A geometric compactification of the moduli space of grafted surfaces

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Abstract

In this thesis, we study the degenerations of complex projective structures on an orientable surface S of genus at least two, aiming to describe a compactification of their moduli space and provide a geometric interpretation of the boundary points. The moduli space $\mathcal{PT}(S)$ of complex projective structures admits a parametrisation due to Thurston via grafting: each structure corresponds to a metric on S that is obtained from a hyperbolic one by grafting, namely inserting, flat parts along a measured lamination. This construction yields a homeomorphism $\mathcal{PT}(S) \cong \mathcal{T}(S) \times \mathcal{ML}(S)$, where $\mathcal{T}(S)$ is Teichmüller space and $\mathcal{ML}(S)$ the space of measured laminations. We refer to the metric surfaces resulting from grafting as grafted surfaces.

We prove that degenerating sequences of grafted surfaces, suitably rescaled, can converge geometrically to half-translation surfaces, that is, Euclidean surfaces with cone singularities. We use the orthogeodesic foliation introduced by Calderon and Farre to analyse this phenomenon, and we construct a bordification of $\mathcal{PT}(S)$ whose boundary at infinity is given by the moduli space $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ of half-translation surfaces up to rotation and rescale. The topology on this bordification is the one induced by a marked version of Gromov-Hausdorff convergence introduced in this work. A preliminary version of these result is already available in a preprint of the author [Mon24].

We also show that $\mathcal{PT}(S)$ embeds into the space of projective geodesic currents and this embedding extends continuously to our bordification too. We describe the whole closure of its image, when embedding $\mathcal{PT}(S)$ into projective currents using a novel l^1-variant of the grafted metric. The boundary in this case is described by mixed structures, which have appeared in different forms in the bordification of other moduli spaces of geometric structures on surfaces. As an application, we describe the limits of so-called generalised stretch rays in Teichmüller space.

Contents

Introduction

Let S be an orientable closed surface of genus g at least 2. Teichmüller space $\mathcal{T}(S)$ parametrizes all the complete hyperbolic metrics on S up to isotopy, or equivalently all the marked Riemann surfaces diffeomorphic to S. Another family of well known geometric structures on S is given by half-translation surfaces. They are surfaces obtained by taking a polygon in $\mathbb{R}^2$ and identifying its sides in pairs via the compositions of translations and possibly rotations by π. A half-translation structure on S is equivalent to the datum of a Riemann surface structure up to isotopy and a holomorphic quadratic differential on it, so the moduli space of such marked structures is described as $\mathcal{QT}(S)$ the bundle of quadratic differentials over $\mathcal{T}(S)$.

Grafting

The main object of study in this thesis will be a third class of geometric structures that endow the surface S with a piecewise hyperbolic and piecewise flat metric. Such surfaces are results of an operation called grafting. In the simplest case, grafting consists of the following: given a hyperbolic surface X, cutting it open along a simple closed geodesic and gluing a flat cylinder in its place (see Figure 2.1). Grafting can be then extended to weighted multicurves, which are collections of disjoint simple closed geodesics, with a positive weight for each component, indicating the height of the cylinder to be inserted in place of each curve of the collection.

Grafting has been studied from various points of view and produced many results in the course of the years, since Thurston's seminal work until more recent times. The main result due to Thurston on grafting is its use to parametrize the moduli space of complex projective structures $\mathcal{PT}(S)$. Weighted multicurves sit inside a larger space $\mathcal{ML}(S)$ of measured laminations, that can be seen as a completion of the space of weighted multicurves. As shown in Thurston's unpublished work and discussed in [Tan97] and [Dum09], a complex projective structure induces a metric on the surface S, called the Thurston metric, that can be interpreted as the metric obtained by grafting a hyperbolic surface X along a measured lamination λ, extending in this way the definition of grafting to measured laminations, and establishing a mapping class group equivariant homeomorphism

$$\mathrm{Gr}\colon \mathcal{T}(S) \times \mathcal{ML}(S) \xrightarrow{\sim} \mathcal{PT}(S).$$

For this reason we can think of $\mathcal{PT}(S)$ as parametrizing grafted surfaces and consider its points as surfaces with the grafted Thurston metric.

Various results show that grafting interacts richly with the geometry of Teichmüller space. By considering the conformal structure of the Thurston metric of grafted surfaces, fixing a lamination λ, $\mathrm{Gr}(\cdot, \lambda)$ induces a homeomorphism of $\mathcal{T}(S)$ [SW02]; fixing a hyperbolic metric X, $\mathrm{Gr}(X, \cdot)$ defines a homeomorphism from $\mathcal{ML}(S)$ to $\mathcal{T}(S)$ [DW08]. Like earthquakes [Thu86], grafting gives paths, called grafting rays, that connect any two points in Teichmüller space. These rays have been studied for their geometric behaviour; for instance, under certain conditions they fellow travel Teichmüller geodesics [CDR10].

Moreover, since grafting can be used to parametrize complex projective structures on a surface, it is then also intimately related to the theory of Kleinian groups. Much more recently, it became apparent that this construction also plays a crucial role in higher Teichmüller theory, which makes it desirable to understand the moduli space of grafted surfaces from the geometric point of view. Indeed, in recent works [BHM25], [Bla+25], grafting has proved to be useful to study the geometry of the moduli space of Hitchin representations. In that setting, a notion of grafting along decorated multicurves is introduced as a deformation of Fuchsian representations into the Hitchin component.

Compactifying moduli spaces via geodesic currents

A central motivation of this thesis is to describe a compactification of the moduli space $\mathcal{PT}(S)$ that captures the geometry of degenerating structures and gives a meaningful interpretation of boundary points. In broad terms, we seek to understand how these geometric structures can degenerate and what asymptotic behaviour arises along diverging sequences. More precisely, our goal is to define a *bordification* of $\mathcal{PT}(S)$: a larger topological space containing $\mathcal{PT}(S)$ as a dense subset, with a boundary at infinity encoding the possible degenerations. When this bordification is compact, it becomes a compactification in the usual sense.

A guiding example for this perspective is Bonahon's reinterpretation of Thurston's compactification of Teichmüller space using geodesic currents [Bon88]. The space of geodesic currents $\mathscr{C}(S)$ is a linear space of measures that encompasses various geometric structures on S. Bonahon showed that each hyperbolic metric X determines a geodesic current L_X, the Liouville current, which encodes the marked length spectrum of X. That is, L_X remembers, for every free homotopy class of closed curves on S, the length of its geodesic representative in the metric X. This construction extends to all negatively curved Riemannian metrics [Ota90], and we adapt it in Chapter 9 to the setting of grafted surfaces.

As a space of measures, $\mathscr{C}(S)$ admits a natural free $\mathbb{R}^+$-action by rescaling. Taking the quotient yields the space of projective geodesic currents $\mathbb{P}\mathscr{C}(S)$, which Bonahon showed to be compact. This allows him to prove the following fundamental result.

Theorem ([Bon88]). *The map $\mathcal{T}(S) \to \mathscr{C}(S)$ given by $X \mapsto L_X$ is a proper* $\mathrm{Mod}(S)$*-equivariant embedding. Projectivising yields a map $\mathcal{T}(S) \to \mathbb{P}\mathscr{C}(S)$, which is also an embedding. The closure of $\mathcal{T}(S)$ in $\mathbb{P}\mathscr{C}(S)$ is a compact disc, with $\mathcal{T}(S)$ as its interior and boundary consisting of projective currents arising from measured laminations.*

Building on Bonahon's approach, Duchin, Leininger and Rafi [DLR10] constructed a Liouville current for the singular Euclidean metrics associated to half-translation surfaces. Since these metrics are invariant under rotation of the half-translation structure, the map $q \mapsto L_q$ descends to a well-defined map $\mathcal{QT}(S)/S^1 \to \mathscr{C}(S)$. They used this to prove the following:

Theorem ([DLR10]). *The map $\mathcal{QT}(S)/S^1 \to \mathscr{C}(S)$ is a proper* $\mathrm{Mod}(S)$*-equivariant embedding, and induces an embedding of the projectivised spaces $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \to \mathbb{P}\mathscr{C}(S)$. The closure of the image consists of projective currents associated to mixed structures.*

Here, a mixed structure consists of a half-translation structure on a subsurface of S together with a measured lamination on its complement. Such structures also appear as boundary points in the compactification of other moduli spaces of surface structures [Bur+21b], [OT21], [Ouy23].

We believe that the map $\mathcal{PT}(S) \to \mathscr{C}(S)$ induced by taking the Liouville current is a proper embedding too, however proving the injectivity is a difficult problem that is still open. It is equivalent to prove marked length spectrum rigidity for grafted surfaces, which means that the metric is determined by its marked length spectrum, which is not known despite the numerous results around that direction [Ota90], [CFF92], [Kim99], [Sun16].[1] What we do instead is following the same idea of Bonahon to achieve a bordification, but using a different ambient space instead of geodesic currents.

[1] Post-submission update: marked length spectrum rigidity holds true for grafted surfaces. A short argument is provided in Appendix A.3, adapting the proof from [CFF92].

Bordification of $\mathcal{PT}(S)$ via marked metrics

We introduce $\mathrm{MMet}(S)$, the space of marked metrics on S, as an ambient space in which to construct our bordification. Elements of $\mathrm{MMet}(S)$ are length metric spaces X that are homeomorphic to the surface S, equipped with a marking, that is, a homotopy class of homeomorphisms $S \to X$. We adapt the notion of Gromov-Hausdorff convergence to this setting of marked metric spaces, and so define a topology on $\mathrm{MMet}(S)$.

Due to its general definition, the space $\mathrm{MMet}(S)$ contains a much wider variety of metrics on S than those arising from grafted or half-translation surfaces, and thus lends itself to the study of various families of metrics. With this in mind, we prove some results that hold in the full generality of $\mathrm{MMet}(S)$. In particular, while Gromov-Hausdorff convergence is generally a weak notion, we show that in our restricted setting, where all spaces are homeomorphic to S, it becomes rather strong. For instance, convergence in $\mathrm{MMet}(S)$ implies a Lipschitz-type convergence of the marked length spectrum.

The space $\mathrm{MMet}(S)$ also carries a natural $\mathbb{R}^+$-action by rescaling the metric, and its projectivisation $\mathbb{P}\,\mathrm{MMet}(S)$ is obtained by taking the quotient. Our main result shows that all singular Euclidean metrics induced by half-translation surfaces can be realized as projective limits of grafted surfaces. This motivates the study of both the moduli spaces of grafted and singular flat metrics as subspaces of $\mathrm{MMet}(S)$. While we prove that the natural inclusion $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is a proper embedding, for $\mathcal{QT}(S)$ the inclusion arises from the quotient $\mathcal{QT}(S)/S^1 \to \mathrm{MMet}(S)$, due to the rotation invariance of the metric. Our main theorem can then be stated as follows.

Theorem A. *Both the inclusions of $\mathcal{PT}(S)$ and $\mathcal{QT}(S)/S^1$ into $\mathrm{MMet}(S)$ are proper $\mathrm{Mod}(S)$-equivariant embeddings. Projectivising yields embeddings of $\mathcal{PT}(S)$ and $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ into $\mathbb{P}\,\mathrm{MMet}(S)$, and identifying them with their images, we obtain*

$$\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S).$$

As mentioned earlier, we do not know whether the map $\mathcal{PT}(S) \to \mathscr{C}(S)$ into geodesic currents is an embedding. However, we show that it extends continuously to our bordification of $\mathcal{PT}(S)$ constructed via marked metrics. By marked length spectrum rigidity for grafted surfaces (see Appendix A.3), this bordification embeds into the space of currents, and its image coincides with the closure of $\mathcal{PT}(S)$ within the subspace of filling geodesic currents.

We conjecture that half-translation surfaces are dense in the boundary of $\mathcal{PT}(S)$ inside $\mathbb{P}\mathscr{C}(S)$, although we do not prove this here. What we do instead is providing a complete description of the closure of $\mathcal{PT}(S)$ in projective geodesic currents, using a different and very natural embedding. This alternative map corresponds to a different type of metric, denoted l^1, that can also be associated to grafting.

The l^1-metrics

Both grafted surfaces and half-translation surfaces can be endowed with a natural alternative metric derived from the l^1 norm on $\mathbb{R}^2$. For half-translation surfaces, this is simply the pull-back of such a metric via the charts. For grafting, we define a metric that in the case of multicurves, is obtained by replacing the Euclidean metric on the inserted cylinders with the l^1 one.

A key advantage of these metrics is that the associated geodesic currents, encoding their length functions, can be described very explicitly:

$$\mathrm{Gr}(X, \lambda) \mapsto L_X + \lambda \quad \text{and} \quad q \mapsto \mathrm{Re}(q) + \mathrm{Im}(q)$$

Moreover, the same argument showing that grafted surfaces can degenerate (up to rescaling) to a half-translation surface applies in the l^1 setting as well. In this case, however, the description of the boundary is considerably simpler thanks to the explicit formulae above. We then prove an analogue of Theorem A for the l^1-metrics.

Theorem B. *The two maps above induce embeddings of $\mathcal{PT}(S)$ and $\mathcal{QT}(S)/\pm 1$ into the space of currents $\mathscr{C}(S)$. Projectivising yields embeddings of $\mathcal{PT}(S)$ and $\mathbb{P}_{\mathbb{R}}\mathcal{QT}(S)$ into $\mathbb{P}\mathscr{C}(S)$. In this setting, the boundary of $\mathcal{PT}(S)$ is formed by what we call* l^1-mixed structures*, and contains $\mathbb{P}_{\mathbb{R}}\mathcal{QT}(S)$ as an open dense subset.*

The l^1-mixed structures we introduce arise as degenerations of the l^1-metrics induced by half-translation surfaces: as in the Euclidean case, the metric is defined on a subsurface and degenerates to a measured lamination on its complement. We establish a correspondence between l^1-mixed structures and geodesic currents that can be written as the sum of two measured laminations. As an application, this analysis of the l^1 variant of grafted surfaces allows us to generalize a result of Papadopoulos [Pap91].

Main arguments

Other similar bordifications of $\mathcal{PT}(S)$ have been studied [Dum06], [Dum07]. However, they differ from ours in the approach, as they are performed just by considering respectively the grafting and the Schwarzian parametrization of $\mathcal{PT}(S)$ and taking a bordification of the relative parameter spaces. On the other hand, in our bordification, points at infinity correspond to surfaces with a metric, and the convergence to such points is not a mere convergence in a parameter space, but Gromov-Hausdorff convergence, up to rescaling. This not only gives a more intrinsic picture of degeneration, but also unifies hyperbolic, grafted and singular flat metrics from half-translation structures in a single moduli space.

The key idea is that, under suitable conditions and after rescaling, the metric of a grafted surface with large grafting measure closely approximates that of a half-translation surface.

Central to our construction is the map $\mathcal{O}\colon \mathcal{PT}(S) \to \mathcal{QT}(S)$ introduced by Calderon and Farre in [CF24b] and [CF24a]. Originally devised to extend Mirzakhani's results [Mir10] and to study dynamical conjugacies between $\mathcal{PT}(S)$ and $\mathcal{QT}(S)$, this map assigns canonically to each grafted surface $\mathrm{Gr}(X,\lambda)$ a half-translation surface $\mathcal{O}(X,\lambda)$. It does so by constructing a measured foliation, called the orthogeodesic foliation, made of piecewise geodesic segments orthogonal to the grafting lamination, and considering the translation surface that has it as horizontal foliation, and λ as vertical foliation. In our setting, this map provides a natural candidate for the limiting geometry of a grafted surface as the grafting area becomes large. Let us first see a simple example of a sequence of grafted surfaces converging, up to rescaling, to a half-translation one.

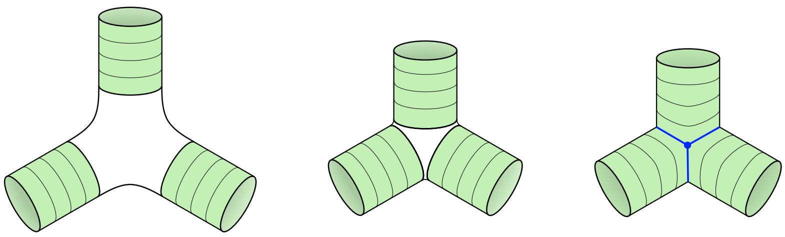

Figure 1: A pair of pants in the grafted surface $\mathrm{Gr}(X_n, n\mu)$ becomes thinner as n increases, eventually collapsing to a graph.

Example . Let X be a hyperbolic surface and $\mu = \sum_i a_i\gamma_i$ a weighted pants decomposition: a collection of disjoint simple closed geodesics γ_i, equipped with weights a_i, and whose complement consists of pairs of pants. Given a length l_i for each γ_i, one can choose a hyperbolic structure on X so that these curves have lengths l_i. The grafted surface $\mathrm{Gr}(X,\mu)$ is obtained by inserting Euclidean cylinders of height a_i along each γ_i. Consider now a sequence of hyperbolic surfaces X_n in which the lengths of the curves in μ are $n \cdot l_i$, then $\mathrm{Gr}(X_n, n\mu)$ is a sequence where the flat cylinders are rescaled by n and the hyperbolic pair of pants become thinner and longer. By rescaling by $1/n$ the metric of $\mathrm{Gr}(X_n, n\mu)$, we have that the cylinders are of constant size, while the negatively curved parts shrink and collapse into graphs. The geometric projective limit is then a surface obtained by gluing the cylinders along graphs, which has a natural half-translation structure.

To make the comparison between $\mathrm{Gr}(X, \lambda)$ and $\mathcal{O}(X, \lambda)$ explicit, we construct a map, called *deflation* $\mathcal{D}\colon \mathrm{Gr}(X, \lambda) \to \mathcal{O}(X, \lambda)$. In the example above, the half-translation surface $\mathcal{O}(X_n, n \cdot \mu)$ is obtained by gluing together only the grafted cylinders of $\mathrm{Gr}(X, \lambda)$. The deflation map $\mathcal{D}\colon \mathrm{Gr}(X_n, n \cdot \mu) \to \mathcal{O}(X_n, n \cdot \mu)$ is a local isometry on the cylinders, while it collapses the hyperbolic parts to graphs. As n goes to infinity, the rescaled grafted surface converges to the half-translation surface. We prove a quantitative version of this approximation in general.

Theorem C. *Consider both* $\mathrm{Gr}(X, \lambda)$ *and* $\mathcal{O}(X, \lambda)$ *rescaled to have unit area. If the systole of* $\mathcal{O}(X, \lambda)$ *is at least* ϵ *and* $\ell_X(\lambda) > 1$*, then the map* $\mathcal{D}\colon \mathrm{Gr}(X, \lambda) \to \mathcal{O}(X, \lambda)$ *is a* $C_\epsilon \cdot (\ell_X(\lambda))^{-1/2}$*-isometry (see Definition* 3.2*), where the constant* C_ϵ *depends only on* ϵ*.*

The deflation map not only allows us to compare the geometry of grafted surfaces with the ones of half-translation surfaces, but also suggests a reverse operation. Namely, we can view a half-translation surface as obtained by collapsing the negatively curved parts of a grafted surface. Reversing this collapse yields a deformation of the half-translation surface in which one-dimensional subspaces are "inflated" into negatively curved hyperbolic surfaces with geodesic boundary. We call this inverse process *inflation*. From this perspective, inflation acts as a counterpart to Thurston's grafting: while grafting inserts flat cylinders into hyperbolic surfaces, inflation introduces negative curved parts into half-translation surfaces.

