

# S4A2 - Graduate Seminar on The Springer Correspondence

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**Overview:** Weyl groups are finite reflection groups that arise naturally in the study of semisimple Lie algebras, algebraic groups, and root systems. They play a central role in representation theory and appear in many different guises across geometry, combinatorics, and algebra. A fundamental example of a Weyl group is the symmetric group  $S_n$ .

In the 1970s, Springer made the remarkable discovery that the representation theory of Weyl groups admits a rich and unexpected geometric interpretation. More specifically, Springer showed that Weyl groups act naturally on the cohomology of certain algebraic varieties, now known as Springer fibers. This construction produces all irreducible representations of Weyl groups in a uniform way and reveals deep connections between geometry, topology, and representation theory.

The goal of this seminar is to understand this geometric construction of Weyl group representations. We will study the geometry of the flag variety, the nilpotent cone, and Springer fibers, and explain how their topology encodes the representation theory of Weyl groups.

Beyond its original scope, Springer theory has become a model for many of the most influential developments in modern representation theory. Its ideas inspired geometric approaches to the representation theory of a wide range of algebraic objects, including (affine) Hecke algebras, finite groups of Lie type, quantum groups, and  $p$ -adic groups, with important applications to the Langlands program. In this sense, Springer theory provides one of the first and most fundamental examples of the powerful philosophy that representation-theoretic phenomena can often be understood through geometry.

The seminar is intended to provide both a conceptual introduction to Springer theory and a concrete understanding of how geometry can be used to construct and study representations.

**Time and location:** Thursdays 12-2pm, room: N0.007

**Main reference:** Representation Theory and Complex Geometry [2]

**Prerequisites:** Basic knowledge of representation theory (finite groups, semisimple Lie algebras) and algebraic geometry (affine varieties). Basic familiarity with cohomology of topological spaces would be helpful but is not necessary.

**Organizational meeting:** Tuesday 10.02.2025 2pm, room N 0.007

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## Talk 1 (16.04): Quasi-projective varieties and quotients

Recall the definition of an affine variety. Define projective and quasi-projective varieties and morphisms between them [6, § 4.1, 4.2]. Recall the definition of the tangent space and that of a smooth variety [6, §1.2,1.3, 1.4]. and define the tangent bundle and cotangent bundle [6, 1.4] (called the tangent fiber space in this book; you may restrict to the affine case and just say that this notion extends to quasi-projective varieties). Point out that over  $\mathbb{C}$ , any smooth variety can also be viewed as a manifold by equipping it with the Euclidean topology. State [6, Chapter 5, Theorem 2]. Define proper morphisms as in [6, Chapter 5, Remark after Theorem 3]. Show that  $\mathbb{A}^1$  does not satisfy [6, Chapter 5, Theorem 2] and thus it is not proper. Describe the quotient of an algebraic group by a closed subgroup [1, Theorem 1.16] (you don't have to do the proof) and state [1, Definition 1.18]. State the following result without proof: If  $H \subset G$  is a subgroup and  $Y$  is an  $H$ -variety then a good quotient  $G \times^H Y = (G \times Y)/H$  exists where  $h \cdot (g, y) = (gh^{-1}, hy)$ . Show that  $SL_2/B \cong \mathbb{P}^1$  where  $B$  is the subgroup of upper triangular matrices (you may use without proof that a bijective morphism of smooth varieties in characteristic 0 is an isomorphism).

Don't forget to do many examples in this talk!!

## Talk 2 (23.04): Flag varieties

Recall/Define the notions of a semisimple algebraic group  $G$ , Borel subgroup  $B$  (you can define this to be a maximal connected solvable subgroup or simply a connected subgroup whose Lie algebra is a Borel subalgebra), maximal torus  $T$ , Weyl group  $W := N_G(T)/T$  and their Lie-algebraic analogues  $\mathfrak{g}$ ,  $\mathfrak{b}$ , and  $\mathfrak{h}$  (see [5]). Describe all of these explicitly for  $G = SL_n$ . Introduce the following objects: the set of Borel subalgebras  $\mathcal{B}$ ; the quotient space  $G/B$ ; and the flag variety  $\mathcal{FL}_n$ . Prove that we can identify  $\mathcal{B} \cong G/B$  [2, Section 3.1.6], and, when  $G = SL_n$ , we have  $\mathcal{B} \cong G/B \cong \mathcal{FL}_n$  [2, Lemma 3.1.15]. State the Bruhat decomposition of the flag variety [2, Theorem 3.1.9] and give a proof for the  $SL_n$ -case [4, § 23.4]. You should explicitly describe the flag varieties when  $G = SL_2$ ,  $G = SL_3$  and  $G = Sp_4$ , and the corresponding Bruhat decompositions. Describe the flag variety of  $SO_n$  and  $Sp_{2n}$  explicitly [4, §23.3, p. 383].

