactly two elements congruent 0 modulo 4 and two entries congruent 1 modulo 4. This is easily verified for the root quadruple $(-3, 5, 8, 8)$. To see that the claim holds in general we only have to verify that this property is preserved when applying the matrices S_1, S_2, S_3 and S_4. Indeed, if suppose $\mathbf{a}$ is as desired. Then we let $\mathbf{a}' = (a_1', a_2, a_3, a_4) = \mathbf{a} \cdot S_1$. We have the two cases

- If $a_1 \equiv 0 \bmod 4$, then $a_2 + a_3 + a_4 \equiv 0 \bmod 2$. Consequently $a_1' = -a_1 + 2(a_2 + a_3 + a_4) \equiv -a_1 \equiv 0 \bmod 4$.
- If $a_1 \equiv 1 \bmod 4$, then $a_2 + a_3 + a_4 \equiv 1 \bmod 4$. We conclude that $a_1' \equiv 2 - a_1 \equiv 1 \bmod 4$.

In both cases the congruence class of the first entry modulo 4 is preserved and the last three entries remain unchanged. Thus $\mathbf{a}'$ has the desired property. The same argument works for the action of S_2, S_3 and S_4. $\square$

We conclude that there are obviously congruence obstructions that can prevent an integer from being a curvature for a given gasket $\mathscr{G}$. These congruence conditions have been studied in [9, Section 6]. There is a deeper reason for these congruence obstructions. Indeed, it can be shown that the reduction $A^{(q)} \subseteq \mathrm{GL}_4(\mathbb{Z}/q\mathbb{Z})$ modulo q of the Apollonian group A agrees with the reduction of $O_q(\mathbb{Z})$ modulo q. Studying the structure of the latter is technical but possible.[8] This was carried out in [7] and allows for a precise determination of $A^{(q)}$. One arrives at a nice structure theorem, see [7, Theorem 1.4], for the orbits $\mathscr{O}_{\mathbf{a}}$ modulo q. To simplify notation let us write

$$\mathscr{O}_{\mathbf{a}}(q) = \{\mathbf{x} \bmod q \colon \mathbf{x} \in \mathscr{O}_{\mathbf{a}}\} \subseteq (\mathbb{Z}/q\mathbb{Z})^4.$$

[8] One can use that the spin double cover of Q_Q satisfies strong approximation.

Note that we assume $\mathcal{O}_{\mathbf{a}}$ is integral and primitive. Then, by the Chinese remainder theorem, we have a natural bijection $\mathcal{O}_{\mathbf{a}}(q_1q_2) = \mathcal{O}_{\mathbf{a}}(q_1) \times \mathcal{O}_{\mathbf{a}}(q_2)$. Furthermore, for primes $p \geq 5$ and $k \in \mathbb{N}$ we have

$$\mathcal{O}_{\mathbf{a}}(p^k) = \{\mathbf{x} \in (\mathbb{Z}/p^k\mathbb{Z})^4 \setminus \{0\} \colon Q(\mathbf{x}) \equiv 0 \bmod p^k\}.$$

It thus turns out that the congruence obstructions must arise from powers of the primes 2 and 3. One can actually show that the relevant powers are 2^3 and 3^1, so that all congruence obstruction become visible modulo 24. We refer to [10, Proposition 2.1] for a complete classification of possible obstructions. Given a primitive integral Apollonian gasket $\mathscr{G}$ we write

$$\kappa(\mathscr{G})^{(24)} = \{\kappa(C) \bmod 24 \colon C \in \mathscr{G}\} \subseteq \mathbb{Z}/24\mathbb{Z}.$$

Taking these into account leads to a conjecture formulated in [9], where it is referred to as strong density conjecture. We refer to [8, Conjecture 1.1].

Conjecture 3.1 (Local to Global Conjecture). *Let $\mathscr{G}$ be a primitive integral Apollonian gasket. Then there is a constant $X = X(\mathscr{G})$ such that for all $n \geq X$ we have that $n \in \kappa(\mathscr{G})$ if and only if $n + 24\mathbb{Z} \in \kappa(\mathscr{G})^{(24)}$.*

Roughly speaking this says that up to finitely many exception all curvatures that are not ruled out by congruence obstructions appear as curvatures of a given primitive integral Apollonian gasket. As pointed out in [8] it is indeed necessary to allow finitely many exceptions.