Structure of the thesis

Chapter 1 recalls classical notions in Teichmüller theory, including hyperbolic and half-translation surfaces, and collects standard results used throughout the thesis. Chapter 2 introduces grafting and presents Thurston's parametrisation of the space of projective structures via grafting.

In Chapter 3, we define the space of marked metrics $\mathrm{MMet}(S)$ and introduce a notion of marked Gromov-Hausdorff convergence, showing in particular that it implies strong convergence of the marked length spectrum. Chapter 4 presents the main result, stated in Theorem A, and proves that the moduli space $\mathcal{PT}(S)$ of grafted surfaces, as well as $\mathcal{QT}(S)/S^1$, embed into $\mathrm{MMet}(S)$. The core argument for the convergence of grafted surfaces to half-translation surfaces is deferred to later chapters. Chapter 5 is somewhat independent of the main result, focusing on quantitative convergence in $\mathrm{MMet}(S)$ for small grafting. Within this simpler setting, it introduces the strategy used in the proof of Theorem C.

Chapter 6 summarizes constructions and results from [CF24b] and [CF24a] concerning the orthogeodesic foliation and the map $\mathcal{O}\colon \mathcal{PT}(S) \to \mathcal{QT}(S)$. These are used to define the deflation map, which plays a key role in the main arguments of the next chapter. In Chapter 7, we study the geometric properties of the deflation map and, in doing so, prove Theorem C, from which we deduce Theorem A. Chapter 8 introduces the notion of inflation, offering a more visual intuition for the degeneration of grafted surfaces to half-translation surfaces.

Chapter 9 constructs Liouville currents for grafted surfaces and compares our bordification via marked metrics with the closure of the image of $\mathcal{PT}(S)$ in the space of projective geodesic currents. Finally, Chapter 10 studies l^1 variants of grafted and half-translation surfaces, and gives a complete description of their compactification via geodesic currents.

1 Preliminaries

We introduce here the classical objects and structures we will use in this thesis. All the statements of this chapter are well known facts in Teichmüller theory and can be found in introductory books such as [Ima92], [FM11], [Mar16].

1.1 Teichmüller space

Let us assume for the entire thesis that S is a closed orientable surface of genus $g \geq 2$. It is known that S admits many hyperbolic structures. Let us denote by $\mathcal{T}(S)$ the Teichmüller space of the surface S, the space where to each point corresponds an isotopy class of a complete hyperbolic metric on S. We remind that thanks to the correspondence between hyperbolic structures and complex structures, $\mathcal{T}(S)$ can be thought of parametrizing either the hyperbolic structures on S up to isotopy, or the Riemann surface structures up to isotopy. Equivalently, a point in $\mathcal{T}(S)$ can be also described as a pair $(X, [f])$ where X is a hyperbolic surface and $[f]$ the homotopy class of $f\colon S \to X$ a diffeomorphism called *marking*. Together they determine, by pull-back, a hyperbolic structure on S, unique up to isotopy.

The mapping class group $\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$ of the surface S acts on $\mathcal{T}(S)$ by pre-composition on the marking, so the quotient $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$ is the moduli space of hyperbolic surfaces up to isometry (or equivalently the space of Riemann surfaces up to biholomorphism) and it is commonly referred to just as *moduli space* of S.

The moduli space is not compact as there exist sequences of hyperbolic surfaces with a closed geodesic whose length goes to zero. Given any $\epsilon > 0$ the subset of $\mathcal{M}(S)$ of surfaces with no closed geodesic of length smaller than ϵ is compact, and its preimage $\mathcal{T}_\epsilon(S)$ in $\mathcal{T}(S)$ is called *ϵ-thick part* of Teichmüller space.

1.2 Half-translation surfaces and quadratic differentials

A half-translation structure on a surface S is an atlas for $S \setminus F$, where $F \subset S$ is finite, made of charts to $\mathbb{R}^2$ such that transition maps are $\pm\,\mathrm{id}$ composed with a translation. Such a structure induces a flat singular metric on S, with finitely many singular points, each with cone angle $k_i\pi$ for some integer $k_i > 2$. By an extension of Gauss-Bonnet theorem for such singular metric it holds $\sum_i (k_i - 2) = 2|\chi(S)|$, in particular the singularities are at most $4g - 4$.

The analogue of Teichmüller space for half-translation structures will be denoted with $\mathcal{QT}(S)$. The space $\mathcal{QT}(S)$ is parametrized by holomorphic quadratic differentials. A holomorphic quadratic differential q on a Riemann surface X is an object that in holomorphic charts reads as $q(z) = \varphi(z)dz^2$ with $\varphi(z)$ holomorphic. The holomorphic quadratic differentials on X form a vector space $Q(X)$ of complex dimension $3g - 3$. If we consider charts of a half-translation surface structure from subsets of $S \setminus F$ to $\mathbb{R}^2 \cong \mathbb{C}$, these induce a complex structure on $S \setminus F$ that one can show extends to the whole surface S. Moreover, the quadratic differential expressed as dz^2 in such charts also extends to a holomorphic quadratic differential on the whole surface. Such correspondence identifies $\mathcal{QT}(S)$ with the vector bundle of quadratic differentials over Teichmüller space. We will make no distinction when talking about a half-translation surface and its associated quadratic differential q.

1.3 Measured laminations

Geodesic laminations are interesting objects that have been very useful in many ways in Teichmüller theory. A *geodesic lamination* λ on a hyperbolic surface X is a compact subset foliated by geodesics. Some geodesic laminations admit a transverse measure, that is the datum of inducing of a measure on each transverse arc α, such that the support is the entire intersection $\alpha \cap \lambda$, and such that it is invariant under homotopies of α preserving its transversality. We will indicate with $i(\lambda, \alpha)$ the total mass of the transverse measure induced on the arc α. The laminations together with such a transverse measure are called *measured laminations*, and the geodesic lamination underlying a measured lamination is called its *support*.

Measured laminations can have both closed and infinite leaves. When a measured lamination has only closed leaves, its support is a *multicurve*, that is a finite union of simple closed geodesics, pairwise non-homotopic. In this case, the transverse measure is determined by a positive weight for each component, and the measured lamination is called *weighted multicurve*.

The measure of the intersection of laminations with suitable sets of curves gives charts that induce a piecewise linear structure on the space $\mathcal{ML}(S)$ of measured laminations, and, in particular, a topology that makes it homeomorphic to $\mathbb{R}^{6g-6}$. According to this topology the set of weighted multicurves is dense in $\mathcal{ML}(S)$. In general when we consider a measured lamination we would assume that it is non-empty; although in some occasion, we will need to consider also the zero lamination. For this reason we will specify with the notation $\mathcal{ML}_0(S)$ when we mean to include the zero lamination.

Geodesic laminations are topological objects, as upon changing the underlying hyperbolic metric on S, it is always possible to straighten them to be geodesic according to the new metric and the topology induced on the space $\mathcal{ML}(S)$ is also independent of the choice of the metric.

There is a continuous function $\ell\colon \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathbb{R}_{\geq 0}$ called *length function*, and that we indicate as $\ell_X(\lambda)$ with the following properties. For λ a unit mass closed geodesic, $\ell_X(\lambda)$ is the length of the geodesic representative of λ according to the hyperbolic metric X; it is linear in λ; and additive for disjoint laminations.

Train tracks Train tracks are very convenient combinatorial object used to study measured laminations. An extensive study of these can be found in [PH91]. A train track τ on S is a smoothly embedded graph on S, where the edges are called *branches* and the vertices *switches*, and at every switch, all the adjacent branches are parallel to a single direction and there is at least one incoming branch for each of the two orientations. For every lamination λ there is a train track τ that *carries* it, meaning that λ can be homotoped onto it (see geometric train track in Chapter 6.5). The transverse measure of λ induces a transverse measure on the branches, which is simply encoded by a *weight system*, that is the assignation of a non-negative weight for each branch, such that the switch condition is satisfied. The switch condition requires that at every switch the sums of the weights of the branches on each of the sides are equal. Vice versa, a train track τ with a weight system always encodes a measured lamination carried by τ. If all the weights are positive, it is said that τ snugly carries λ.

Hausdorff topology The Hausdorff topology on the space of geodesic laminations is given by fixing a hyperbolic structure X for S and taking then the induced Hausdorff distance between geodesic laminations as compact subsets of X. We observe that if we have a sequence of measured laminations λ_n converging as measures to another measured lamination λ, this does not necessarily imply the Hausdorff convergence of their support. The Hausdorff limit, for n going to infinity, of the support of λ_n will be a geodesic lamination containing λ, but the containment can be strict, as there can be leaves of λ_n whose mass converge to 0.

Nevertheless, even requiring convergence of the supports, any measured lamination can still be approximated by weighted multicurves. In [CEG06, Section 4.2] it is shown that every measured lamination is uniquely decomposed as a finite union of minimal components. Every minimal component is such that every leaf is dense in it and the supported transverse measures form a finite dimensional simplicial cone; moreover it is approximable in both the measure and in the Hausdorff sense by a single weighted closed geodesic. This implies that for every $\lambda \in \mathcal{ML}(S)$ there exists a sequence of weighted multicurves $(\mu_n)_n$ that converges to λ in the measure topology and whose supports converge to the support of λ with respect to the Hausdorff topology. Or in general, if τ is a train track that carries snugly λ, then a sequence of measured laminations converging to λ as measure, converges to λ in the Hausdorff topology if and only if λ_n is eventually carried by τ (see [CF24a, Lemma 15.1]).

1.4 Measured foliations

To a half-translation surface structure $q \in \mathcal{QT}(S)$ is associated a pair of singular foliations on S, $(\mathrm{Re}(q), \mathrm{Im}(q))$, called respectively *vertical* and *horizontal foliation*, as they are defined as the two measured foliations whose leaves are respectively vertical and horizontal in the charts given by q. The transverse measure is the one given by the Euclidean metric in the horizontal and vertical directions respectively.

There is a canonical identification (see [Lev83]) between the space of measured laminations $\mathcal{ML}(S)$ and the space $\mathcal{MF}(S)$ of measured foliations on S up to homotopy and Whitehead moves (operations that do not change essentially the foliation, except for merging or separating two singularities connected by a leaf).

The association of the pair of vertical and horizontal foliations $(\mathrm{Re}(q), \mathrm{Im}(q))$ to the half-translation surface q induces an embedding $\mathcal{QT}(S) \hookrightarrow \mathcal{MF}(S) \times \mathcal{MF}(S)$. The image of the embedding is given by all the pairs of foliations that can be realized as transverse foliations (meaning with same singular points, and transverse everywhere else). Let us denote by $q(\eta, \lambda)$ the quadratic differential associated to the pair of measured foliations (η, λ). Given $\lambda \in \mathcal{MF}(S)$, we denote by $\mathcal{MF}(\lambda)$ the space of transverse measured foliations transverse to λ. It is known, from the work of Hubbard and Masur [HM79], that $\mathcal{MF}(\lambda)$ is homeomorphic to $\mathcal{T}(S)$.

2 Grafting and projective structures

In this chapter we will introduce grafting first as a geometric operation to produce a surface with piecewise hyperbolic and piecewise flat metric. Then we will introduce complex projective structures on surfaces, and show how grafting can be described as a deformation of such structures. We briefly present Thurston's result on the parametrization of the moduli space of complex projective structure using grafting, and show the equivalence between the projective and geometric points of view.

2.1 Geometric grafting

Let us first see the easiest example, also called *simple grafting* where the inserted flat part is only one cylinder. Later on we will extend the definition to a more general setting. Let X be a hyperbolic surface, γ a simple closed geodesic in it and $a > 0$ a real number. We will perform a grafting of X along γ with weight a. For easiness of notation, here as in the rest of the paper, we will use the same symbol to indicate both the parametrization and the image of curves and arcs.

Definition 2.1. Let us consider a cylinder $\gamma \times [0, a]$ endowed with the Euclidean product metric, so that it has circumference of length $\ell_X(\gamma)$ and height a. Let us cut X open along γ, obtaining a hyperbolic surface X' with two geodesic boundary components γ_1, γ_2, each of length $\ell_X(\gamma)$, with parametrizations compatible with the cut, meaning that for every t, the points $\gamma_1(t), \gamma_2(t)$ both correspond to $\gamma(t)$ before the cut. Then glue the Euclidean cylinder to X' along the boundary with the isometries $\varphi_1 \colon \gamma \times \{0\} \to \gamma_1$, and $\varphi_2 \colon \gamma \times \{a\} \to \gamma_2$. The result is called *grafted surface* and we denote it with $\mathrm{Gr}_{a\gamma}(X)$.

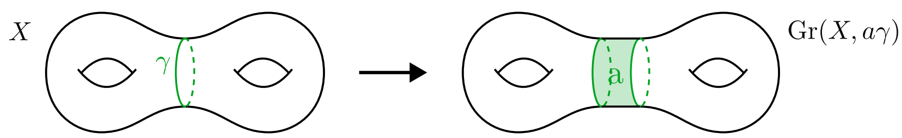

Figure 2.1: Simple grafting along γ with weight a.

Note that the obtained grafted surface has a well-defined differentiable structure diffeomorphic to S. Since the gluing is performed via isometries along the boundary components, the metric of the grafted surface is Riemannian with C^1 regularity, but not C^2, as we notice that the curvature is discontinuous, as it jumps from locally constant -1 in the hyperbolic part to locally constant 0 in the flat cylinder.

There is a natural homotopy equivalence from $\mathrm{Gr}_{a\gamma}(X)$ to X that consists in collapsing the grafted cylinder to the curve γ. This homotopy equivalence allows us, given a marking $f\colon S \to X$ to induce one on $\mathrm{Gr}_{a\gamma}(X)$, so grafting can be also seen as an operation on marked structures.

Moreover, we observe that the grafting datum is only topological, as the curve γ can be seen just as a free homotopy class of simple closed curves on S. In order to perform the grafting on X we just need to push forward γ to X via the marking f, take in its homotopy class the representative that is geodesic, and perform the grafting operation along it. Therefore, grafting is well-defined on the pairs $(X, a\gamma)$ where $X \in \mathcal{T}(S)$ and $a\gamma$ is a weighted free homotopy class of simple closed curves on S.

Grafting easily extends to weighted multicurves. So given $X \in \mathcal{T}(S)$ and a weighted multicurve $\mu = a_1\gamma_1 + \cdots + a_k\gamma_k$, the grafted surface $\mathrm{Gr}_\mu(X)$ is obtained by inserting one cylinder of height a_i for each component γ_i of μ. We call *hyperbolic pieces* the connected components of the complement of the cylinders in $\mathrm{Gr}_\mu(X)$. They are the interior of hyperbolic surfaces with geodesic boundary.

2.2 Complex projective structures

Let us consider the Riemann sphere $\mathbb{CP}^1$. The projective automorphisms of $\mathbb{CP}^1$ are the maps induced by the tautological linear action of $\mathrm{GL}_2\,\mathbb{C}$ on $\mathbb{C}^2$, that on $\mathbb{CP}^1$ acts with kernel $\mathbb{C}^*\,\mathrm{id}$. So the automorphisms group is $\mathrm{PSL}_2\,\mathbb{C} = \mathrm{GL}_2\,\mathbb{C}/\mathbb{C}^*\,\mathrm{id}$, and its elements are also known as *Möbius transformations.*

Definition 2.2. A complex projective structure on S is a maximal atlas for S of charts to $\mathbb{CP}^1$, where the transition maps are restrictions of Möbius transformations.

We will often omit the word *complex* and with *projective structures* we always mean the complex projective ones. A map between surfaces with projective structures is called *projective* if in charts it is read as the restriction of a Möbius transformation.

The definition of projective structures is a particular case (with $X = \mathbb{CP}^1$ and $G = \mathrm{PSL}_2\,\mathbb{C}$) of the more general concept of (G, X)-structure on a smooth n-dimensional manifold, where X is a real analytic manifold of dimension n, and G is a Lie group acting faithfully and analytically on X. The theory of (G, X)-structures is developed in [CEG06, Chapter I.1]. We will state here some properties of projective structures, that are commonly known for hyperbolic structures, which coincide with $(\mathrm{PSL}_2\,\mathbb{R}, \mathbb{H}^2)$-structures, and true in general for any (G, X)-structure.

Given a projective structure Z, we recall that the universal cover $\tilde{S}$ of S also inherits naturally from Z a projective structure just by pre-composition of the atlas for S with the covering map $\pi\colon \tilde{S} \to S$. We call $\tilde{Z}$ the surface $\tilde{S}$ with such an inherited projective structure.

Proposition 2.3. *For every projective structure Z, there exists a homomorphism $\rho_Z\colon \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C}$ called* holonomy representation*, and a ρ_Z-equivariant projective map $\mathrm{dev}_Z\colon \tilde{Z} \to \mathbb{CP}^1$ called* developing map*. The pair (ρ_Z, dev_Z) is unique, up to the action of Möbius transformations. This action of $\varphi \in \mathrm{PSL}_2\,\mathbb{C}$ is given by conjugation by φ on ρ_Z and post-composition by φ on dev_Z.*

Definition 2.4. The moduli space $\mathcal{PT}(S)$ of marked projective structures on S, similarly to the case of marked hyperbolic structures, can equivalently be defined as:

1. the set of projective structures on S up to isotopy;
2. the set of marked projective surfaces

$$\mathcal{PT}(S) = \{(Z, [f]) \mid Z \text{ projective surface},\ f\colon S \to Z \text{ marking}\} / \sim$$

   where $(Z, [f]) \sim (Z', [f'])$ if there exists a projective diffeomorphism $\varphi\colon Z \to Z'$ such that $\varphi \circ f$ is homotopic to f'.

3. the set of pairs of developing map and holonomy, up to isotopy

$$\mathcal{PT}(S) = \{(\mathrm{dev}, \rho) \mid \rho\colon \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C},\ \mathrm{dev}\colon \tilde{S} \to \mathbb{CP}^1\ \rho\text{-equiv.}\ \}/\,\mathrm{Diff}_0^+(S);$$

The last one also allows us to endow $\mathcal{PT}(S)$ with a topology. Indeed, the set of all possible developing maps from $\tilde{S}$ to $\mathbb{CP}^1$ has naturally the compact open topology, or in other words, the topology given by the uniform convergence on compact subsets. We can take the quotient by the action of the group $\mathrm{Diff}_0^+(S)$ and so endow the space $\mathcal{PT}(S)$ with the quotient topology.

Remark . A complex projective structure induces also a complex structure, as the Möbius transformations are holomorphic. Thus, we can define a forgetful map $\pi\colon \mathcal{PT}(S) \to \mathcal{T}(S)$ that remembers only the complex structure.

There is a parametrization of $\mathcal{PT}(S)$, called Schwarzian parametrization (see [Dum07]), that gives a homeomorphism $\mathcal{PT}(S) \to \mathcal{QT}(S)$ compatible with forgetful maps from the spaces to $\mathcal{T}(S)$. This implies for example that $\mathcal{PT}(S)$ is finite dimensional. Although, we will not use this parametrization, but rather a more geometrical one using grafting and introduced by Thurston.

Remark . We observe that $\mathcal{T}(S)$ embeds naturally into $\mathcal{PT}(S)$. Indeed, the hyperbolic plane $\mathbb{H}^2$ can be modelled as the upper half plane $\mathcal{H}$ of $\mathbb{C} \subset \mathbb{CP}^1$, with isometry group given by the real Möbius transformations $\mathrm{PSL}_2\,\mathbb{R} < \mathrm{PSL}_2\,\mathbb{C}$. So the developing map and holonomy of a hyperbolic structure are also developing map and holonomy for a projective structure.