## Talk 3 (30.04): The universal resolution

Recall the statement of [2, Lemma 3.1.15]. Define  $\tilde{\mathfrak{g}}$  for  $SL_n$  and prove [2, Claim 3.1.19 and Lemma 3.1.20] (you don't have to define  $\mathfrak{g}^{rs}$  in the general case yet, just define it as the set of all diagonalizable matrices in  $\mathfrak{g} = \mathfrak{sl}_n$  with  $n$  distinct eigenvalues). Define the abstract Weyl group  $\mathbb{W}$  and the abstract Cartan  $\mathfrak{H}$  following [2, 3.1.22] and prove [2, 3.1.29]. Define the 'universal resolution' as in [2, 3.1.31]. Show that  $\tilde{\mathfrak{g}} \rightarrow \mathcal{B}$  is a vector bundle and  $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  is proper. Define regular elements ([2, Definition 3.1.3]) and state [2, Lemma 3.1.4]. Then prove [2, 3.1.36]. State Chevalley restriction theorem [2, Theorem 3.1.39] without proof, if there is time you can sketch a proof in the  $\mathfrak{sl}_n$ -case.

## Talk 4 (07.05): The Springer resolution

Recall the definition of universal resolution [2, 3.1.31]. Then prove [2, Lemma 3.1.42]. Recall the Jordan decomposition in semisimple Lie algebras and discuss examples for  $\mathfrak{sl}_n$ . Define the nilpotent

cone and the Springer resolution [2, §3.2]. Point out that for  $SL_n$  the orbits in  $\mathcal{N}$  are indexed by partitions (by Jordan normal form). Make sure to draw a picture of the Springer resolution for the case of  $G = SL_2$ . Show that  $\tilde{\mathcal{N}} \cong T^*\mathcal{B}$  (no need to do [2, 3.2.3]). Discuss [2, Proposition 3.2.5]. Then prove that the Springer resolution is an isomorphism over the regular nilpotent orbit [2, 3.2.8-3.2.14] (you may assume that there are only finitely many nilpotent orbits). You should discuss some explicit examples of Springer fibers, in particular an example where the Springer fiber is not smooth (e.g. describe the Springer fibers of  $SL_3$  explicitly).

## Talk 5 (21.05): The Steinberg variety

Define the Steinberg variety [2, §3.3] and describe it explicitly for  $SL_2$ . Prove that the Steinberg variety is a union of conormal bundles [2, Proposition 3.3.4]. Use this to compute the irreducible components of the Steinberg variety [2, Corollary 3.3.5]. Continue with [2, Lemma 3.3.8] (you may skip 3.3.6 and 3.3.7). State without proof that [2, Lemma 3.3.8] is in fact an equality (this is essentially the content of [2, Lemma 3.3.6]). Discuss [2, Lemma 3.3.15] and prove [2, Corollary 3.3.20, 3.3.24, 3.3.27, 3.3.28]. Recall the notion of a Young tableaux and sketch a proof that for  $SL_n$  the irreducible components of the Steinberg variety are indexed by standard Young tableaux [7, Theorem 1.1]. Deduce from [2, Corollary 3.3.27] that there is a bijection between  $S_n$  and the set of all pairs of standard Young tableaux of the same shape. This is a well-known correspondence in combinatorics known as the Robinson-Schensted correspondence.

## Talk 6 (11.06): Borel-Moore homology I

Survey the basic properties of (singular) cohomology [3, B.1] including homotopy invariance, the long exact sequence and the Künneth formula. Then move on to define Borel-Moore homology following [3, B.2] and prove that it is independent of the choice of embedding ([3, B.2 Lemma 1]) and prove [3, B.2 Lemma 2]. Construct the the proper push-forward and the long exact sequence [3, B.2]. Prove the Künneth formula [2, 2.6.19]. Define the fundamental class following [3, B.3]. Prove [3, B.3 Lemma 6] and use this to compute the Borel-Moore homology of the flag variety.

See also: [2, §2.6]

## Talk 7 (18.06): Borel-Moore homology II

Recall the definition of Borel-Moore homology. Define the intersection pairing [2, (2.6.16)]. Point out that it depends on the ambient smooth space  $M$ . Illustrate this with the example of  $[\mathbb{P}^1] \cap [\mathbb{P}^1]$  first for the ambient smooth space  $M = \mathbb{P}^1$  and then in  $M = T^*\mathbb{P}^1$  (where we embed  $\mathbb{P}^1$  via the zero-section). In the first case, you can check directly from the definition that you get again  $[\mathbb{P}^1]$ . In the second you may use without proof that for some  $u \in H^2(T^*\mathbb{P}^1, T^*\mathbb{P}^1 \setminus \mathbb{P}^1)$  you have  $u \cup u \neq 0$  and deduce that  $[\mathbb{P}^1] \cap [\mathbb{P}^1] = c[pt]$  for some  $c \neq 0$ . Continue with the definition of the restriction with support map [2, 2.6.21]. State the existence and main properties of the smooth pullback and the Gysin pullback from [2, 2.6.26]. Discuss the special case where the fibration comes from restriction of a vector bundle with smooth base (the "locally trivial oriented fibration case"). State the projection formula [2, (2.6.29)] and the proper base change theorem [2, Propositions 2.7.22] without proof. Define the specialization map [2, 2.6.30]. Show that this map is independent of choices and compatible with restriction and the intersection pairing.