The following theorem roughly says that the local to global conjecture holds for almost all integer and thus gives some evidence for the truth of the conjecture.

Theorem 3.1 (Bourgain-Kontorovich 2013). *Let $\mathscr{G}$ be a primitive integral Apollonian gasket. Then there is an effective constant $\eta > 0$ such that*

$$\sharp\{1 \leq n \leq X \colon n + 24\mathbb{Z} \in \kappa(\mathscr{G})^{(24)} \text{ and } n \notin \kappa(\mathscr{G})\} \ll_{\mathscr{G}} X^{1-\eta}.$$

The proof utilizes the so called orbital circle method and giving details goes beyond the scope of this talk. Let us however end this section by presenting some milestones on the way to Theorem 3.1:

1. In its first form the local to global conjecture appeared in [9]. In loc. cit. it was also shown that
$$\{1 \leq n \leq X \colon n \in \kappa(\mathscr{G})\} \gg X^{\frac{1}{2}}.$$
2. An argument by Sarnak sketched in [19] sharpens the previous lower bound significantly to
$$\{1 \leq n \leq X \colon n \in \kappa(\mathscr{G})\} \gg \frac{X}{\log(X)^{\frac{1}{2}}}.$$
In her 2010 Princeton University Thesis Fuchs was able to slightly improve the power of the logarithm from 0.5 to $0.150\ldots$.
3. In [1] the so called positive density conjecture was established. This is the estimate
$$\{1 \leq n \leq X \colon n \in \kappa(\mathscr{G})\} \gg X.$$
4. Finally, Theorem 3.1 was established in [4, Theorem 1.2].

4 Counterexamples

In a surprising turn of events counter examples to the local to global conjecture were discovered by Haag, Kertzer, Rickards and Stange in [10]. We have for example the following result stated in [10, Theorem 1.6].

Theorem 4.1 (Haag-Kertzer-Rickards-Stange 2024)**.** *The Apollonian gasket $\mathscr{G}_0$, whose curvatures are generated by the quadruple $(-3, 5, 8, 8)$ admits no square curvatures.*

One can go further and also show that $\kappa(C)$ with $C \in \mathscr{G}_0$ can not be of the form $6n^2$. In the notation of [10, Theorem 2.4] the gasket $\mathscr{G}_0$ is of type $(6, 5, -1)$.

Proof. Recall that all curvatures of circles appearing in $\mathscr{G}_0$ are congruent to 0 or 1 modulo 4 and are all non-zero. We now start by constructing an function $\chi_2 \colon \mathscr{G}_0 \to \{\pm 1\}$. Indeed, we set

$$\chi_2(C) = \left(\frac{\kappa(C')}{\kappa(C)}\right),$$

where $C' \in \mathscr{G}_0$ is any circle tangent to C with $(\kappa(C), \kappa(C')) = 1$. Here $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.[9] Since the $\mathscr{G}_0$ is primitive we can always find such a circl C', but we still have to prove that χ_2 is well defined. Since $\kappa(C) \equiv 0, 1 \bmod 4$ we need to show that there is $A_C \in \mathbb{Z}/\kappa(C)\mathbb{Z}$ such that $\kappa(C') \equiv A_C \cdot x^2 \bmod \kappa(C)$. From Proposition 2.2 we recall that $\kappa(C') \equiv f_C(x, y) \bmod \kappa(C)$ for some $(x, y) = 1$. However, since $\kappa(C) \mid \mathrm{disc}(f_C)$ we find that $[f_C]$ contains a representative which is congruent to $A_C \cdot x^2$ modulo $\kappa(C)$. This establishes well definedness.