2.3 Projective grafting

As we pointed out in the previous section, Teichmüller space is naturally embedded in $\mathcal{PT}(S)$, meaning that every hyperbolic surface induces a projective structure too. We will see now how grafting can be seen also as a deformation of such projective structures, and see how Thurston used it to parametrize the space $\mathcal{PT}(S)$.

For this scope, let us introduce the following model for cylinders with a projective structure. Consider, for $a < 2\pi$, V_a the sector of the complex plane defined as

$$V_a = \{re^{i\theta} \mid r > 0,\, \theta \in (0, a)\}.$$

It has a natural projective structure given by the inclusion in $\mathbb{C} \subset \mathbb{CP}^1$. The quotient $V_{a,l}$ of V_a by the $\mathbb{Z}$-action generated by the Möbius transformation $z \mapsto e^l z$, with $l \in \mathbb{R}^+$, is a cylinder endowed with the quotient projective structure. For $a > 2\pi$ we can construct $V_{a,l}$ as a concatenation, that is by gluing along their boundaries n copies of projective cylinders $V_{a/n,l}$ for n large enough so that $a/n < 2\pi$. Now we can define projective grafting totally analogously to how we defined the geometric one.

Definition 2.5. Given $X \in \mathcal{T}(S)$ and $\mu = \sum_i a_i\gamma_i$ a weighted multicurve, we cut X along the geodesic representative of μ and for every term $a_i\gamma_i$ we insert a projective cylinder $V_{a_i,\ell_X(\gamma_i)}$ in place of the closed geodesic γ_i. It is easy to verify, by comparing the developing maps, that the projective structure on the hyperbolic pieces and the cylinders match nicely and form a projective structure on the whole surface obtained.

This operation defines a grafting map $\mathrm{Gr}\colon \mathcal{T}(S) \times \mathscr{M}(S) \to \mathcal{PT}(S)$ where $\mathscr{M}(S)$ is the set of weighted multicurves. This leads to Thurston parametrization theorem of $\mathcal{PT}(S)$

using grafting.

Theorem 2.6 (Thurston). *The projective grafting defined above extends to a mapping class group equivariant homeomorphism*

$$\mathrm{Gr}\colon \mathcal{T}(S) \times \mathcal{ML}_0(S) \to \mathcal{PT}(S).$$

This means that grafting completely parametrizes the space of projective structures. And vice versa, for every grafting datum $(X, \lambda) \in \mathcal{T}(S) \times \mathcal{ML}_0(S)$ there is a projective structure $\mathrm{Gr}(X, \lambda)$ associated to it. The goal of the next section is to view $\mathrm{Gr}(X, \lambda)$ as geometric object, and try to understand its geometry also in the case when λ is not a weighted multicurve.

2.4 Thurston metric

In this section we explain why grafting and projective grafting are essentially the same. Thurston showed that any projective structure $\mathrm{Gr}(X, \lambda)$ on S induces a metric that, in the case of a structure obtained by projective grafting along a multicurve, coincides with the geometrically grafted metric $\mathrm{Gr}_\lambda(X)$ of the section above. The definition of such a metric given by Thurston, valid in the general case, is an adaptation to the projective case of the Kobayashi metric.

Definition 2.7 (Thurston metric). Given $Z \in \mathcal{PT}(S)$, the Thurston metric associated to it is a norm on the tangent bundle of S defined as follows. For every $p \in S$ and $v \in T_pS$

$$\|v\|_{Th} = \inf_j \|j^* v\|_{hyp}$$

where the infimum is taken over all projective immersions $j\colon \Delta \to Z$ from the complex unit disc Δ, containing p in their image, and where $\|\cdot\|_{hyp}$ is the hyperbolic metric on Δ.

We recall that the complex unit disc $\Delta = \{z \in \mathbb{C} | z\overline{z} < 1\}$ can be both seen as the Poincaré disc model for the hyperbolic plane $\mathbb{H}^2$ and as $\Delta \subset \mathbb{C} \subset \mathbb{CP}^1$ inside the Riemann sphere, which equips it with a canonical projective structure.

A discussion on the Thurston metric, in a more general setting, can be found in [KP94], where in particular they show the following.

Theorem 2.8 ([KP94]). *Let Z be a surface with a complex projective structure. The associated Thurston metric is a complete Riemannian metric g_Z of class $C^{1,1}$ (meaning*

C^1 with Lipschitz derivatives), compatible with Z, i.e., it is conformal to $\pi(Z)$. Moreover, the Lipschitz constant of its derivatives is bounded locally in $\mathcal{PT}(S)$.

Our goal is now to understand better the geometry of projective surfaces with the Thurston metric. For this scope we need to extrapolate some results from Thurston's proof of Theorem 2.6. The idea of his proof goes in the opposite direction of the grafting map: showing a construction to retrieve, given a projective structure Z, the grafting datum (X, λ) and proving that this depends continuously on Z. At the core of the construction is producing a map $\kappa\colon Z \to X$ that will play the role of the generalization of the collapsing map we mentioned in the previous section. A more detailed discussion of Thurston's argument can be found in [Dum09] and [Tan97]. Here we summarize in the following statement only some particular geometric information that emerges from Thurston's argument.

Theorem 2.9 (Thurston). *Given a projective structure $\mathrm{Gr}(X, \lambda) \in \mathcal{PT}(S)$, there exists a homotopy equivalence $\kappa\colon \mathrm{Gr}(X, \lambda) \to X$ such that*

i. $\kappa^{-1}(\lambda)$ is a lamination on $\mathrm{Gr}(X, \lambda)$, geodesic with respect to the Thurston metric;

ii. κ maps isometrically each complete geodesic contained in $\kappa^{-1}(\lambda)$ to a leaf of λ;

iii. for every connected component A of $X \setminus \lambda$, κ restricts to an isometry between $\kappa^{-1}(A)$ and A.

Moreover, in the case when λ is a weighted multicurve, the surface $\mathrm{Gr}(X, \lambda)$ equipped with the Thurston metric coincides with the geometrically grafted surface $\mathrm{Gr}_\lambda(X)$ obtained from X inserting Euclidean cylinders along λ, and the map $\kappa\colon \mathrm{Gr}(X, \lambda) \to X$ is the map collapsing the cylinders.

For this reason from now on, we will unify the notation, and with $\mathrm{Gr}(X, \lambda)$ we will mean a surface equipped with both the projective structure and the Thurston metric. In the general case, when λ is not a weighted multicurve, the behaviour of the map κ, which we will still call *collapsing map*, is more complicated. See for example Appendix A.2 for a better understanding of the local behaviour. Nevertheless, the following result holds true.

Corollary 2.10. *The collapsing map $\kappa\colon \mathrm{Gr}(X, \lambda) \to X$ is 1-Lipschitz.*

Proof. It is 1-Lipschitz in the case when λ is a weighted multicurve as it is piecewise a local isometry or the projection map of a cylinder to one of its boundary component. The Thurston metric of $\mathrm{Gr}(X, \lambda)$ depends continuously on the grafting data X and λ, and so does the collapsing map κ, so by continuity κ is 1-Lipschitz for any λ measured lamination. □

2.5 Continuity of grafting

According to the definition, each point Z in $\mathcal{PT}(S)$ is an equivalence class of projective structures, that is it determines a projective structure only up to the action of elements of $\mathrm{Diff}_0^+(S)$. So also the associated Thurston metric induced on S by Z is determined only up to isotopy. In order to overcome this ambiguity we have the following.

Lemma 2.11. *The projection to the quotient by the action of* $\mathrm{Diff}_0^+(S)$ *from the space of all projective structures on* S *to* $\mathcal{PT}(S)$ *admits a continuous section.*

Proof. For every $Z \in \mathcal{PT}(S)$ we need to pick a representative, and make sure the choice depends continuously on Z. We will follow the same idea of [Wol89], where harmonic maps are used to relate any hyperbolic surface with a fixed one, in order to identify $\mathcal{T}(S)$ with a precise set of Riemannian metrics on a fixed surface.

More precisely, let us fix a hyperbolic structure $X \in \mathcal{T}(S)$ on S. Then given a projective structure Z on S we choose its representative $\hat{Z}$ with respect the action of $\mathrm{Diff}_0^+(S)$ such that the identity map on S is harmonic between X and $\pi(\hat{Z}) \in \mathcal{T}(S)$, where here $\pi(\hat{Z})$ is meant as the hyperbolic metric conformal to the complex structure induced by Z. This can be achieved by simply choosing $\hat{Z}$ to be the pull-back of Z via the unique harmonic map between X and $\pi(Z)$ compatible with the markings.

One can check that this section $Z \mapsto \hat{Z}$ is continuous with respect to the topology on $\mathcal{PT}(S)$ as the forgetful map $\pi\colon \mathcal{PT}(S) \to \mathcal{T}(S)$ is continuous and the harmonic map depends continuously on the target metric $\pi(Z)$.

□

From now on we will imply the choice of a fixed section and by a point $Z \in \mathcal{PT}(S)$ we mean a precise projective structure on the surface S. The associated Thurston metric g_Z can be then considered as a metric on the very same surface S. For this reason by Z we will mean the surface S equipped with the Thurston metric g_Z. In this way it will make sense for example to consider points x, y on S and compare their distances $d_{Z_n}(x, y)$ with respect to the metrics induced by different structures Z_n.

We now have the following result, essentially a consequence of the regularity result in Theorem 2.8, and that can be found in [Dum09, Section 4.3].

Theorem 2.12. *The Thurston metric* g_Z *on* S *depends continuously on the projective structure* $Z \in \mathcal{PT}(S)$*, with respect to the topology of the uniform convergence on the space of* C^1 *Riemannian metrics on* S*.*

3 Moduli space of marked metrics

In this chapter we introduce the space of marked metrics on the surface S and the notion of marked Gromov-Hausdorff convergence. We will use this as an ambient space in which to embed both the spaces of grafted surfaces and of metrics coming from half-translation surfaces. Let us first recall some basics on the Gromov-Hausdorff convergence.

Gromov-Hausdorff convergence A very common notion of convergence of compact metric spaces is the so called Gromov-Hausdorff convergence, introduced by Gromov in [Gro81]. The convergence is induced by a metric that is called Gromov-Hausdorff distance, which is defined as follows.

Definition 3.1 (Gromov-Hausdorff distance). Given two compact metric spaces (X, d_X) and (Y, d_Y), their Gromov-Hausdorff distance $d_{GH}(X, Y)$ is defined as

$$d_{GH}(X, Y) = \inf\{d_H(X, Y) \mid d \text{ distance on } X \cup Y,\ d|_X = d_X,\ d|_Y = d_Y\}$$

where d_H is the Hausdorff distance induced by the distance d on $X \cup Y$.

We remind that the Hausdorff distance is a metric on the space of closed subsets of a topological space. The Hausdorff distance between two closed subspaces is the infimum over the values of δ such that each of the two spaces is contained in the δ-neighbourhood of the other. Gromov showed that the Gromov-Hausdorff metric is a well-defined metric, in particular, the topology that it induces is Hausdorff.

A more practical approach to Gromov-Hausdorff distance, is offered by the concept of ϵ-isometry and the following lemma.

Definition 3.2. An ϵ-isometry is a map $f\colon X \to Y$ between metric spaces that is an isometry up to an additive error, or more precisely such that

i. for all $x_1, x_2 \in X$,

$$d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) + \epsilon;$$

ii. for every $y \in Y$ there is $x \in X$ with $d_Y(f(x), y) \leq \epsilon$.

Lemma 3.3 ([BBI01, Corollary 7.3.28]). *Let X, Y be two compact metric spaces. If there is an ϵ-isometry between X and Y, then $d_{GH}(X, Y) \leq 2\epsilon$. Vice versa, if $d_{GH}(X, Y) \leq \epsilon$ then there exists a 2ϵ-isometry between X and Y.*

This notion of geometric convergence is in general very weak, as it allows two metric spaces to be very close with respect to the Gromov-Hausdorff distance, yet rather different both metrically and topologically. However, we will use this convergence only on a rather restricted class of metric spaces, where it will be much stronger. But beside adding more restrictions on the metric spaces we consider, like for example them being surfaces, we are interested also in a notion of convergence that can keep track of the markings, for this reason we introduce a notion of *marked* Gromov-Hausdorff convergence.

3.1 Marked Gromov-Hausdorff convergence

In this section we introduce the notion of marked Gromov-Hausdorff convergence for a rather large class of metrics on a surface and show some of its properties. A quick comparison with the notion of equivariant Gromov-Hausdorff convergence introduced by Paulin in [Pau88] is discussed later on.

We remind that a *length metric space*, or *path metric space*, is a metric space where the distance between any two points is the infimum of the lengths of the paths connecting them. Then we define the following.

Definition 3.4. The *moduli space of marked metrics on S* is defined as

$$\mathrm{MMet}(S) = \{(X, [f]) \mid X \text{ length metric space, } f\colon S \to X \text{ homeomorphism}\}/\sim$$

where f is called *marking*, and where $(X, [f]) \sim (X', [f'])$ if there exists an isometry $\varphi\colon X \to X'$ compatible with the markings, i.e., such that $[\varphi \circ f] = [f']$.

As for Teichmüller space, the mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{MMet}(S)$ by precomposition on the marking. We can define then $\mathrm{Met}(S)$ the *moduli space of (unmarked) metrics* on S as the space obtained from $\mathrm{MMet}(S)$ by forgetting the markings and considering only the underlying spaces up to isometry. As a set it is then the quotient of $\mathrm{MMet}(S)$ by the action of the group $\mathrm{Mod}(S)$.

For any $(X, [f])$, the space X is homeomorphic to S, then by pulling back the metric via the marking, we can interpret each element of $\mathrm{MMet}(S)$ as the topological manifold S

equipped with a length metric inducing the topology of S. Sometimes, when the marking is not particularly relevant, we will write just $X \in \mathrm{MMet}(S)$ and leave the marking implicit.

Remark . Being S compact, then any space $X \in \mathrm{MMet}(S)$ is a complete length space, so it is also a *geodesic space*, meaning that for every pair of points x, y there is a geodesic segment connecting them and that realizes their distance.

The topology we give to the space $\mathrm{MMet}(S)$ is the one induced by what we define here as marked Gromov-Hausdorff convergence.

Definition 3.5 (Marked Gromov-Hausdorff convergence). Let $(X_n)_{n\in\mathbb{N}}$ and X be marked metric surfaces in $\mathrm{MMet}(S)$. We say that the sequence X_n converges to X in the *marked Gromov-Hausdorff* sense if there exist $\varphi_n\colon X_n \to X$ homotopy equivalences that are marking preserving and ϵ_n-isometries, where ϵ_n goes to 0 as n goes to infinity.

Remark . One can verify that given two points in $\mathrm{MMet}(S)$, the infimum of the ϵ for which there is a map φ as above that is an ϵ-isometry, gives an asymmetric metric on $\mathrm{MMet}(S)$ (the only non-trivial part is to check non-degeneracy of the metric, and this follows from Proposition 3.9). It is not clear a priori if such a metric is also symmetric. Anyway, we will not make use of this asymmetric metric, as it does not have overall good global properties. For example, it is not a path metric, and for instance, the distance between $(X, [f])$ and $(X, [\varphi \circ f])$ is always at most the diameter of X, independently of $\varphi \in \mathrm{Mod}(S)$. However, it is convenient to use it to define a local basis for the induced topology.

Remark . Given $(X, [f]) \in \mathrm{MMet}(S)$, the subset $V_{(X,[f])}(\epsilon)$ defined as

$$V_{(X,[f])}(\epsilon) = \{(Y, [g]) \mid \exists \varphi\colon Y \to X \text{ homotopy equivalence and } \epsilon\text{-isometry}\}$$

is a neighbourhood of $(X, [f])$. The family of such ϵ-neighbourhoods is, by definition of marked Gromov-Hausdorff convergence, a local basis for the topology of $\mathrm{MMet}(S)$.

We endow $\mathrm{Met}(S) = \mathrm{MMet}(S)/\,\mathrm{Mod}(S)$ with the quotient topology.

Proposition 3.6. *The topology on* $\mathrm{Met}(S)$ *is the same as the one induced by the Gromov-Hausdorff convergence.*

Proof. A priori there is a subtle difference between the two topologies: Gromov-Hausdorff and the quotient of our marked Gromov-Hausdorff. The latter trivially implies the former just by ignoring the markings and applying Lemma 3.3. To prove their equivalence we need to show that for any sequence $(X_n)_n$ in $\mathrm{Met}(S)$ converging with respect to the Gromov-Hausdorff topology to $X \in \mathrm{Met}(S)$, there exist markings f_n such that $(X_n, [f_n]) \to (X, [f])$ in $\mathrm{MMet}(S)$. In other words, we need to show that if there are ϵ_n-isometries $X_n \to X$, with ϵ_n going to zero, then we can find homotopy equivalences that are ϵ'_n-isometries, with ϵ'_n also going to zero.

We split the proof of this fact in two separate statements, that we show separately below. The first, Lemma 3.7, provides us a sequence $f_n \colon X_n \to X$ of continuous ϵ'_n-isometries with $\epsilon'_n \to 0$. Then Theorem 3.8 shows that with n large enough so that ϵ'_n is small enough, the maps f_n must be automatically homotopy equivalences. □

Here is a lemma, that extends Lemma 3.3, showing that when two spaces have small Gromov-Hausdorff distance, not only there is an ϵ-isometry, but under enough regularity conditions for the spaces, the ϵ-isometry can be chosen to be continuous.

Lemma 3.7 ([BBI01, Exercise 7.5.8]). *If $(X_n)_{n\in\mathbb{N}}, X$ are metric spaces homeomorphic to closed manifolds and $X_n \to X$ in the sense of Gromov-Hausdorff, then there are continuous maps $\varphi_n \colon X_n \to X$ that are ϵ'_n-isometries with $\epsilon'_n \to 0$ for n going to infinity.*

Sketch of the proof: Since we have Gromov-Hausdorff convergence, then by Lemma 3.3 there exists $f_n \colon X_n \to X$ that are ϵ_n-isometries, with ϵ_n going to zero. The idea is that such maps, even if not continuous, can be approximated with a continuous map. Indeed, if we find continuous maps φ_n such that $d_X(\varphi_n(x), f_n(x)) \leq \delta_n$, then one can easily see that φ_n is a $(\epsilon_n + 2\delta_n)$-isometry. The idea is then to prove that X has this property: any ϵ isometry $Y \to X$ can be approximated with a continuous function φ distant at most δ from it. And δ goes to zero for ϵ going to zero.

First, one can show this for X being the Euclidean space $\mathbb{R}^n$. In this case one can just choose a locally finite ϵ-net in Y, take a partition of unity of Y where every partition function has support in a ball of radius ϵ centred in a point of the ϵ-net. Then define φ simply by averaging (we can do that because $X = \mathbb{R}^n$) the values of f on the ϵ-net by weights given by the partition functions.

Then one can show the same statement for X a Riemannian manifold smoothly and isometrically embedded in $\mathbb{R}^n$. In this case one can repeat the same process as before, but the averaging will yield us a function φ with image in $\mathbb{R}^n$. One can check that up to choosing a denser ϵ net, such an image lies in a small tubular neighbourhood of X, so we can compose with the projection to X, and check that the composition map still well-approximates f.