## Talk 8 (25.06): Convolution

Start by discussing the convolution of functions on finite sets [2, 2.7.1]. Skip the discussion on differential forms. Continue with the definition of convolution in [2, 2.7.5] and discuss the examples in [2, 2.7.10]. Prove that convolution is associative [2, 2.7.18]. Prove the compatibility of convolution with specialization [2, 2.7.23]. Discuss the convolution algebra and its modules [2, § 2.7.40] and the dimension properties in [2, 2.7.48, 2.7.48, 2.7.50]. As a key example, compute the algebra structure on  $H(Z)$  where  $Z$  is the Steinberg variety of  $SL_2$ . Recall for this from talk 5 that  $Z$  has two components, namely  $\overline{Z}_s = \mathbb{P}^1 \times \mathbb{P}^1$  and  $Z_e = \Delta_{T^*\mathbb{P}^1}$ .  $[Z_e]$  is the identity, so you only need to compute  $[\overline{Z}_s] \star [\overline{Z}_s]$ . For this you need to compute  $p_{12}^*[\overline{Z}_s] \cap p_{23}^*[\overline{Z}_s] = [\mathbb{P}^1 \times \mathbb{P}^1 \times T^*\mathbb{P}^1] \cap [T^*\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1]$  in  $(T^*\mathbb{P}^1)^3$ . By the Künneth formula this comes down to computing the cap product of  $[\mathbb{P}^1]$  with itself or with  $[T^*\mathbb{P}^1]$  which was explained in the last talk. From this you should be able to prove that  $[\overline{Z}_s] \star [\overline{Z}_s] = c[\overline{Z}_s]$  for some  $c \neq 0$ . Deduce that  $H(Z) \cong \mathbb{C}[S_2]$  where  $[\overline{Z}_s] \mapsto \frac{c}{2}(s+1)$ . Conclude by stating the base locality property [2, 2.7.45].

## Talk 9 (02.07): A geometric realization of the Weyl group

The main goal of this talk is to prove the isomorphism  $H(Z) \cong \mathbb{C}[W]$  from [2, Theorem 3.4.1]. Start by recalling the varieties  $\tilde{\mathfrak{g}}, \tilde{\mathcal{N}}, \mathcal{N}, \tilde{\mathfrak{g}}^{rs}, \dots$  and the definition of convolution defined in the previous talks. Define the 'graphs'  $\Lambda_w^h$  and prove [2, Lemma 3.4.5]. Show that  $[\Lambda_{yw}^h] = [\Lambda_y^{w(h)}] \star [\Lambda_w^h]$ . Then go through the proof of [2, Theorem 3.4.1].

## Talk 10 (09.07): The Weyl group action on the homology of Springer fibers

Recall the isomorphism  $H(Z) = \mathbb{C}[W]$  from the last talk. Explain the  $H(Z)$ -module structure on  $H_*(\mathcal{B}_x)$ . Prove [2, Lemmas 3.5.1, 3.5.2, 3.5.3]. Compute some examples of component groups (e.g. for  $SL_n$ ). Prove [2, Lemma 3.6.3]. Use this to conclude that for  $G = SL_n$  only the trivial component group representation appears in  $H_*(\mathcal{B}_x)$  (since the  $C_{SL_n}(x)$ -action factors through the  $C_{GL_n}(x)$ -action). Prove [2, Lemma 3.6.11] and use this to prove [2, Claim 3.5.5] (leave [2, Theorem 3.6.9] to the next talk). Discuss the Weyl group action on  $H_*(\mathcal{B})$  induced by the map  $G/T \rightarrow \mathcal{B}$  [2, 3.6.13]. Then prove [2, Claim 3.6.17]. If there is time, give an example of a Springer fiber where

the action of  $C(x)$  is non-trivial (e.g.  $G = Sp_4$ ,  $x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ ).

## Talk 11 (16.07): The Springer correspondence

The main objective of this talk is to prove [2, Theorem 3.5.7]. Discuss the notion of a transversal slice [2, § 3.2.18] including [2, 3.2.20, 3.2.21] (you can skip the proofs if you need more time for the rest of the talk). State the Jacobson-Morozov theorem [2, Theorem 3.7.1] without proof. Define the Slodowy slice [2, Proposition 3.7.15] and prove [2, 3.7.19]. Prove [2, 3.5.8 and 3.5.9] (where you should pick  $S$  to be the Slodowy slice  $S := \mathcal{N} \cap (e + Z_{\mathfrak{g}}(f))$ ). Compute the dimension of  $\tilde{S}$  and

continue with [2, 3.5.13]. Then prove [2, 3.5.7]. Recall from last time that for  $SL_n$  only the trivial  $C(x)$  representation appears in  $H_*(\mathcal{B}_x)$ . Use this to deduce [2, Theorem 3.6.2].

## References

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