Next we observe that if C and C' are tangent circles with co-prime curvatures, then

$$\chi_2(C)\chi_2(C') = \left(\frac{\kappa(C')}{\kappa(C)}\right)\left(\frac{\kappa(C)}{\kappa(C')}\right) = 1.$$

In the last step we have used quadratic reciprocity and the observation that at $\kappa(C) \equiv 1 \bmod 4$ or $\kappa(C') \equiv 1 \bmod 4$.[10] It turns out that this implies that χ_2 is actually constant on $\mathscr{G}_0$. Indeed, given any two circles $C, C' \in \mathscr{G}_0$ we can find a (finite) sequence circles

$$C = C_0, C_1, \ldots, C_N = C'$$

such that for all $i = 1, \ldots, N$ we have C_i and C_{i-1} are tangent and $\kappa(C_i)$ and $\kappa(C_{i-1})$ are co-prime. To see this we argue in several steps:

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[9] Let $n \neq 0$ and $m = \pm p_1^{e_1} \cdots p_r^{e_r} \neq 0$, then we define

$$\left(\frac{n}{m}\right) = \left(\frac{n}{\pm 1}\right) \cdot \prod_{s=1}^{r} \left(\frac{n}{p}\right)^{e_r}.$$

Here we have the usual Legendre symbols $\left(\frac{\cdot}{p}\right)$ for odd p. Further, we set $\left(\frac{n}{1}\right) = 1$, $\left(\frac{n}{-1}\right) = \mathrm{sgn}(n)$ and

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } 2 \mid n, \\ 1 & \text{if } n \equiv \pm 1 \bmod 8, \\ -1 & \text{if } n \equiv \pm 3 \bmod 8. \end{cases}$$

For $m \not\equiv 2 \bmod 4$ one checks that, if $n_1 \cdot n_2 > 0$ and $n_1 \equiv n_2 \bmod m$, then $\left(\frac{n_1}{m}\right) = \left(\frac{n_2}{m}\right)$.

[10] For the classical Legendre symbol the law of quadratic reciprocity is due to Gauß. It extends to the Kronecker symbol and reads

$$\left(\frac{n}{m}\right)\left(\frac{m}{n}\right) = (-1)^{\frac{n'-1}{2}\frac{m'-1}{2}},$$

for co-prime integers $n = 2^e n'$ and $m = 2^f m'$ that are not both negative. Here e and f are chosen such that n' and m' are odd.

1. Take a quadruple $\mathbf{a} = (a_1, a_2, a_3, a_4)$ satisfying $Q(\mathbf{a}) = 0$ and let C_i be tangent circles with $\kappa(C_i) = a_i$ for $i = 1, 2$. Then we claim that

$$\{\kappa(C) \colon C \text{ tangent to } C_1 \text{ and } C_2\} = \{(a_1 + a_2)x^2 - (a_1 + a_2 + a_3 - a_4)x + a_3 \colon x \in \mathbb{Z}\}.$$

   Let $f(x) = (a_1 + a_2)x^2 - (a_1 + a_2 + a_3 - a_4)x + a_3$ and note that $f(0) = a_3$ and $f(1) = a_4$. We define

$$\mathbf{a}_k = (a_1, a_2, f(2k), f(2k+1)) \text{ for } k \in \mathbb{Z}.$$

   In particular $\mathbf{a}_0 = \mathbf{a}$. Using induction we check that $\mathbf{a}_k = \mathbf{a} \cdot (S_3S_4)^k$ for $k \in \mathbb{Z}$. Indeed, for $k > 0$ one needs to verify

$$2a_1+2a_2+2f(2k+1)-f(2k) = f(2k+2) \text{ and } 2a_1+2a_2+2f(2k+2)-f(2k+1) = f(2k+3).$$

   Similar identities hold for $k < 0$ after noting that $(S_3S_4)^k = (S_4S_3)^{-k}$.

2. We claim that given tangent circles $C_1, C_2 \in \mathscr{G}_0$ there is a circle C'' tangent to C_1 and C_2 such that $(\kappa(C_1), \kappa(C'')) = (\kappa(C_2), \kappa(C'')) = 1$. To see this one simply observes that for $p \mid \kappa(C_1)\kappa(C_2)$ the function $f(x)$ constructed in the previous step takes values co-prime to p. Note that here it is used that the packing $\mathscr{G}_0$ is primitive.