Finally, if X is any metric space homeomorphic via $h\colon X \to X'$ to a smooth manifold X', we can embed X' in $\mathbb{R}^n$ for some n by Whitney theorem and endow X' with the restriction of the Riemannian metric of $\mathbb{R}^n$ to X'. Notice that the homeomorphism h, being X and X' compact, is uniformly continuous. So if we well-approximate as above the map $h \circ f\colon Y \to X'$ with a continuous map φ, then $h^{-1} \circ \varphi$ will well-approximate f, up to replacing δ with the one given by the modulus of continuity of h. □

For this second result we will need a lemma that we show in the next subsection. Let us denote with $\mathrm{sys}(X)$ the length of the *systole* of X, which is the shortest non homotopically trivial closed curve in X.

Theorem 3.8. *If X, Y are surfaces in $\mathrm{Met}(S)$ and $f\colon X \to Y$ is a continuous ϵ-isometry with $\epsilon < \mathrm{sys}(Y)/8$, then it is a homotopy equivalence.*

Proof. It will be enough to show that the homomorphism induced by f between the fundamental groups of X and Y is surjective. Both fundamental groups are isomorphic to $\pi_1(S)$, and it is known that any surjective self homomorphism of $\pi_1(S)$ is an isomorphism: the fundamental group of a surface of finite type is finitely generated and residually finite (see [Lop94]) and as a consequence, Hopfian, meaning that any surjective self homomorphism is an isomorphism. So f induces an isomorphism at the level of the π_1, which, for our closed surfaces of genus at least 2, is enough to conclude that f is a homotopy equivalence, by Whitehead theorem.

So let us show that given any closed curve γ in Y, there is a curve η in X whose image under f is homotopic to γ. Let us call $r = \mathrm{sys}(Y)/2$. In particular, we have that $\epsilon < r/4$.

Let us subdivide γ in the concatenation of segments $\{\gamma|_{[t_i,t_{i+1}]}\}_{i=1}^N$ of length $d < r - 4\epsilon$. For every point $\gamma(t_i)$, thanks to the second property of an ϵ-isometry, there is a point $x_i \in X$ such that $d_Y(f(x_i), \gamma(t_i)) < \epsilon$. Connect now each x_i to x_{i+1} with the shortest geodesic segment and call, realizing their distance, and call it η_i. We now show that the concatenation η of all such segments is the sought curve.

Observe that the points x_i and x_{i+1} are close to each other. More precisely

$$\begin{aligned} d_X(x_i, x_{i+1}) &\leq d_Y(f(x_i), f(x_{i+1})) + \epsilon \\ &\leq d_Y(f(x_i), \gamma(t_i)) + d_Y(\gamma(t_i), \gamma(t_{i+1})) + d_Y(\gamma(t_{i+1}), f(x_{i+1})) + \epsilon \\ &\leq \epsilon + d + \epsilon + \epsilon \\ &= 3\epsilon + d. \end{aligned}$$

Then every point in η_i lies at distance at most $d + 3\epsilon$ from x_i. The map f then sends η_i to an arc $f(\eta_i)$ connecting $f(x_i)$ to $f(x_{i+1})$ that lies entirely inside the ball of centre x_i

and radius $d_X(x_i, x_{i+1}) + \epsilon \leq d + 4\epsilon < r$. Let us compare $f(\eta_i)$ with the concatenation of the shortest segment between $f(x_i)$ and $\gamma(t_i)$, the arc $\gamma|_{[t_i,t_{i+1}]}$ and the shortest segment between $\gamma(t_{i+1})$ and $f(x_{i+1})$. This concatenation has length at most $2\epsilon + d < r$ and connects the same endpoints $f(x_i), f(x_{i+1})$, so by Lemma 3.12 is homotopic to $f(\eta_i)$.

By concatenating the homotopies, we obtain that $f(\eta)$ is homotopic to γ as we wanted. □

Moreover, we have the following result, showing that even if the space $\mathrm{MMet}(S)$ is very large, its topology is still well-behaved. This will allow us to show in the next chapter that the mapping class group acts properly on $\mathrm{MMet}(S)$.

Proposition 3.9. *The space* $\mathrm{MMet}(S)$ *is first-countable and Hausdorff.*

Proof. The neighbourhoods $V_{(X,[f])}(1/n)$ are a countable local basis for the topology of $\mathrm{MMet}(S)$.

Suppose now that $(X, [f])$ and $(Y, [g])$ are such that for every ϵ there is an ϵ-isometry $\varphi_\epsilon \colon X \to Y$ that is a homotopy equivalence compatible with the markings. By Arzelà-Ascoli theorem, there is a sequence $\epsilon_n \to 0$, such that φ_{ϵ_n} converges uniformly to $\varphi \colon X \to Y$, which then must be an isometry. We only need to show that φ is also compatible with the markings, so that by definition of $\mathrm{MMet}(S)$ we will have that $(X, [f]) \sim (Y, [g])$. For this it is enough to show that for every n large enough, φ is homotopic to φ_{ϵ_n}. Or equivalently, that for any closed curve γ in X, its images under φ and φ_{ϵ_n} are homotopic. By an argument analogue to the one in the previous proof, one can see that choosing n large enough so that by uniform convergence $d_Y(\varphi(x), \varphi_{\epsilon_n}(x)) < \mathrm{sys}(Y)/4$, is enough to achieve that. □

3.2 Marked length spectrum

Note that grafted surfaces, as we defined them, and singular flat metrics arising from quadratic differentials are considerably more regular than a generic metric in $\mathrm{MMet}(S)$. In fact, we will see in the next chapter that on such metrics, marked Gromov-Hausdorff convergence satisfies even stronger properties. However, in Chapter 10 we will deal with less regular metrics. For this reason, in this section we work again in the full generality of $\mathrm{MMet}(S)$, to ensure that our results apply in that broader context as well.

Let $\mathcal{C}(S)$ be the set of free homotopy classes of non homotopically trivial closed curves on S. There is a natural correspondence between $\mathcal{C}(S)$ and the conjugacy classes of elements in the $\pi_1(S)$.

Given $X \in \mathrm{MMet}(S)$, the marking also induces a correspondence between $\mathcal{C}(S)$ and the set of free homotopy classes of curves on X. So given $\gamma \in \mathcal{C}(S)$, we call *length function* $\ell_X(\gamma)$ the length in X of the shortest curve in the homotopy class of γ.

Proposition 3.10. *For any $X \in \mathrm{MMet}(S)$ and $\gamma \in \mathcal{C}(S)$, the length function $\ell_X(\gamma)$ is well-defined and positive.*

Proof. By definition, X is a geodesic metric space homeomorphic to a smooth manifold. Indeed, we can take $\tilde{X}$ the metric universal cover of X. The group of covering automorphisms can be identified with $\pi_1(X)$ and under this identification the free homotopy class of γ corresponds to a conjugacy class in $\pi_1(X)$. Consider then any point $x \in \tilde{X}$. Its orbit under the action of the elements in the conjugacy class of γ is discrete. Then there is another orbit point, distinct from x, that realizes the minimal distance from x. Such a distance depends continuously on x. If we let x vary in a compact fundamental domain for the action of deck transformations, we get that the distance attains a positive minimum. Since $\tilde{X}$ is a geodesic space, such a minimal distance is realized by a geodesic arc. The image of such an arc will be a closed curve realizing the minimum length $\ell_X(\gamma)$ in the homotopy class of γ. □

The collection of the lengths of all the non-trivial homotopy classes of closed curves $(\ell_X(\gamma))_{\gamma\in\mathcal{C}}$ is called *marked length spectrum.* See Chapter 9 for a discussion on the marked length spectrum of grafted surfaces. Observe that the minimum of the lengths in the length spectrum is the systole length. Additionally, we also have that like in the hyperbolic case, also in the generality of metrics in $\mathrm{MMet}(S)$, the length spectrum is well-behaved in the following sense.

Lemma 3.11. *Given $X \in \mathrm{MMet}(S)$, and $L > 0$, there are only finitely many homotopy classes of curves such that $\ell_X(\gamma) < L$.*

Proof. Le us fix $D \subset \tilde{X}$ a compact fundamental domain. As we discussed above, for every free homotopy class γ there is a point $x \in D$ and γ' conjugate of γ such that $d(x_0, \gamma' x_0) = \ell_X(\gamma)$. If $\ell_X(\gamma) < L$, then this means that $\gamma' D$ lies at a distance less than L from D. Take then K the compact metric L-neighbourhood of D. Since the action of $\pi_1(S)$ is properly discontinuous, the number of translates $\gamma' D$ of D intersecting K is finite, and so is the number of γ such that $\ell_X(\gamma) < L$. □

The injectivity radius of a Riemannian surface Y is defined as the largest radius such that, at every point of Y, the exponential map is injective on the ball of such a radius in the tangent space. With Y non-positively curved, the injectivity radius is always positive.

Given any two points in Y non-positively curved, then every geodesic arc connecting them is unique in its homotopy class, and if their distance is less than the injectivity radius r, then there is only one geodesic arc connecting them with length at most r.

For non-positively curved Riemannian surfaces, one can see that the injectivity radius equals half of the systole length. In general, for a surface $X \in \mathrm{MMet}(S)$ we now show nevertheless that the constant $\mathrm{sys}(X)$ plays a similar role to the one of the injectivity radius.

Lemma 3.12. *Given two points $x, y \in X$, any two arcs connecting x to y, and lying inside $B_x(r)$, are homotopic.*

Proof. Equivalently we can prove that, if $r < \mathrm{sys}(X)/2$, then for any $x \in X$, any loop based at x, which is entirely contained in the ball $B_x(r)$ is homotopically trivial in X. Note that this statement is non-trivial, because the ball $B_x(r)$ is not necessarily contractible. Let us assume by contradiction that there exist loops based in x and fully contained in $B_x(r)$ which are non homotopically trivial in X. By definition of systole, their length is at least $\mathrm{sys}(X)$.

Let $\gamma\colon [0,1] \to X$ be one of such loops, parametrized with constant speed. Since X is a geodesic metric space, there is a geodesic segment α connecting $x = \gamma(0)$ to $\gamma(\frac{1}{2})$ and realizing their distance. Since α is length minimizing, and $\gamma(\frac{1}{2}) \in B_x(r)$, its length is less than r and it then is also entirely contained in $B_x(r)$. Consider then the two loops given by the concatenations $\gamma|_{[0,\frac{1}{2}]} * \overline{\alpha}$ and $\alpha * \gamma|_{[\frac{1}{2},1]}$, where $\overline{\alpha}$ is the arc α run in the opposite orientation. At least one of them has to be also non homotopically trivial, as their concatenation yields γ which is non-trivial. Let us call such a loop γ'. We observe that since $\mathrm{Len}(\gamma) \geq \mathrm{sys}(X)$

$$\mathrm{Len}(\gamma') = \mathrm{Len}(\gamma) - \mathrm{Len}(\gamma)/2 + \mathrm{Len}(\alpha) < \mathrm{Len}(\gamma) - \mathrm{sys}(X)/2 + r$$

But then for any non homotopically trivial loop γ in $B_x(r)$ we could find another such a loop γ', shorter than γ by at least $\mathrm{sys}(X)/2 - r$. This gives a contradiction with the fact that their length must always be more than $\mathrm{sys}(X)$.

□

We will now use this fact repeatedly in the next statements to show that marked Gromov-Hausdorff convergence implies convergence of the marked length spectrum. This result will be useful in Chapters 9, 10.

Lemma 3.13. *If $X_n \to X$ in $\mathrm{MMet}(S)$, then there exists a positive uniform lower bound for the length of the systole of X_n.*

Proof. Let us take $\gamma_n\colon [0,1] \to X_n$ the systole of X_n. Note that γ_n lies in the ball of radius $\mathrm{sys}(X_n)/2$ around $\gamma_n(0)$. If there was a point $\gamma_n(t)$ further away from $\gamma_n(0)$, then we would have $\mathrm{sys}(X_n) = \mathrm{Len}(\gamma_n) \geq 2d_{X_n}(\gamma_n(0), \gamma_n(t)) > 2\,\mathrm{sys}(X_n)/2$.

Call $\varphi_n\colon X_n \to X$ the ϵ_n-isometries realizing the marked Gromov-Hausdorff convergence. Since φ_n is an ϵ_n-isometry, then $\varphi_n(\gamma_n)$ lies inside the ball of radius $r = \mathrm{sys}(X_n)/2 + \epsilon_n$ around $\varphi_n(\gamma_n(0))$. But φ_n is a homotopy equivalence, so the curve $\varphi_n(\gamma_n)$ is not homotopically trivial, as γ_n is not. If we had $r < \mathrm{sys}(X)/2$, then by Lemma 3.12 we would reach a contradiction. So we can conclude that

$$\mathrm{sys}(X_n)/2 + \epsilon_n = r \geq \mathrm{sys}(X)/2, \tag{3.1}$$

from which for n large enough, so that $\epsilon_n < \mathrm{sys}(X)/2$, we obtain a positive uniform lower bound for $\mathrm{sys}(X_n)$. □

We can now finally prove that the marked Gromov-Hausdorff convergence implies convergence of the marked length spectrum.

Proposition 3.14. *If $X_n \to X$ in $\mathrm{MMet}(S)$, then for every closed curve γ we have that $\ell_{X_n}(\gamma) \to \ell_X(\gamma)$.*

Proof. By definition of marked Gromov-Hausdorff convergence there exist $\varphi_n\colon X_n \to X$ homotopy equivalences that are ϵ_n-isometries with $\epsilon_n \to 0$ for $n \to \infty$ and compatible with the markings. By Lemma 3.13, there exists a uniform constant r smaller than $\mathrm{sys}(X)/2$ and of $\mathrm{sys}(X_n)/2$ for every n.

Take γ any homotopy class of closed curves and $\overline{\gamma}$ a length minimizing geodesic representative in X. Take N an integer such that

$$\frac{2\ell_X(\gamma)}{r} < N \leq \frac{4\ell_X(\gamma)}{r}.$$

There exists such an integer because $r < \mathrm{sys}(X)/2 \leq \ell_X(\gamma)/2$ by our assumption and so $2\,\mathrm{Len}(\overline{\gamma})/r \geq 4$. Then we can subdivide $\overline{\gamma}$ in N geodesic segments $\{\overline{\gamma}|_{[t_i,t_{i+1}]}\}_{i=1}^N$, each of length $\ell_X(\gamma)/N < r/2$. We notice that each of these segments realizes the distance between their endpoints $\overline{\gamma}(t_i), \overline{\gamma}(t_{i+1})$. Indeed, if that was not the case, then the length minimizing arc between the endpoints would be shorter, and in particular contained in the ball $B_{\overline{\gamma}(t_i)}(r/2)$ and thus, by Lemma 3.12, homotopic to $\overline{\gamma}|_{[t_i,t_{i+1}]}$, against the hypothesis of $\overline{\gamma}$ being length minimizing in its homotopy class.

Fixed n, let us now take points $x_i \in \varphi_n^{-1}(\overline{\gamma}(t_i))$. Since φ_n is an ϵ_n-isometry

$$d_{X_n}(x_i, x_{i+1}) \leq d_X(\overline{\gamma}(t_i), \overline{\gamma}(t_{i+1})) + \epsilon_n \leq r/2 + \epsilon_n.$$

Then let us join all the pairs x_i, x_{i+1} with the shortest geodesic segment connecting them, and form in this way a closed piecewise geodesic curve γ_n in X_n. The curve γ_n is in the same homotopy class of γ, indeed we can show that its image $\varphi_n(\gamma_n)$ is homotopic to $\overline{\gamma}$. The segment of γ_n between x_i and x_{i+1} lies in the ball $B_{x_i}(r/2+\epsilon_n)$, so its image under φ_n will lie in $B_{\overline{\gamma}(t_i)}(r/2+2\epsilon_n)$, so provided that n is large enough such that $r/2+2\epsilon_n < r$, again by Lemma 3.12, it is homotopic to $\overline{\gamma}|_{[t_i,t_{i+1}]}$. This holds for every i, and thus provides a homotopy between $\varphi_n(\gamma_n)$ and $\overline{\gamma}$.

Then we have the following.

$$\begin{aligned}
\ell_{X_n}(\gamma) \leq \mathrm{Len}(\gamma_n) &= \sum_{i=1}^{N} d_{X_n}(x_i, x_{i+1}) \\
&\leq \sum_{i=1}^{N} d_X(\overline{\gamma}(t_i), \overline{\gamma}(t_{i+1})) + N\epsilon_n \\
&= \mathrm{Len}(\overline{\gamma}) + N\epsilon_n \\
&= \ell_X(\gamma) + N\epsilon_n \\
&\leq \ell_X(\gamma) + \frac{4\ell_X(\gamma)}{r}\epsilon_n \\
&= \ell_X(\gamma)\left(1 + \frac{4\epsilon_n}{r}\right)
\end{aligned}$$

From which, taking the lim sup, we obtain $\limsup_n \ell_{X_n}(\gamma) \leq \ell_X(\gamma)$. As a consequence we also have that for n large enough $\ell_{X_n}(\gamma) \leq 2\ell_X(\gamma)$

Let us call $\overline{\gamma}_n$ the geodesic representative of γ in X_n. Take now N' integer such that

$$\frac{4\ell_X(\gamma)}{r} < N' \leq \frac{8\ell_X(\gamma)}{r}$$

Then we can subdivide $\overline{\gamma}_n$ in the concatenation of N' geodesic segments $\{\overline{\gamma}_n|_{[t_i,t_{i+1}]}\}_{i=1}^{N'}$ of length

$$\frac{\mathrm{Len}(\overline{\gamma}_n)}{N'} = \frac{\ell_{X_n}(\gamma)}{N'} \leq \frac{2\ell_X(\gamma)}{N'} < r/2.$$

Once again by Lemma 3.12, each of such segments $\overline{\gamma}_n|_{[t_i,t_{i+1}]}$ is realizing the distance between its endpoints $\overline{\gamma}_n(t_i), \overline{\gamma}_n(t_{i+1})$, and hence lies in the closed ball of radius $r/2$ centred in $\overline{\gamma}_n(t_i)$. The image $\varphi_n(\overline{\gamma}_n)$ is a curve in X in the homotopy class of γ, because φ_n preserves markings. Let us now take $\overline{\gamma}'_n$ in X the curve obtained as concatenation of the shortest geodesic segments connecting the endpoints $\varphi_n(\overline{\gamma}_n(t_i))$ and $\varphi_n(\overline{\gamma}_n(t_{i+1}))$. We now show that $\overline{\gamma}'_n$ is homotopic to $\varphi(\overline{\gamma}_n)$. Since φ_n is an ϵ_n-isometry, the arc $\varphi_n(\overline{\gamma}_n|_{[t_i,t_{i+1}]})$

is contained in the ball of radius $r/2+\epsilon_n$ centred in $\varphi_n(\overline{\gamma}_n(t_i))$, and, for n large enough such that $\epsilon_n < r/2$, by Lemma 3.12 it is homotopic to $\varphi(\gamma|_{[t_i,t_i+1]})$. This holds for every i, and thus gives a homotopy between $\varphi(\overline{\gamma}_n)$ and $\overline{\gamma}'_n$. We then have the following bound.

$$\begin{aligned}\ell_X(\gamma) \leq \mathrm{Len}(\overline{\gamma}'_n) &= \sum_{i=1}^{N} d_X(\varphi_n(\overline{\gamma}_n(t_i)), \varphi_n(\overline{\gamma}_n(t_{i+1}))) \\ &\leq \sum_{i=1}^{N} d_{X_n}(\overline{\gamma}_n(t_i), \overline{\gamma}_n(t_{i+1})) + N\epsilon_n \\ &= \mathrm{Len}(\overline{\gamma}_n) + N\epsilon_n \\ &= \ell_{X_n}(\gamma) + N\epsilon_n \\ &\leq \ell_{X_n}(\gamma) + \frac{8\ell_X(\gamma)}{r}\epsilon_n\end{aligned}$$

In particular we get

$$\left(1 - \frac{8\epsilon_n}{r}\right)\ell_X(\gamma) \leq \ell_{X_n}(\gamma),$$

where taking the lim inf yields $\ell_X(\gamma) \leq \liminf_n \ell_{X_n}(\gamma)$. □

Notice that in the previous proof it was enough to choose $r < \mathrm{sys}(X)/2$, in particular setting $r = \mathrm{sys}(X)/4$ the previous proof gives the following quantitative convergence result.