3. We can now establish the claim by noting that, by the general definition of Apollonian gaskets, there must be a path of tangent circles from C to C'. If a step in this path features circles with not co-prime curvatures, then we can fix this by inserting an additional circle as constructed in the step above.

We are almost done. Indeed, recall that all curvatures are generated by the quadruple $(-3, 5, 8, 8)$. Thus there is a circle $C \in \mathscr{G}_0$ with $\kappa(C) = 5$, which is tangent to a circle $C' \in \mathscr{G}_0$ with curvature 8. Since $5 \equiv -3 \bmod 8$ we apply the definition of the Kronecker symbol and find

$$\chi_2(C) = \left(\frac{5}{8}\right) = -1. \tag{1}$$

Since χ_2 is constant on $\mathscr{G}_0$ we have $\chi_2(C) = -1$ for all $C \in \mathscr{G}_0$. Suppose now that there $C \in \mathscr{G}_0$ with $\kappa(C) = k^2$. Then we must necessarily have

$$-1 = \chi_2(C) = \left(\frac{*}{k^2}\right) = \left(\frac{*}{k}\right)^2 = 1.$$

This is a contradiction. □

In the remarkable paper [10] they have studied many more packings and established the existence of certain quadratic and even quartic obstructions. While the basic idea is already visualized in the proof given above carrying out the details is slightly more technical.

The natural next step is to ask for an updated version of the local to global conjecture. It seems reasonable to believe that the aforementioned obstructions are the only ones. This expectation was formally put forward in [10, Conjecture 1.5].

Conjecture 4.1. *Let $\mathscr{G}$ be a primitive integral Apollonian gasket. Then all but finitely many natural numbers that are not ruled out by congruence obstructions or by the quadratic and quartic obstructions given in* [10, Theorem 2.4] *appear in $\kappa(\mathscr{G})$.*

5 Concluding remarks

Even though the counter examples given in [10] can be constructed essentially elementarily, they came quite unexpectedly. It is in general an interesting question to seek a better structural understanding of these obstructions and maybe extend them to different settings. Let us linger a bit on the latter question.

The local to global conjecture for Apollonian gaskets can be put into a very general setting. Indeed, suppose we have a thin integer set $\Gamma \subseteq \mathrm{Mat}_{n\times n}(\mathbb{Z})$. We can think of this as thin (semi)-group such as the Apollonian group A that appeared above. Further we take a affine linear map $F\colon \mathrm{Mat}_{n\times n}(\mathbb{Q}) \to \mathbb{Q}$ such that $F(\Gamma) \subseteq \mathbb{Z}$. In the setting above we take $F(\gamma) = \langle e_1, \mathbf{a}\cdot\gamma\rangle$ where e_1 is the first standard unit vector and $\langle\cdot,\cdot\rangle$ is the euclidean inner product. We are interested in the (asymptotic) structure of $F(\Gamma)$.

In general there will be certain congruence obstructions, which one can hope to understand for reasonable F and Γ. We call $n \in \mathbb{Z}$ admissible if $F(\gamma) \equiv n \bmod q$ has solutions for all $q \in \mathbb{N}$. Naively one could expect that every sufficiently large admissible integer actually appears in $F(\Gamma)$. Note that this is way to optimistic in general but under favorable circumstances many number theorists (including myself) were probably inclined to believe such a statement prior to the work in [10]. It is very likely that obstructions structurally similar to those presented in loc. cit. appear in many instances of this general local to global philosophy.

While the set-up we have described is very abstract it covers many interesting situations such as Zaremba's conjecture. See [5] for details.[11] Indeed, in this and also in several other related set-ups Bourgain and Kontorovich can put their orbital circle method to good use and establish almost all versions of these local to global conjectures. We refer to [14] for a nice survey in this direction.

It is very interesting to see if and how the obstructions in [10] appear in these more general settings. A particular interesting set-up is the one related to Zaremba's conjecture. Indeed certain generalizations of Zaremba's conjecture have been disproven in [18] using reciprocity obstructions that are related to those discussed here.

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