Corollary 3.15. *If $X, Y \in \mathrm{MMet}(S)$ and there is an ϵ-isometry which is a homotopy equivalence between them, with $\epsilon < \mathrm{sys}(X)/4$, then*

$$L^{-1}\ell_X(\gamma) \leq \ell_{X_n}(\gamma) \leq L\ell_X(\gamma)$$

with $L = 1 + \frac{4\epsilon}{r} = 1 + \frac{16}{\mathrm{sys}(X)}\epsilon$.

Notice that the constants are not optimal, and the numbers appearing are only by-product of our choices: for convenience whenever we could choose any constant greater than 1, we chose 2.

4 Bordification of $\mathcal{PT}(S)$

In this chapter we present the main result of the thesis and so define a bordification of the space of marked grafted surfaces, where the points of the boundary are identified with marked half-translation surfaces. Let us first define properly what we mean by bordification.

Definition 4.1. Given a topological space A, the space B together with the map $A \to B$ is called a *bordification* if the map is an embedding, and the image of A is dense in B.

Essentially by seeing A as included in B, we have that B adds some points to the space A, that become limit of some specific sequences that in A were diverging. The set of the additional points is indicated with $\partial A = \overline{A} \setminus A = B \setminus A$ and is referred to as boundary or boundary at infinity.

Definition 4.2. A *compactification* C together with a map $A \to C$ is defined as a bordification of A where, in addition, the space C is compact.

In this case every diverging sequence in A will have, up to passing to a subsequence, a limit in C; while in the case of a bordification B only some special diverging sequences of A will have a limit in B.

A standard technique to obtain a bordification of a moduli space is embedding it in a much larger ambient space and taking the closure of its image there. Note that by embedding we always mean only topological embedding, as the target space $\mathrm{MMet}(S)$ does not have any smooth structure. Our strategy will be the following. We consider the natural map from $\mathcal{PT}(S)$ into $\mathrm{MMet}(S)$, given by considering the Thurston metric and show it is a proper embedding; then we compose it with the projection to the projective space $\mathbb{P}\,\mathrm{MMet}(S)$, and show that $\mathcal{PT}(S)$ embeds there too; finally we take the closure of its image there and show that it coincides with the addition of the metrics coming from half-translation surfaces.

4.1 Embeddings into the space of marked metrics

The Thurston metric for grafted surfaces and the singular Euclidean metrics coming from quadratic differentials, are families of rather regular metrics, compared to the generic metric in MMet(S). The curvature is not defined everywhere on them, as they are not smooth. Nevertheless, they still exhibit typical properties of non-positively curved spaces. Indeed, a weaker notion of non-positive curvature, is satisfied by grafted surfaces and the singular flat metrics of half-translation structures.

Definition 4.3. A geodesic metric space (X, d) is said to be CAT(0) if for any geodesic triangle and for any pair of points p and q on two sides of the triangle, the distance $d(p, q)$ is less than or equal to the distance between the corresponding points $\overline{p}$ and $\overline{q}$ on the comparison triangle in Euclidean space $\mathbb{R}^2$ with the same side lengths.

The definition essentially says that the triangles in a CAT(0) space are thinner then the corresponding ones, with sides of the same length, in the Euclidean plane. This property is global and does not allow the space to have closed geodesics. The local version of the CAT(0) property is instead a good generalization of non-positive curvature.

Definition 4.4. A metric space (X, d) is said to be locally CAT(0) if every point $x \in X$ has a neighbourhood U such that $(U, d|_U)$ is a CAT(0) space with the induced metric.

Let us then introduce the following subspace of MMet(S). We will show that grafted and half-translation surfaces lie in this subspace, where the marked Gromov-Hausdorff topology is stronger.

Definition 4.5. Let us call NPC(S) $\subset$ MMet(S) the subspace formed by the pairs $(X, [f])$ where X is a locally CAT(0) space.

Proposition 4.6. *Grafted surfaces with the Thurston metric and half-translation surfaces with the singular Euclidean metric are locally CAT(0).*

Proof. According to [BH, Theorem II.11.1], gluing locally CAT(0) spaces along a geodesic boundary yields still a CAT(0) space. So, half-translation surfaces are trivially locally CAT(0) away from the singularities, while the neighbourhood of a cone singularity of angle $k\pi$ is modelled on the gluing of k half planes, thus by the theorem above is locally CAT(0) as well.

Grafted surfaces, where the grafting lamination is a multicurve, are locally isometric to either $\mathbb{R}^2$, $\mathbb{H}^2$ or to a half Euclidean plane glued to a half hyperbolic plane, which again by the theorem above is locally CAT(0). For the general case we will use a density argument.

By [BH, Theorem II.3.10], the local CAT(0) property passes through Gromov-Hausdorff limit for complete metric spaces. This implies that NPC(S) is a closed subspace of MMet(S). Then we conclude as the subset of surfaces grafted along weighted multicurves is dense in $\mathcal{PT}(S)$ and, by Theorem 2.12, the Thurston metric depends continuously on the point in $\mathcal{PT}(S)$. □

Restricting ourselves to the subspace NPC(S), makes the Gromov-Hausdorff convergence much stronger. In [Nag02] it is shown that being locally CAT(0) (or in general CAT(k)), under mild non degeneracy conditions, allows to promote the Gromov-Hausdorff convergence to a version of Lipschitz convergence. In brief, generally for two spaces to have small Gromov-Hausdorff distance means that there is an ϵ-isometry between them. Nagano showed in [Nag02] that such a map can also be assumed to be $(1+\epsilon')$-bilipschitz away from a small neighbourhood.

In particular, with this, they deduce the following result of volume convergence. We rewrite here their main theorem, which is way more general, translating it in our far more restricted setting. For this translation we just need to observe that their assumption on the lower bound of what they call CAT(0)-radius is implied by a uniform lower bound on the systole of the surfaces, which we obtain from the convergence in Met(S), and them being locally CAT(0).

Theorem 4.7 ([Nag02, Theorem 1.1]). *If a sequence of metric surfaces $(X_n)_n$ in* NPC(S) *converges in the Gromov-Hausdorff sense to $X \in$ NPC(S), then the area, meant as 2-dimensional Hausdorff measure, of X_n converges to the area of X.*

In other words, the area functional Area: NPC(S) $\to \mathbb{R}^+$ is continuous.

Remark . There exists in literature a notion of *equivariant Gromov-Hausdorff convergence*, introduced by Paulin in [Pau88]. It is based on a convergence on compact subsets of the universal cover of the surface, compatibly with the action of the deck transformations; one could see it as a convergence of geometric actions of $\pi_1(S)$ on metric spaces.

The main difference with ours is that in the equivariant Gromov-Hausdorff convergence the metric space that is acted upon is not required to be the universal cover of a surface, but can degenerate. For example hyperbolic surfaces can degenerate into geometric actions on $\mathbb{R}$-trees (see [Bes88]). The advantage is that since all kind of possible degenerations are accounted for, such a convergence is suited for describing a compactification, while with our method will yield only a bordification. The advantage of avoiding degenerations is that the added points at infinity are still surfaces homeomorphic to S, in particular the action of the mapping class group will extend to a proper action on the bordification too. Indeed, we can show the following.

Proposition 4.8. *The action of* $\mathrm{Mod}(S)$ *on* $\mathrm{MMet}(S)$ *by pre-composition on the marking is continuous and proper.*

Sketch of proof. Let us notice that from the definition, the topology on $\mathrm{MMet}(S)$ is invariant under the action of $\mathrm{Mod}(S)$, which means that $\mathrm{Mod}(S)$ acts by homeomorphisms on $\mathrm{MMet}(S)$. The very same proof of the properness of the action of $\mathrm{Mod}(S)$ on Teichmüller space in [FM11, Section 12.3.3] adapts directly to our wider context.

The key observations to extend the argument are the following. We showed in Lemma 3.11 that for surfaces in $\mathrm{MMet}(S)$ it is true that for every $L > 0$, only finitely many curves have length less than L. Moreover, we showed that, provided ϵ_n small enough, thanks to Corollary 3.15, every point in $\mathrm{MMet}(S)$ have a neighbourhood, where we have a uniform Lipschitz control on all the length functions. □

Proposition 4.9. *The map* $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ *given by considering the Thurston metric is a* $\mathrm{Mod}(S)$*-equivariant proper embedding.*

Proof. The map is continuous, as by Theorem 2.12 the Thurston metric depends continuously on the projective structure. Uniform convergence of the Riemannian metric easily implies convergence in the marked Gromov-Hausdorff sense.

We now show that this is an injective map, as it is possible to recover the grafting data from the Thurston metric. Indeed, the interior of the hyperbolic part is given by the set of points of the grafted surface with a neighbourhood locally isometric to $\mathbb{H}^2$. The metric on such pieces and the relative position of such pieces is enough to determine the hyperbolic structure X (see Theorem 6.3). On the other side, the complement of the hyperbolic part will be a geodesic lamination λ, possibly with foliated cylinders in place of isolated leaves (see Section 6.5). The transverse measure is uniquely determined by taking arcs crossing orthogonally the lamination, and looking and the length deposited in the intersection with the lamination.

Thanks to Lemma A.3, in order to show that the continuous inclusion $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is an embedding, it is enough to show that the map is proper. The inclusion $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is $\mathrm{Mod}(S)$-equivariant and the action is proper on both space by Proposition 4.8 and because the homeomorphism $\mathcal{PT}(S) \cong \mathcal{T}(S) \times \mathcal{ML}_0(S)$ is $\mathrm{Mod}(S)$-equivariant and the action is proper on $\mathcal{T}(S)$. Then, by Lemma A.5, it is enough to show the properness of the map at the quotient $\mathcal{PM}(S) \to \mathrm{Met}(S)$.

We observe that the subset of $\mathcal{PM}(S)$ described by pairs (X, λ) where X is ϵ-thick and $\ell_X(\lambda)$ is bounded, is compact as it is a bundle over the ϵ-thick part of the moduli space, which is compact, and where every fibre is also compact. Let us consider Z_n a diverging sequence in $\mathcal{PM}(S)$. There are two cases: either $\ell_X(\lambda)$ is going to infinity, thus, by Proposition 4.12, also the area of the grafted surface is going to infinity; or $\ell_X(\lambda)$ is bounded and the metric X has systole going to 0, which, by the collar lemma, implies that the diameter is going to infinity. Observe that the diameter is a continuous function on $\mathrm{Met}(S)$, and also the area, by Theorem 4.7, is a continuous function on the image of $\mathcal{PM}(S)$ in $\mathrm{Met}(S)$. Hence, the sequence Z_n maps to a diverging sequence in $\mathrm{Met}(S)$. So the inclusion $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is a proper $\mathrm{Mod}(S)$-equivariant embedding. $\square$

Let us now consider half-translation surfaces. We remind that half-translation structures are encoded by quadratic differentials, so their moduli space is identified with the bundle $\mathcal{QT}(S)$ of quadratic differentials over Teichmüller space. Considering the singular Euclidean metric associated to every $q \in \mathcal{QT}(S)$ induces a map $\mathcal{QT}(S) \to \mathrm{MMet}(S)$.

Remark . Observe that this map is not injective as the pair of metric surface with marking associated to q is unique only up to the S^1 action by multiplying q with unit complex numbers, corresponding to a rotation of the half-translation structure. Indeed, the metric is enough to determine the period coordinates of q, only up to a rotation factor.

Proposition 4.10. *The natural map induced on the quotient* $\mathcal{QT}(S)/S^1 \to \mathrm{MMet}(S)$ *is a* $\mathrm{Mod}(S)$*-equivariant proper embedding.*

Proof. The map $\mathcal{QT}(S)/S^1 \to \mathrm{MMet}(S)$ is continuous as the singular flat metric depends continuously on q, and is also injective as we noticed in the previous remark. We want now to show that its inverse is also continuous, showing that period coordinates depend continuously on the metric.

Assume that the sequence q_n converges to q in $\mathrm{MMet}(S)$. Then by definition of marked Gromov-Hausdorff convergence, there are ϵ_n-isometries $f_n\colon q_n \to q$. We show that eventually f_n maps singularities close to singularities. Let $r > 0$ be smaller than the length of the shortest saddle connection in q, and smaller than $\mathrm{sys}(q)/2$. Notice that in q_n, around every singularity p we can find a completely degenerate equilateral triangle of sides of length r: one just needs to take as vertices x, y, z, three points that are connected to p via segments of length $r/2$ and that form with each other angles of at least π. Up to passing to a subsequence, being q compact, we can assume that the images of via f_n of all such points converge to a quadruple $\overline{p}, \overline{x}, \overline{y}, \overline{z}$, and being f_n an ϵ_n-isometry, with ϵ_n going to zero, such points must have the same distance patterns as p, x, y, z in q_n. By the choice of r small enough, this implies that the shortest geodesic segment between any two of the vertices $\overline{x}, \overline{y}, \overline{z}$ must contain $\overline{p}$, so they form again a degenerate equilateral triangle, hence the centre $\overline{p}$ must be a cone singularity for q. This shows that, for n large enough, every singularity has to be mapped by f_n close to a singularity, and being f_n an ϵ_n-isometry with ϵ_n going to zero, this implies that the length of all the saddle connections of q_n must also converge to the ones of q, and so $q_n \to q$ in $\mathcal{QT}(S)$. Therefore, the map is an embedding.

It is proper because a diverging sequence of quadratic differentials has area going to infinity or systole going to zero (see [MT02, Proposition 3.6]), and we showed that the systole is a continuous positive function on $\mathrm{Met}(S)$, and the area converges for converging sequences in $\mathrm{NPC}(S)$. □

Corollary 4.11. *By mapping class group equivariance, we obtain that also the maps* $\mathcal{PM}(S) \to \mathrm{Met}(S)$ *and* $\mathcal{QM}(S)/S^1 \to \mathrm{Met}(S)$ *are also embeddings.*

4.2 Area normalization

The space $\mathrm{MMet}(S)$ naturally supports a free $\mathbb{R}^+$-action by rescaling the metric. So we can projectivise by taking the quotient by this action $\mathbb{P}\,\mathrm{MMet}(S) = \mathrm{MMet}(S)/\mathbb{R}^+$. Since the area is continuous on $\mathrm{NPC}(S)$, we can identify the subset $\mathbb{P}\,\mathrm{NPC}(S) \subset \mathbb{P}\,\mathrm{MMet}(S)$ with the subspace of $\mathrm{NPC}(S)$ made of surfaces with unit area. For grafted surfaces we can compute the area more precisely.

Proposition 4.12. *For any* $\mathrm{Gr}(X,\lambda) \in \mathcal{PT}(S)$ *we have that*

$$\mathrm{Area}(\mathrm{Gr}(X,\lambda)) = 2\pi|\chi(S)| + \ell_X(\lambda).$$

Proof. Both the area and the left-hand side are continuous on $\mathcal{PT}(S)$ so it is enough to show that the statement holds true for surfaces $\mathrm{Gr}(X,\mu)$ where $\mu = \sum_i a_i\gamma_i$ is a weighted multicurve.

In this case, the area of the hyperbolic part is the area of the hyperbolic surface X, which is $2\pi|\chi(S)|$ by Gauss-Bonnet theorem, while the flat part is made of a finite union of cylinders whose area is given exactly by the product of their height and circumference $a_i\ell_X(\gamma_i)$, which by linearity add up to $\ell_X(\mu)$. □

Remark . While we can compute the area of the hyperbolic part as the integral of the volume form induced by the Thurston metric on the grafted surface, because it is an open subset; the same does not hold for the grafted region. Indeed, in the generic case of a measured lamination with no closed leaves, the grafted region has empty interior, although the integral of the volume form on the hyperbolic part is strictly less than the integral on the whole grafted surface. A working notion of area is given instead by the volume measure induced by the volume form on the grafted surface. The mass of the grafted region with respect to such a measure is $\ell_X(\lambda)$, as it is the difference between the total area we computed before and the one of the hyperbolic part.

We now want to consider the composition of the embedding $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ with the projection to the quotient $\mathrm{MMet}(S) \to \mathbb{P}\,\mathrm{MMet}(S)$. As the image lands in $\mathbb{P}\,\mathrm{NPC}(S)$, and we said that this can be identified with the subspace of $\mathrm{NPC}(S)$ of the unit area surfaces, let us do the following. Let us consider from now on, each grafted surface $\mathrm{Gr}(X, \lambda)$ to be rescaled to have unit area, meaning that its Thurston metric tensor is rescaled by a multiplicative factor k^{-2} such that

$$k^2 = 2\pi|\chi(S)| + \ell_X(\lambda)$$

This means that instead of a hyperbolic part, $\mathrm{Gr}(X, \lambda)$ now has a part with constant negative curvature $-k^2$, of area

$$\frac{2\pi|\chi(S)|}{2\pi|\chi(S)| + \ell_X(\lambda)}$$

We notice that for $\ell_X(\lambda)$ going to infinity, and so k^2 too, the negatively curved part of $\mathrm{Gr}(X, \lambda)$ will have curvature going to $-\infty$ and area going to 0. Since the total area is fixed to be one by the renormalization, this means that, on the other side, the area of the flat grafted part, for $\ell_X(\lambda)$ going to infinity, will tend to 1. This suggests that the larger the grafting parameter $\ell_X(\lambda)$, the more the area-normalized grafted surface will look flat.

Proposition 4.13. *The composition* $\mathcal{PT}(S) \to \mathrm{MMet}(S) \to \mathbb{P}\,\mathrm{MMet}(S)$ *is also an embedding.*

Proof. For the consideration above about the area-normalization, the map is given by taking the Thurston metric of $\mathrm{Gr}(X, \lambda)$ and rescaling it by $k^{-2} = 2\pi|\chi(S)| + \ell_X(\lambda)$ as above. The map is clearly continuous and injective.

We now show that the map is proper onto its image. The strategy is the same that we used in the proof of Proposition 4.10. Take a sequence of grafted surfaces $\mathrm{Gr}(X_n, \lambda_n)$ whose area is diverging, so $k_n^2 = \ell_{X_n}(\lambda_n)$ is diverging, and suppose that their area-normalized metrics are converging in $\mathrm{MMet}(S)$ to a point $X \in \mathrm{MMet}(S)$. Then there exists r a uniform lower bound for $\mathrm{sys}(\mathrm{Gr}(X_n, \lambda_n))/2$. Observe that in every hyperbolic surface with geodesic boundary there exists an embedded ideal triangle, so in every area-normalized grafted surface $\mathrm{Gr}(X_n, \lambda_n)$ we can find an embedded ideal triangle of constant curvature $-k_n^2$. Inside this we can find a compact equilateral triangle, that is small enough so that it is isometrically embedded in the surface. To achieve this, thanks to Lemma 3.12, it is enough to choose it with sides of length r. Observe that since $-k_n^2$ is going to $-\infty$, then for n larger and larger, such a triangle is thinner and thinner, so the distance of its centre from the vertices will converge to half of the length of the sides. As in the proof of Proposition 4.10, up to subsequence, we can assume that the images under the ϵ_n-isometries between the area-normalized surfaces $\mathrm{Gr}(X_n, \lambda_n) \to X$ of such quadruples of points (vertices and centre) converge. Then they must converge to a quadruple of vertices and centre of a completely degenerate triangle, and this shows that X must have a cone singularity, in particular it is not in the image of $\mathcal{PT}(S)$.

□

On the other hand, by rescaling a quadratic differential q, the area of the associated half-translation surface is rescaled accordingly. In particular, the closed embedding $\mathcal{QT}(S)/S^1 \to \mathrm{MMet}(S)$ is equivariant with respect to the rescaling action by $\mathbb{R}^+$, so it descends to a closed embedding at the quotient, meaning from $\mathcal{QT}(S)/S^1/\mathbb{R}^+ \cong \mathcal{QT}(S)/\mathbb{C}^* = \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ to $\mathbb{P}\,\mathrm{MMet}(S)$.

Proposition 4.14. *The induced map* $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \to \mathbb{P}\,\mathrm{MMet}(S)$ *is a* $\mathrm{Mod}(S)$*-equivariant closed embedding.*

Our main result is then the following.

Theorem 4.15. *By identifying* $\mathcal{PT}(S)$ *and* $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ *with the images of their embeddings into* $\mathbb{P}\,\mathrm{MMet}(S)$*, we have that*

$$\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$$

This means that every marked unit-area half-translation surface is adherent to $\mathcal{PT}(S)$. In particular, there is a sequence of marked area-normalized grafted surfaces converging to it with respect to the marked Gromov-Hausdorff topology. Moreover, by $\mathrm{Mod}(S)$-equivariance, the same holds at the quotient, where we call $\mathbb{P}\,\mathrm{Met}(S)$ the projectivisation of $\mathrm{Met}(S)$.

Corollary 4.16. *By considering* $\mathcal{PM}(S)$ *and* $\mathbb{P}_{\mathbb{C}}\mathcal{QM}(S)$ *as subspaces of* $\mathbb{P}\,\mathrm{Met}(S)$*, we have that*

$$\overline{\mathcal{PM}(S)} = \mathcal{PM}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QM}(S)$$

5 Quantitative control on small grafting

We showed continuity of grafting, meaning in particular also that the grafted surface $\mathrm{Gr}(X, \lambda)$ converges in the sense of Gromov-Hausdorff to $\mathrm{Gr}(X, 0) = X$ as λ tends to the zero lamination. We show in this chapter a quantitative control on the convergence, in particular we show that it is uniform in the ϵ-thick part of Teichmüller space.

Proposition 5.1. *If $X \in \mathcal{T}_\epsilon(S)$ then there is a constant C_ϵ depending only on ϵ such that*

$$d_{GH}(\mathrm{Gr}(X, \lambda), X) \leq C_\epsilon \, \ell_X(\lambda).$$

Proof. Let $\kappa\colon \mathrm{Gr}(X, \lambda) \to X$ be the collapsing map introduced by Thurston (see Theorem 2.9). We will show that κ distorts the metric by at most an additive constant proportional to $\ell_X(\lambda)$. Then Lemma 3.3 will apply and conclude the proof. We know by Corollary 2.10 that κ is 1-Lipschitz, so we are left to show that the map κ does not shrink too much the distance between any two points.

Let x, y be points in $\mathrm{Gr}(X, \lambda)$. Let us consider their images $\kappa(x), \kappa(y)$ in X and the geodesic arc α joining them, realizing their distance on X. We remind that with a slight abuse of notation we denote with α both the parametrization and the image of the arc as subset of the surface. Our goal is to pick an arc β joining x, y, such that $\kappa(\beta) = \alpha$ and show that

$$\mathrm{Len}(\beta) \leq \mathrm{Len}(\alpha) + C_\epsilon \ell_X(\lambda) \tag{5.1}$$

for a suitable constant C_ϵ. Indeed, this will suffice to conclude that

$$d(\kappa(x), \kappa(y)) = \mathrm{Len}(\alpha) \geq \mathrm{Len}(\beta) - C_\epsilon \ell_X(\lambda) \geq d(x, y) - C_\epsilon \ell_X(\lambda)$$

We have to distinguish here two cases: either α is contained in a leaf of λ, or it is transverse to λ, possibly with endpoints on λ.

First case. If α is contained in a leaf of λ, then either the leaf has no atomic measure, in which case κ restricts to an isometry between $\kappa^{-1}(\alpha)$ and α, and we can just choose $\beta = \kappa^{-1}(\alpha)$; or α lies in a closed leaf γ of λ that is the image under κ of a flat cylinder in $\mathrm{Gr}(X, \lambda)$. We choose in this case β in $\kappa^{-1}(\alpha)$ joining x, y such that it is the concatenation of a geodesic segment β_1 parallel to γ and β_2 orthogonal to β_1. It is easy to observe that $\mathrm{Len}(\beta_1) = \mathrm{Len}(\alpha)$ while $\mathrm{Len}(\beta_2)$ is less than the height h of the cylinder. By additivity of the length function we have $h \cdot \ell_X(\gamma) \leq \ell_X(\lambda)$, and by assumption on the thickness of X, we also have that $\ell_X(\gamma) \geq \epsilon$, so

$$\mathrm{Len}(\beta_2) \leq h \leq \ell_X(\lambda)/\ell_X(\gamma) \leq \epsilon^{-1}\ell_X(\lambda).$$

This implies (5.1) with $C_\epsilon = \epsilon^{-1}$.

Second case. We will now show that, if α is transverse to λ, then $\kappa^{-1}(\alpha)$ is an arc, so we can choose $\beta = \kappa^{-1}(\alpha)$, and then we will also show that

$$\mathrm{Len}(\beta) = \mathrm{Len}(\alpha) + i(\alpha, \lambda), \tag{5.2}$$

where we remind that $i(\alpha, \lambda)$ is defined as the transverse measured deposited by λ on the transverse arc α. This, together with Lemma 5.2 will imply (5.1) as we wanted. Note that actually if an endpoint of α, say $\kappa(x)$, is on a closed leaf of λ, then $\kappa^{-1}(\alpha)$ might contain x in its interior instead of having it as an endpoint. But even in this case $d(x, y) \leq \mathrm{Len}(\beta)$, so the argument still holds.

Let us show that the preimage $\kappa^{-1}(\alpha)$ of the arc α is also an arc in $\mathrm{Gr}(X, \lambda)$. Let us remind that κ is a local homeomorphism everywhere in the domain except on flat cylinders, each coming from the grafting along an isolated leaf γ of λ with positive mass, which is hence a closed geodesic. An intersection point $\alpha(t_0)$ of α with such a closed curve γ has as preimage a straight segment in the corresponding cylinder, orthogonal to the boundary of it, that extends and joins the arcs $\kappa^{-1}(\alpha(t < t_0))$ and $\kappa^{-1}(\alpha(t > t_0))$. Note that this happens only on the intersection with on closed leaves of λ, which are finitely many and compact.

We now want to show that (5.2) holds true. We know that κ is a local isometry on the hyperbolic pieces of $\mathrm{Gr}(x, \lambda)$, so the arc $\kappa^{-1}(\alpha)$ is basically α with some additional pieces inserted along the grafting locus. It is clear to see that in the case when λ is a weighted multicurve $\lambda = \sum_i a_i\gamma_i$, the inserted grafted parts have lengths exactly the sum of the heights of the cylinders crossed by α (with multiplicity), that is $\sum_i a_i i(\alpha, \gamma_i) = i\left(\alpha, \sum a_i\gamma_i\right) = i(\alpha, \lambda)$. So equation (5.2) holds for weighted multicurves. The case for λ general measured lamination follows from the continuity of grafting thanks to Theorem 2.12, since weighted multicurves are dense in $\mathcal{ML}(S)$ and one can also argue (with an Arzelà-Ascoli argument) that once two points x, y are fixed on S, the arc α minimizing their length with respect to the metric $\mathrm{Gr}(X, \lambda)$ changes continuously with respect to the metric. □

We separated the statement of the following lemma from the previous proof, in that it is independent of grafting, and of independent interest.

Lemma 5.2. *Let $X \in \mathcal{T}_\epsilon(S)$ be an ϵ-thick hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ a measured lamination. If α is a geodesic arc in X transverse to λ, and it is distance minimizing between its endpoints, then*

$$i(\alpha, \lambda) \leq C_\epsilon \cdot \ell_X(\lambda)$$

where the constant C_ϵ depends only on ϵ.

Proof. Let us consider for every point $\alpha(t)$ in $\alpha \cap \lambda$, the geodesic segment U_t, which is an open neighbourhood of radius $\epsilon/4$ of $\alpha(t)$ along the leaf of λ containing $\alpha(t)$. Note that it is a segment, i.e. it does not form a closed curve since it would have length less than $\epsilon/2$, but we know by assumption that X is ϵ-thick. Moreover, all such segments U_t are disjoint. Suppose by contradiction that U_t intersects $U_{t'}$, with wlog $t < t'$. Then they lie on the same leaf of λ and so $\alpha(t)$ and $\alpha(t')$ will also lie on the same leaf of λ at distance strictly less than $\epsilon/2$. Since the injectivity radius in an ϵ-thick surface is $\epsilon/2$, the segment on λ joining $\alpha(t)$ and $\alpha(t')$ is length minimizing. But being α length minimizing, this implies that that segment lies in α, which contradicts the hypothesis α transverse to λ.

Consider now U the subset of λ formed by all the (disjoint) segments U_t we considered

$$U := \bigcup_{t \in \alpha^{-1}(\lambda)} U_t.$$

We remind that the length $\ell_X(\lambda)$ for a measured lamination λ can be seen as the total mass of the measure supported on λ given locally by the product $d\lambda \times d\ell$ where $d\lambda$ is the transverse measure associated to λ and $d\ell$ is the Lebesgue measure along the leaves induced by the hyperbolic metric X. Now since every segment U_t has by construction length ϵ, one can compute

$$\begin{aligned}
\int_U d\lambda \times d\ell &= \int_U d\lambda(t) \times d\ell_{U_t} \\
&= \int_\alpha d\lambda(t) \cdot \epsilon/2 \\
&= i(\alpha, \lambda) \cdot \epsilon/2
\end{aligned}$$

And since trivially $U \subset \lambda$, we also have

$$i(\alpha, \lambda) \cdot \epsilon/2 = \int_U d\lambda \times d\ell \leq \int_\lambda d\lambda \times d\ell = \ell_X(\lambda)$$

That will complete the proof with $C_\epsilon = 2\epsilon^{-1}$.

□

We chose to give a proof of the previous lemma that does not make use of grafting, since its statement does not mention grafting as well. Although, the proof above has a very geometric interpretation involving grafting.

Remark . From the proof above, consider the preimage U' of U through the collapsing map $\kappa\colon \mathrm{Gr}(X, \lambda) \to X$. Then U' is a subset of the grafted region. In particular, it is the subset of the grafted region, made of points that lies at distance $\epsilon/4$ along λ from the arc $\kappa^{-1}\alpha$. We remind that the transverse measure $i(\lambda, \alpha)$ gives exactly the total grafted height along all the leaves crossed by α. So one can easily compute the area of U as the product $i(\lambda, \alpha) \cdot \epsilon/2$. And by containment this must be less than the total grafted area, which is $\ell_X(\lambda)$ (see Proposition 4.12). Leading to the exact same computation as in the proof above.

We invite the reader to remember this interpretation as it will be the main idea to prove the complementary result Lemma 7.8 for the deflation of grafted surfaces.

6 The orthogeodesic foliation

In this chapter we proceed to introduce the orthogeodesic foliation: a measured foliation of a hyperbolic surface, transverse to a given measured lamination and whose construction depends on the hyperbolic metric of the surface. It was already used in [CDR10], but has been studied more deeply by Calderon and Farre in [CF24b] and [CF24a]. We will report some results about it, and then in the last subsection we perform our own construction of a map, called deflation, that we will use in the next chapter to compare grafted surfaces with half-translation ones.

6.1 Construction of the orthogeodesic foliation

Definition 6.1. Given $X \in \mathcal{T}(S)$ a hyperbolic surface and λ a measured lamination on it, the *orthogeodesic foliation* $\mathcal{O}_\lambda(X)$ is the piecewise smooth singular foliation where the leaves are the fibres of the closest point projection $p\colon X \to \lambda$. Moreover, $\mathcal{O}_\lambda(X)$ is endowed with a transverse measure induced by the hyperbolic length along the leaves of λ.

Let us realize first the same construction on the universal cover $\tilde{X} \cong \mathbb{H}^2$ of the hyperbolic surface X. Let us consider the lift $\tilde{\lambda}$ of λ in $\mathbb{H}^2$. It is a collection of disjoint geodesic lines. We observe that the nearest point projection p is not well-defined everywhere, since there are points equidistant from two or more leaves of $\tilde{\lambda}$. It is easy to observe that all such points form a geodesic spine, i.e. a graph where edges are geodesics and are also allowed to have only one endpoint. More precisely, observe that such a spine $\tilde{\mathrm{Sp}}$ has as vertices the points equidistant to at least three leaves of $\tilde{\lambda}$ and its edges are geodesic segments connecting two vertices or infinite geodesic rays emanating from one vertex. The whole construction is invariant under the covering automorphisms, so passing to the quotient it gives a spine $\mathrm{Sp} \subset X$.

Example . When the support of λ is a multicurve, Sp is a (finite) graph. Infinite edges are possible only when there is at least one region in the complement of λ with infinite diameter with respect to the induced path metric.

Example . When λ is maximal, so its complementary regions are all ideal triangles, Sp is a disjoint union of tripods: for every ideal triangle we have a vertex of Sp in the centre and one geodesic ray for each ideal vertex.

In the complement of $\tilde{\mathrm{Sp}}$ in $\mathbb{H}^2$ the closest point projection to λ is well-defined and smooth, then its fibres determine a smooth foliation of $\mathbb{H}^2 \setminus \tilde{\mathrm{Sp}}$. Such a foliation projects down to a foliation of $X \setminus \mathrm{Sp}$. There are a few verifications needed to check that this foliation matches nicely to give a foliation on the entire surface X and that the transverse measure is well-defined. The details are discussed in [CF24b]. We summarize the result in the following proposition.

Proposition 6.2 ([CF24b, Section 5.2]). *The foliation described on $X \setminus \mathrm{Sp}$ extends (after a small homotopy of $\mathbb{H}^2$ supported on a small neighbourhood of* Sp*) to a well-defined smooth measured foliation $\mathcal{O}_\lambda(X) \in \mathcal{MF}(\lambda)$ on X. Moreover, $\mathcal{O}_\lambda(X)$ meets λ orthogonally and the transverse measure it induces along the leaves of λ coincides with the hyperbolic length.*

Let us remind that with $\mathcal{MF}(\lambda)$ we mean the subspace of $\mathcal{MF}(S)$ formed by measured foliation that can be realized transversely to the measured foliation associated to λ (see Section 1.3).

Remark . Notice that, by construction, the orthogeodesic foliation $\mathcal{O}_\lambda(X)$ intersects λ orthogonally, but this does not imply a priori that the pair of foliations $(\mathcal{O}_\lambda(X), \lambda)$ can be realized transversely, meaning that $O_\lambda(X) \in \mathcal{MF}(\lambda)$. However, this is checked in [CF24b, Lemma 5.8].

6.2 Main properties of the orthogeodesic foliation

The orthogeodesic foliation determines completely the hyperbolic metric of the surface, indeed the following holds.

Theorem 6.3 ([CF24b, Theorem D]). *For any measured lamination λ, the map $\mathcal{O}_\lambda \colon \mathcal{T}(S) \to \mathcal{MF}(\lambda)$ is a* $\mathrm{Mod}(S)$*-equivariant homeomorphism.*

And in general by combining all the maps $\mathcal{O}_\lambda$ for all $\lambda \in \mathcal{ML}(S)$, one obtains the following mapping

$$\mathcal{O} \colon \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{QT}(S); \quad \mathcal{O}(X, \lambda) = q(\mathcal{O}_\lambda(X), \lambda).$$

Remark . Since $\mathcal{QT}(S)$ can be seen as the subset of $\mathcal{MF}(S) \times \mathcal{MF}(S)$ made of pairs of transverse foliations, thanks to the previous theorem and the natural correspondence between measured laminations and measured foliations, it follows immediately that $\mathcal{O}$ is a bijection. Moreover, due to its construction of geometric nature, $\mathcal{O}$ is mapping class group equivariant, meaning that for any $\varphi \in \mathrm{Mod}(S)$, one has $\mathcal{O}(\varphi_* X, \varphi_* \lambda) = q(\varphi_* \mathcal{O}_\lambda(X), \varphi_* \lambda) = \varphi_* q(\mathcal{O}_\lambda(X), \lambda)$.

We point out that the mapping is not a morphism of bundles, as the metric on the half-translation surface $q(\mathcal{O}_\lambda(X), \lambda)$ will not be conformal to X. The mapping $\mathcal{O}$ is not even continuous. Although not globally continuous, the same authors in [CF24a], showed that $\mathcal{O}$ has many large domains on which it is continuous, with continuous inverse.

Theorem 6.4 ([CF24a, Theorem A]). *Suppose that $\lambda_n \to \lambda$ in the measure topology, and the supports of λ_n also converge to the one of λ in the Hausdorff topology. Then $(X_n, \lambda_n) \to (X, \lambda)$ in $\mathcal{T}(S) \times \mathcal{ML}(S)$ if and only if $\mathcal{O}(X_n, \lambda_n) \to \mathcal{O}(X, \lambda)$ in $\mathcal{QT}(S)$.*

6.3 Crowned surfaces

Definition 6.5. A hyperbolic surface with *crowned boundary* is a complete, finite-area hyperbolic surface with totally geodesic boundary, with punctures on the boundary such that each boundary component is either a closed geodesic, if it does not have punctures, or a hyperbolic crown. This means that the segments between the punctures are infinite geodesics, and the two segments adjacent to a puncture are asymptotic and the puncture correspond to their common point at infinity.

Hyperbolic surfaces with crowned boundary, or shortly *crowned surfaces*, arise as the metric completion of connected components of the complement of a geodesic lamination on a hyperbolic surface. Theorem 6.3 tells us that after having fixed λ, the vertical foliation of $\mathcal{O}(X, \lambda)$ determines the hyperbolic structure X, so, in particular, it determines the metric on every component of $S \setminus \lambda$. A better description of the hyperbolic structure of each of those crowned surface is showed in [CF24b, Section 6]. Indeed, the proof of Theorem 6.3 consists in showing first that the orthogeodesic foliation determines the metric on each component of $X \setminus \lambda$, and then that it encodes also the shear between those, which means, roughly speaking, how the different components are put together.

If we look at a component A of $X \setminus \lambda$, the information of the hyperbolic metric kept by the orthogeodesic foliation A can be condensed to just the embedding of the spine Sp in A and the transverse measure induced by the measured foliation $\mathcal{O}_\lambda(X)$ on each of the edges of Sp. These data are enough to reconstruct the hyperbolic metric on A. See for example [CF24b, Section 6], [Mon09], [AC22].

Remark . Moreover, we also notice that A deformation retracts onto Sp, that is, A is a ribbon graph modelled on $\mathrm{Sp}(S)$. Hence, there is a correspondence between crowned surfaces with a hyperbolic metric and their metric ribbon graphs $\mathrm{Sp}(A)$.

This is stated more precisely in [AC22], but only for compact surfaces with geodesic boundary (non-crowned). An equivalent discussion in the general case of crowned surfaces is carried in [CF24b, Section 6], but there instead of metric ribbon graphs, a dual datum, given by arc systems, is used. The correspondence between the two points of view with metric ribbon graphs and weighted arc systems is discussed in [Mon09].

6.4 Deflation

The main construction we will use to prove our main result is what we call deflation map: a map between the grafted surface $\mathrm{Gr}(X, \lambda)$ and the half-translation surface $\mathcal{O}(X, \lambda)$ that allows us to relate their geometries. In [CF24b] they describe already a version of this map and show a property of it.

Proposition 6.6 ([CF24b, Proposition 5.10]). *Given a marked hyperbolic structure $(X, [f]) \in \mathcal{T}(S)$ and $\lambda \in \mathcal{ML}(S)$, let $(Z, [g]) = \pi(q(\mathcal{O}_\lambda(X), \lambda)) \in \mathcal{T}(S)$ be the marked complex structure on which $q(\mathcal{O}_\lambda(X), \lambda)$ is holomorphic. There is a map*

$$D \colon X \to Z$$

that is a homotopy equivalence restricting to an isometry between Sp *with its metric induced by integrating the edges against $\mathcal{O}_\lambda(X)$ and the graph of horizontal saddle connections of $q(\mathcal{O}_\lambda(X), \lambda)$ with the induced path metric. Moreover, $D \circ f \simeq g$ and $D_*\mathcal{O}_\lambda(X) = \mathrm{Re}(q)$ and $D_*\lambda$ is equivalent to* $\mathrm{Im}(q)$ *as measured foliations.*

They show that $D_*\lambda$ is equivalent, as measured foliation, to $\mathrm{Im}(q)$. Let us remind though, that for example if λ is a multicurve, it has a finite number of leaves, while $\mathrm{Im}(q)$ has uncountably many, so $D_*(\lambda)$ will only be equivalent to $\mathrm{Im}(q)$ via the laminations - foliation correspondence.

What we will do is to replicate the construction, but replacing X with the grafted surface $\mathrm{Gr}(X, \lambda)$ and λ with $\hat{\lambda}$, a specific lamination on $\mathrm{Gr}(X, \lambda)$ that is equivalent to λ. In this case it is possible to define a different map $\mathcal{D} \colon \mathrm{Gr}(X, \lambda) \to \mathcal{O}(X, \lambda)$, that we will still call deflation, that actually sends $\hat{\lambda}$ to $\mathrm{Im}(q)$, leaf by leaf. More precisely, we will show the following.

Proposition 6.7. *Given $(X,\lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$, let $q = \mathcal{O}(X,\lambda)$ be the associated half-translation surface. There is a homotopy equivalence*

$$\mathcal{D}\colon \operatorname{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$$

compatible with the markings of $\operatorname{Gr}(X,\lambda)$ *and of* $\mathcal{O}(X,\lambda)$*, such that* $\mathcal{D}_*\mathcal{O}_\lambda(X) = \operatorname{Re}(q)$ *and* $\mathcal{D}_*\hat{\lambda} = \operatorname{Im}(q)$*.*

6.5 Construction of the deflation

The idea for the construction of $\mathcal{O}(X,\lambda)$ and the deflation map $\mathcal{D}$ is in essence the same as in [CB88], where given a pair of transverse geodesic measured laminations on a hyperbolic surface, the half-translation surface associated to the pair is constructed. Here the starting surface will be a grafted surface $\operatorname{Gr}(X,\lambda)$ and the pair we consider is made of a measured foliation and a geodesic measured lamination, corresponding respectively to $O_\lambda(X)$ and to λ.

The orthogeodesic foliation on the grafted surface We constructed $\mathcal{O}_\lambda(X)$ as a foliation on X. This induces naturally an equivalent foliation on $\operatorname{Gr}(X,\lambda)$ just by taking the preimage of $\mathcal{O}_\lambda(X)$ via Thurston's collapsing map $\kappa\colon \operatorname{Gr}(X,\lambda) \to X$ and equip it with the pull-back measure. Indeed, since $O_\lambda(X)$ intersects λ orthogonally in X, one can see that its preimage under κ is still a smooth foliation, equivalent to $O_\lambda(X)$.

The measured lamination on the grafted surface The collapsing map κ allows us to pull-back λ to a measured lamination $\kappa^*\lambda$ on $\operatorname{Gr}(X,\lambda)$. We will now define $\hat{\lambda}$ a geodesic representative for $\kappa^*\lambda$, as, due to the piecewise flat metric, laminations do not have necessarily a unique representative which is geodesic with respect to the grafted metric.

For example if $\lambda = a\gamma$ a simple closed geodesic on X with mass $a > 0$, then $\operatorname{Gr}(X,\lambda)$ contains a flat cylinder of height a corresponding to the curve γ. We then choose $\hat{\lambda}$ to be the lamination foliating the cylinder by parallel copies of γ with a uniform transverse measure, that is the one given by the Euclidean metric of the cylinder along the direction orthogonal to the foliation. This naturally extends to any multicurve $\mu = \sum_i a_i\gamma_i$, by putting $\hat{\mu} = \sum_i \widehat{a_i\gamma_i}$.

In the general case λ can always be decomposed as $\lambda = \mu + \lambda'$ with μ a weighted multicurve and λ' a measured lamination with no closed leaves (see the decomposition into minimal components in [CEG06, Section 4.2]). Then we can just define $\hat{\lambda} = \hat{\mu} + \kappa^*\lambda'$, where $\kappa^*\lambda'$ is simply the preimage $k^{-1}(\lambda')$ with the pull-back measure, as κ is injective on $\kappa^{-1}(\lambda')$. For simplicity of notation, from now on, we will also call λ the above defined measured lamination $\hat{\lambda}$ when it's clear that we are referring to the lamination on the grafted surface.

Multicurve case We now have on the grafted surface $\mathrm{Gr}(X, \lambda)$ a measured lamination λ and a singular measured foliation $\mathcal{O}_\lambda(X)$ transverse to it. We want to construct $\mathcal{O}(X, \lambda) = q(\mathcal{O}_\lambda(X), \lambda)$ the half translation surface with the pair of vertical and horizontal foliation equivalent to $\mathcal{O}_\lambda(X)$ and λ. Let us consider now the easier case where λ is a weighted multicurve on X.

In this case $\mathrm{Gr}(X, \lambda)$ is just a gluing of finitely many cylinders and compact hyperbolic surfaces with geodesic boundary. The lamination λ and the foliation $\mathcal{O}_\lambda(X)$, restricted to each cylinder, give a pair of transverse foliations, geodesic with respect to the Euclidean metric of the cylinder, the first one parallel to the boundary and the second one orthogonal to it. The interior of each hyperbolic piece is instead only foliated by $\mathcal{O}_\lambda(X)$ and does not contain any leaf of λ. What we will do is completely collapsing each hyperbolic piece along the orthogeodesic foliation. More precisely this means taking the quotient by the following equivalence relation. Two points in the same hyperbolic piece are equivalent if they lie in the same (possibly singular) leaf of $\mathcal{O}_\lambda(X)$ restricted to the hyperbolic piece.

Proposition 6.8. *The quotient above yields a half-translation surface obtained by gluing the cylinders along maps between their boundaries which are piecewise isometries.*

Proof. Let us focus on a single hyperbolic piece A, and let us call for easiness of notation η the orthogeodesic foliation restricted to A. The foliation η has a finite number of singularities, from which finitely many singular leaves emanate. By the definition of orthogeodesic foliation, each of those singular leaves will be a geodesic segment between a singularity and the boundary of A, and orthogonal to it. Let us call Γ the union of all the singular leaves. The boundary ∂A is cut by the endpoints of Γ in finitely many segments that we will call a_i.

We now observe that each connected component of $A \backslash \Gamma$ is foliated by a family of regular leaves of η, that, by definition of orthogeodesic foliation, are arcs with the two endpoints on ∂A. This implies that each of those connected components is, like in Figure 6.1 a band connecting two segments a_i, a_j on ∂A. We also observe that it is made of two isometric pieces separated by the spine: the reflection along the spine induces a symmetry of the band, as by definition the segment of spine in the band is the locus of points equidistant from a_i and a_j. In particular, the two segments a_i, a_j have the same hyperbolic length.

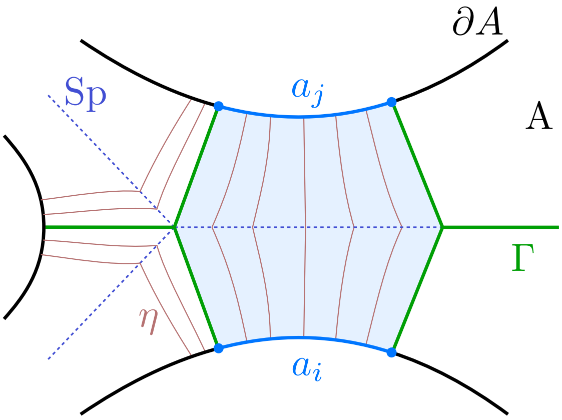

Figure 6.1: A band foliated by η, encoding the gluing between a_i and a_j.

Indeed, by definition, the transverse measure of the orthogeodesic foliation η induces the hyperbolic length along ∂A, so the pairing between a_i and a_j given by the leaves of η gives an isometry between the two segments.

Let us now consider singular leaves. For every singularity, collapsing the k-pronged star of singular leaves through it, is equivalent to removing it and gluing its k endpoints on ∂A together. But this is equivalent to considering, in the gluing described above, the segments a_i to be closed, that is with endpoint included.

In conclusion the quotient of $\mathrm{Gr}(X, \lambda)$ obtained by collapsing the leaves of η is equivalent to removing the interior of the hyperbolic piece A and then gluing its boundary (which lies on the boundary of the cylinders adjacent to the hyperbolic piece A) along isometries between the segments $(a_i)_i$ as we described.

By repeating this construction for each of the finitely many hyperbolic pieces, we are left with just a collection of cylinders with their boundary components glued by piecewise isometries. The resulting surface q is a half-translation surface. Indeed, every cylinder can be realized as a quotient of a rectangle in $\mathbb{R}^2$ with vertical sides identified. Then the gluing above is realized by piecewise isometries between their horizontal sides. □

The deflation map $\mathcal{D}\colon \mathrm{Gr}(X, \lambda) \to \mathcal{O}(X, \lambda)$ in this case is just the map collapsing the hyperbolic pieces of the grafted surface along the orthogeodesic foliation, essentially collapsing them onto their spine. By construction of $\mathcal{O}_\lambda(X)$ and λ on the grafted surface, $\mathcal{D}$ maps them to the pair of vertical and horizontal foliations on $\mathcal{O}(X, \lambda)$. And its restriction to the spine $\mathcal{D}|_{\mathrm{Sp}}$ is a homeomorphism to the gluing locus between the cylinders, that is to the diagram of horizontal separatrices of $\mathcal{O}(X, \lambda)$.

Geometric train track and rectangle decomposition For the general case we will essentially follow along the lines the proof of [CF24b, Proposition 5.10], adding more details in order to better understand the additional metric properties of our version of the deflation map $\mathcal{D}$. The construction is based on Thurston's geometric train track construction (also explained in [CF24b, Construction 5.6]) and uses a rectangle decomposition introduced in [CDR10]. In this last work, like in this thesis, the starting surface is the grafted one, however the map produced is different as what they seek to obtain is a quasi-conformal map, while we are interested in obtaining an ϵ-isometry.

Let us consider $\delta > 0$ small enough so that the metric δ-neighbourhood $N_\delta(\lambda)$ of λ in $\mathrm{Gr}(X, \lambda)$ is topologically stable. The orthogeodesic singular foliation restricts to a regular foliation η on the subset $N_\delta(\lambda)$ of $\mathrm{Gr}(X, \lambda)$. The boundary of $N_\delta(\lambda)$ is not geodesic, but it is by construction a level set of the distance function from a leaf of λ, and thus orthogonal to the leaves of the orthogeodesic foliation as they realize the paths of minimum distance to λ. Hence, $N_\delta(\lambda)$ is also called *geometric train track* for λ and the restriction of the leaves of $\mathcal{O}_\lambda(X)$ to it are called *ties* of the train track. The geometric train track $N_\delta(\lambda)$ deformation retracts to a train track $\tau \subset N_\delta(\lambda)$ by collapsing each tie. We also observe that, by construction, τ snugly carries λ, meaning that the transverse measure induced by λ on each branch of τ is strictly positive.

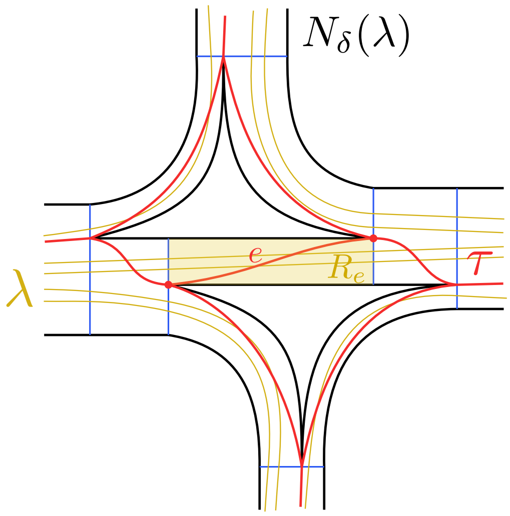

Figure 6.2: Example of geometric train track $N_\delta(\lambda)$ with its decomposition in rectangles $R(e)$'s.

We remind that τ has finite combinatorics, meaning a finite amount of switches and branches. To each branch e of τ corresponds a region $R(e)$ of $N_\delta(\lambda)$, which we may think as a rectangle (although only the sides along the ties are geodesics), which is the union of ties crossing e. In this way we divided $N_\delta(\lambda)$ into rectangles, which are pairwise either disjoint or adjacent along a tie through a switch of τ.

We partitioned the grafted surface $\mathrm{Gr}(X,\lambda)$ in finitely many rectangles inside $N_\delta(\lambda)$ and finitely many connected regions in the complement of $N_\delta(\lambda)$. We will now explicitly construct the half-translation surface $\mathcal{O}(X,\lambda)$ and the deflation map $\mathcal{D}\colon \mathrm{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$.

The deflation map locally Let us first consider one rectangle $R(e)$. As it is a union of ties, we can define its length l as the total transverse measure induced by the orthogeodesic foliation, and its height h as the total mass given by the transverse measure of λ to any of the ties in the interior of $R(e)$. The height is well-defined as all such ties are transverse to λ and transversely homotopic one to the other.

Let us now consider a Euclidean rectangle $Q(e)$ with same length l and height h. We can map $R(e)$ to it, such that the transverse measures with respect to λ and $\mathcal{O}_\lambda(X)$ are mapped to the vertical and horizontal coordinates in the rectangle. More precisely we define the map

$$f_e\colon R(e) \to Q(e) = [0,l] \times [0,h]$$

as follows. We choose one of the two sides of $R(e)$ along the ties, call it a. The orientation of the surface together with the choice of a induces an orientation for the ties. Then we define f_e as in Figure 6.3.

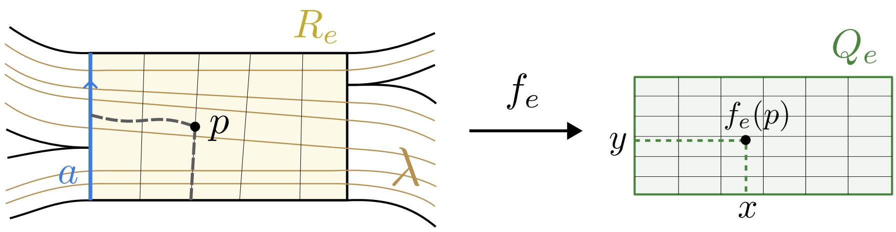

Figure 6.3: Any point p in $R(e)$ is mapped to the point (x,y) in $Q(e)$, where x is the transverse measure with respect to the orthogeodesic foliation given to an arc joining p to a and transverse to the ties; and y is the transverse measure given by λ to the subsegment of the tie going from the boundary of $R(e)$ to p following the tie in the positive direction.

We observe that the map f_e is surjective as both $\mathcal{O}_\lambda(X)$ and λ no not have atomic leaves: the first one by definition, the second one thanks to our choice of the representative

$\hat{\lambda}$ for λ in $\mathrm{Gr}(X, \lambda)$. We also notice that by invariance of the transverse measure upon transverse homotopy, each tie is mapped to a vertical segment in $Q(e) = [0, l] \times [0, h]$ and similarly each leaf of λ is mapped to a horizontal segment. In other words, $\mathcal{O}_\lambda(X)$ and λ are mapped to the vertical and horizontal measured foliation of the Euclidean rectangle.

Remark . This is possible as a measured geodesic lamination, is either a multicurve or has uncountably many leaves. In our case, since we replaced each atomic leaf with a foliated cylinder, λ is such that in every rectangle $R(e)$ it has uncountably many leaves.

We also observe that f_e is not in general a homeomorphism as it is not necessarily injective. Indeed, if λ is not a foliation in $R(e)$, then the restriction of f_e to a tie is constant on each subsegment disjoint from λ. In particular, along a tie the map f_e behaves very similarly to a Cantor function (see Appendix A.2).

Remark . We notice, though, that when λ has a closed leaf with positive mass, so $\mathrm{Gr}(X, \lambda)$ has an inserted cylinder, the map f_e restricted to the intersection between R_e and the cylinder is an isometry, as λ and $\mathcal{O}_\lambda(X)$ already form a pair of orthogonal foliations on the cylinder. In particular, this shows that this construction extends the one we described above in the special case of λ multicurve.

Gluing the rectangles Let us observe that the rectangles $R(e)$ are adjacent to each other along ties corresponding to switches of τ. Let us consider the collection of Euclidean rectangles $Q(e)$ for all branches e of τ and glue them along their vertical sides following the pattern induced by the adjacency of the respective rectangles $R(e)$. This means that for a segment a of the tie separating $R(e)$ from $R(e')$, we glue $Q(e)$ to $Q(e')$ by attaching $f_e(a)$ to $f_{e'}(a)$ so that $f_e, f_{e'}$ patch together to form a continuous map from $R(e) \cup R(e')$ to $Q(e) \cup Q(e')$.

This gluing is induced by piecewise isometries between boundaries of the rectangles $Q(e)$. Indeed, any subsegment of a, is mapped by f_e and $f_{e'}$ to a segment of length equal to the transverse measure of a given by λ. Since, by construction, the maps $(f_e)_e$ patch together continuously, we can call $f \colon N_\delta(\lambda) = \bigcup_e R_e \to \bigcup_e Q(e)$ the resulting continuous map, where we mean the rectangles $(Q(e))_e$ glued along the vertical sides as described.

In order to get a half-translation surface, we just need to perform the gluing of the horizontal sides of the rectangles $Q(e)$'s. For this we proceed very similarly to the multicurve case above. Let us consider one connected component A of the complement of $N_\delta(\lambda)$. The orthogeodesic foliation $\mathcal{O}_\lambda(X)$ restricted to A is a singular foliation with leaves orthogonal to the boundary, as we showed before, in particular, transverse to it. Exactly as in the multicurve case, the singular foliation induces a gluing pattern on the boundary of A. We now transfer this gluing pattern via the map f to the Euclidean rectangles $(Q(e))_e$. If a segment a on $\partial A \subset \partial N_\delta(\lambda)$ is paired by $\mathcal{O}_\lambda(X)$ to the segment b also on ∂A via the bijection $\varphi\colon a \to b$, then we glue $f(a)$ to $f(b)$ so that $f|_a = f|_b \circ \varphi$. This gluing is induced by isometries on the boundary of $\bigcup_e Q(e)$ as φ preserves the transverse measure with respect to $\mathcal{O}_\lambda(X)$ and f, along the boundary of $N_\delta(\lambda)$ sends transverse measure to Euclidean length.

The result of all the gluing operations of the rectangles $(Q(e))_e$ is a half-translation surface, because obtained by gluing rectangles along their boundaries via piecewise isometries, vertical sides with vertical sides and horizontal with horizontal ones, hence the vertical and horizontal foliations on each rectangle extend to two orthogonal singular foliations on the entire surface.

We can now finally define the deflation map $\mathcal{D}$.

Definition 6.9. On every rectangle $R(e)$ we define $\mathcal{D}$ to coincide with f_e. With this definition the map is continuous on $N_\delta(\lambda)$ as along each tie separating two adjacent rectangles $R(e)$ and $R(e')$, f_e and $f_{e'}$ match by construction. On every connected component in the complement of $N_\delta(\lambda)$, we define $\mathcal{D}$ to send each - possibly singular - leaf l to the same point, determined by the image of $\mathcal{D}$ on the endpoints of l, which we already defined as they lie on the boundary of $N_\delta(\lambda)$.

It is easy to see that by construction the pair of foliation and lamination $(\mathcal{O}_\lambda(X), \lambda)$ are sent by $\mathcal{D}$ to the vertical and horizontal foliation on the resulting half-translation surface, which is then $\mathcal{O}(X, \lambda)$.

7 Geometric control of deflation

At the core of the arguments lies Theorem 7.1, estimating the constant ϵ for which the deflation map $\mathcal{D}\colon \operatorname{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$ is an ϵ-isometry. For this scope we will choose a different renormalization of the metric. More precisely, we want to rescale both $\operatorname{Gr}(X,\lambda)$ and $\mathcal{O}(X,\lambda)$ with the same constant k^{-1}, so that $\mathcal{O}(X,\lambda)$ has unit area. This means that $k^2 = \ell_X(\lambda)$ (see equation (8.1)). Thus, the grafted surface $\operatorname{Gr}(X,\lambda)$ rescaled by $k^{-1} = \ell_X(\lambda)^{-1/2}$ will have the flat part of unit area. We remind that for this chapter we will consider only surfaces grafted along a non-zero lamination, corresponding to the domain $\mathcal{T}(S) \times \mathcal{ML}(S) \subset \mathcal{T}(S) \times \mathcal{ML}_0(S) \cong \mathcal{PT}(S)$ of the map $\mathcal{O}$.

Remark . For large grafting, that is for $\ell_X(\lambda)$ large, the area-normalization and the renormalization obtained dividing by $k = \sqrt{\ell_X(\lambda)}$, are asymptotic. Therefore, showing convergence to a half-translation surface in this renormalization is equivalent to using the total unit-area one.

Remark . This normalization hints to the dual point of view we will discuss in Chapter 8, where grafted surfaces can be viewed as unit area half-translation surfaces where some negatively curved parts have been inserted, and the deflation map just undoes this operation that we will call *inflation*, exactly as Thurston's collapsing map undoes grafting. In view of this, we can call the grafted and the negatively curved part of $\operatorname{Gr}(X,\lambda)$ as respectively the flat and the *inflated region.*

Let us then assume from now on, that all the grafted surfaces $\operatorname{Gr}(X,\lambda)$ we consider are rescaled to have the flat part of unit area, and the half-translation surfaces $\mathcal{O}(X,\lambda)$ also have unit area. This is the technical result that will imply our main bordification theorem.

Theorem 7.1. *Let* $\operatorname{Gr}(X,\lambda)$ *be a grafted surface. Assume* $\mathcal{O}(X,\lambda)$ *has systole longer than* ϵ *and that* $\ell_X(\lambda) > 1$. *Then there exists* C *depending only on* ϵ *(and on the topology of* S*) such that the deflation map* $\mathcal{D}\colon \operatorname{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$ *is a* $C \cdot (\ell_X(\lambda))^{-1/2}$*-isometry compatible with the markings. In particular*

$$d_{GH}(\operatorname{Gr}(X,\lambda), \mathcal{O}(X,\lambda)) \leq C \cdot (\ell_X(\lambda))^{-1/2}.$$

In general the constants in our estimates will often depend on the topology of the surface S. However, we will think of S as fixed once for all, so we will omit to mention the dependency of constants from the topology of S. We will also prove in Section 7.5 an improved version that provided L large enough, replacing the dependency of the constant C from the systole of $\mathcal{O}(X,\lambda)$, with the one only from the systole of $\mathrm{Gr}(X,\lambda)$.

7.1 Pull-back arcs

In this section we prove fundamental properties of the deflation map $\mathcal{D}$. First we show that the deflation map, similarly to Thurston's collapsing map, preserves the metric in the grafted region while collapsing the inflated, that is the negatively curved, pieces. This is clear in the case of a multicurve, as the combinatorics of the grafted surface is finite and the deflation map simply restricts to a local isometry in the flat part and collapses the inflated part along the orthogeodesic foliation. In the general case we have to make a more precise meaning of "preserves the metric", as the grafted region can have empty interior, so $\mathcal{D}$ will not restrict to a local isometry there. In order to control the metric, instead, we will use the length of geodesic arcs, as we did in the geometric control of grafting.

Construction of pull-back arcs Given a geodesic arc α in $\mathcal{O}(X,\lambda)$, its pull-back will be a corresponding arc β in $\mathrm{Gr}(X,\lambda)$ that we will use to compare distances on the two surfaces. The construction of β, similarly to what we did in Chapter 5, is obtained essentially by taking the preimage via $\mathcal{D}$ of α with some extra care where α is parallel to the horizontal foliation or crosses singularities.

Remark . Let us first observe that from the construction of $\mathcal{D}$, it follows immediately that the preimage of a point $p \in \mathcal{O}(X,\lambda)$ is either

- a single point when p does not lie on a horizontal separatrix;
- a segment of a leaf of the orthogeodesic foliation contained in a hyperbolic piece when p lies on a horizontal separatrix;
- a union of those, when p is a singularity.

Lemma 7.2. *Given $\alpha\colon (0,1) \to \mathcal{O}(X,\lambda)$ a geodesic arc disjoint from singularities and transverse to the horizontal foliation, then its preimage $\mathcal{D}^{-1}(\alpha)$ is an arc in* $\mathrm{Gr}(X,\lambda)$.

Proof. It follows easily from the definition of $\mathcal{D}$, especially from the interpretation of the deflation viewed as collapsing the inflated pieces to the spine Sp.

We can observe that if λ has closed leaves, then, as we showed, the cylinders corresponding to them are mapped isometrically by $\mathcal{D}$ to $\mathcal{O}(X, \lambda)$. Let us call C the union of the interior of the images in $\mathcal{O}(X, \lambda)$ of the grafted cylinders. The preimages of subsegments of α that lie in C are isometric copies of them in the grafted cylinders.

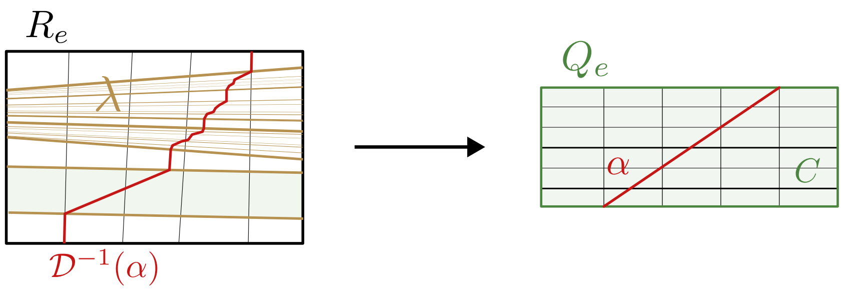

Figure 7.1: The preimage via $\mathcal{D}$ of a non-horizontal geodesic arc α.

Meanwhile, subsegments of α in the complement of C have preimages that are arcs where each intersection point of α with $\mathcal{D}(\mathrm{Sp})$ is inflated to a segment of the orthogeodesic foliation contained in the negatively curved part.

□

Let us now construct the pull-back of geodesic arcs in the general case.

Lemma 7.3. *Given $\alpha\colon [0,1] \to \mathcal{O}(X, \lambda)$ a geodesic arc, there exists an arc β in $\mathrm{Gr}(X, \lambda)$ such that $\alpha = \mathcal{D} \circ \beta$ and such that the intersection β_{infl} of β with the interior of the inflated part is a union of segments of leaves of the orthogeodesic foliation of $\mathrm{Gr}(X, \lambda)$.*

Proof. The geodesic arc α can go through at most finitely many singularities, in order $v_1, \ldots, v_{n-1}$, splitting α in a concatenation of finitely many consecutive geodesic segments $\alpha_1, \ldots, \alpha_n$, each not containing a singularity in its interior. Since $\mathcal{D}$ is a homotopy equivalence, there exists in $\mathcal{D}^{-1}(\alpha)$ an arc β such that $\mathcal{D} \circ \beta = \alpha$. As it may be not unique, we will explicitly construct it piece by piece, that is, we will choose first for the interior of each α_i an open arc β_i such that $\alpha_i = \mathcal{D} \circ \beta_i$, then for every i we will join the arcs β_i to β_{i+1} with an arc in $\mathcal{D}^{-1}(v_i)$. For simplicity of notation, from now on, by α_i we mean its interior, so with endpoints excluded. Each segment α_i can be horizontal, or transverse to the horizontal foliation λ. We will deal with the two different cases separately, and then deal with the singularities.

Horizontal segments. If α_i is horizontal, either it lies on a non-singular leaf, in which case $\mathcal{D}|_{\alpha_i}$ is invertible and is an isometry, so there is only one possible choice for β_i; or α_i lies on a horizontal separatrix. In this last case then $\mathcal{D}^{-1}(\alpha_i)$ is a portion of an inflated piece (regularly) foliated by the orthogeodesic foliation. In this case we choose β_i to be on the boundary of such a region (see Figure 7.2), that is on λ, so its length will be the same as the one of α_i. Notice that we have two choices here. Both choices will equally work for our scopes. For coherence, we could also consistently always choose the one on the right (this is well-defined from the orientation of α and the orientation of the surface).

Non-horizontal segments. If α_i is not horizontal, which means it is transverse to the horizontal foliation, then we showed in the previous lemma that its preimage $\mathcal{D}^{-1}(\alpha_i)$ is already an arc, so we simply choose β_i to be $\mathcal{D}^{-1}(\alpha)$.

Vertices. The preimage $\mathcal{D}^{-1}(v_i)$ of a singular point v_i of order m is, by construction of the orthogeodesic foliation, the union of m segments of the same length, sharing one endpoint. So for every singularity in the interior of α, we can join the two arcs β_i, β_{i+1} by following two of the m segments in $\mathcal{D}^{-1}(v_i)$ as in Figure 7.2. Which segments to follow depends on the choice for the horizontal segments we discussed above, but this will not have an impact on the length as the m segments have all the same length.

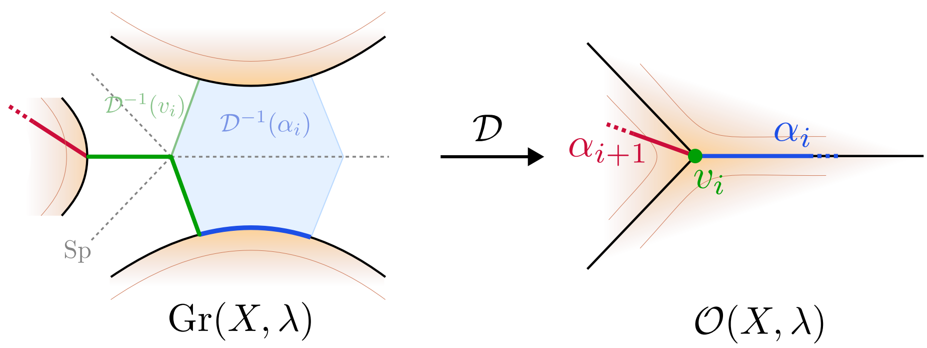

Figure 7.2: Summary of the construction of the pieces of the pull-back arc β: in green a vertex and the selected path in its preimage; in blue a horizontal segment and the selected path as pull-back in its preimage; and in red a segment transverse to the horizontal foliation and its pull-back.

This concludes the construction of the pull-back arc β meeting the requirements. □

Definition 7.4. With the notation above, we call β the pull-back of α and write $\beta = \mathcal{D}^*\alpha$.

The idea of the proof of the main theorem is, as in the case of small grafting, to show good control of the length of β in comparison to α in order to prove that $\mathcal{D}$ is an ϵ-isometry. The argument is split into multiple lemmas that we prove here below, and combine at the end of the chapter to deduce the main theorem.

7.2 Approximation by multicurves

We want to show that, as we heuristically depicted, the increment of length from α to β is given exactly by the length of the inflated part β_{infl} of β. This is simple in the case of a multi curve, but less trivial for the general case.

Let us then first observe the continuous dependency of our constructions from the grafting data, so that we can deal only with the case of λ multicurve, and deduce the general case by a density argument. Let us approximate λ by a sequence of multicurves μ_n such that their supports also converge in the Hausdorff sense to the support of λ (see Chapter 1.3). We recall that as we discussed after Lemma 2.11, we are considering each grafted surface $Z = \mathrm{Gr}(X, \lambda)$ as the same smooth surface S equipped with a specific representative (in the $\mathrm{Diff}_0^+(S)$ orbit) of the Thurston metric g_Z. In this setting, as stated in Theorem 2.12, the metrics of the grafted surfaces $\mathrm{Gr}(X, \mu_n)$ converge uniformly to the one of $\mathrm{Gr}(X, \lambda)$. Then we can also observe that the grafted part with respect to the metric g_Z depend continuously on Z. More precisely, let us first call $Z_n = \mathrm{Gr}(X, \mu_n)$ and $Z = \mathrm{Gr}(X, \lambda)$, and also $\overline{\mu_n}$ the representative of μ_n on S geodesic with respect to g_{Z_n} and $\overline{\lambda}$ the representative of λ on S, geodesic with respect to the metric g_Z. Then $\overline{\mu_n}$ converges in the Hausdorff sense to $\overline{\lambda}$, as geodesics converge to geodesics.

On the side of flat surfaces, we have that, by Theorem 6.4, $\mathcal{O}(X, \mu_n)$ converges to $\mathcal{O}(X, \lambda)$. More precisely for n large enough, since, in X, the support of μ_n will lie close to the support of λ, the combinatorics of the rectangle decomposition in the construction of $\mathcal{O}(X, \mu_n)$ will be the same as the one for $\mathcal{O}(X, \lambda)$ and only the sizes of the Euclidean rectangles $(Q_e)_e$ will be different, but converging to the ones of the rectangles of $\mathcal{O}(X, \lambda)$. So for every n large enough, we can identify $\mathcal{O}(X, \mu_n)$ topologically with the surface $\mathcal{O}(X, \lambda)$ but endowed with a slightly different metric where every rectangle is linearly rescaled horizontally and vertically by factors that approach 1 for n going to infinity.

If α is a geodesic arc in $\mathcal{O}(X, \lambda)$, call α_n the arc joining the endpoints of α but geodesic with respect to the metric of $\mathcal{O}(X, \mu_n)$. Then one can see that α_n converges to α, due to the convergence of the metric and the uniqueness of the length minimizing arc between two points.

If we now see $\mathcal{D}_n \colon \mathrm{Gr}(X, \lambda) \to \mathcal{O}(X, \lambda)$ through the identifications above, as a map between the same surfaces $\mathcal{D}_n \colon S \to \mathcal{O}(X, \lambda)$, which do not depend on n, then because of the convergence of $\overline{\mu_n}$ to $\overline{\lambda}$, the maps $\mathcal{D}_n$ converge to $\mathcal{D}$ and the pull-back arcs $\beta_n = \mathcal{D}_n^* \alpha_n$ converge to $\beta = \mathcal{D}^* \alpha$ (up to making consistent choices in their construction as we mentioned in the previous section).

We can now use the approximation by multicurves to prove the two following two statements.

Lemma 7.5. *If α is a geodesic arc in $\mathcal{O}(X, \lambda)$ and $\beta = \mathcal{D}^* \alpha$ its pull-back in* $\mathrm{Gr}(X, \lambda)$*, then*

$$\mathrm{Len}(\beta) = \mathrm{Len}(\alpha) + \mathrm{Len}(\beta_{\mathrm{infl}}).$$

Proof. We first prove it in the case when λ is a multicurve. In this case the grafted surface is a union of finitely many rescaled hyperbolic pieces and finitely many cylinders. So β is a concatenation of finitely many arcs, each one contained either in the interior of the inflated part or in a cylinder, considered closed, meaning boundary included. The segments in the interior of the inflated part constitute β_{infl} by definition, while the arcs in the cylinders are mapped isometrically to $\mathcal{O}(X, \lambda)$, so the sum of their length is equal to $\mathrm{Len}(\alpha)$.

The general case where λ is any measured lamination will follow by a continuity argument. Notice first that in this general case we can still interpret $\mathrm{Len}(\beta_{\mathrm{infl}})$ as length in the classical sense, as β_{infl}, in the general case, is a countable union of sub arcs of β. We cannot say the same for the complementary $\beta \setminus \beta_{\mathrm{infl}}$, as this is a positive-measured Cantor set along β. By length in this case we will mean the one dimensional Lebesgue measure induced by the length along β. Luckily we can just compute its length as $\mathrm{Len}(\beta) - \mathrm{Len}(\beta_{\mathrm{infl}})$ which in the case when λ is a multicurve, equals the length of the image $\mathcal{D}(\beta) = \alpha$.

We now use the approximation by multicurves as above, noticing that also the inflated part of β_n converges to the inflated part of β, and so the equation passes to the limit. □

Lemma 7.6. *The deflation map $\mathcal{D}$ is 1-Lipschitz.*

Proof. Let us start with the case when λ is a weighted multicurve. Then the restriction of $\mathcal{D}$ to the grafted cylinders is a local isometry, while the restriction to a negatively curved part is a contraction for the following reason. From the construction, we can see $\mathcal{D}$ locally as the composition of the closest point projection to the lamination, which in negative curvature is a contraction, which is then mapped isometrically into $\mathcal{O}(X, \lambda)$.

In the general case, it follows by the approximation by multicurve and continuity discussed above. □

Although we could deal only with multicurves and deduce the whole result of Theorem 7.1 by approximation with multicurves, we will carry out the rest of the arguments in the maximum generality with λ any measured laminations because it is actually not much more complicated than replacing sums with series and has the advantage of giving a better understanding of the geometry of grafted surfaces in the general case.

7.3 Upper bound for the inflated length

Since what we will want to show is that the metric of the grafted surface converges to the one of the half-translation surface, what we aim to prove is an upper bound for $\mathrm{Len}(\beta_{\mathrm{infl}})$.

The first key ingredient is to show that under certain conditions the inflated part is uniformly slim. Let us remind that the inflated parts have negative constant curvature $-k^2$. We remind the normalization we are currently using is the one for which the flat part of $\mathrm{Gr}(X, \lambda)$ has area 1, so the scaling factor is $k^{-1} = \ell_X(\lambda)^{-1/2}$.

Proposition 7.7. *Let* $\mathrm{Gr}(X, \lambda)$ *be a grafted surface with* $\ell_X(\lambda) > 1$ *and systole longer than* δ*. Then there exists a constant* D_δ*, depending only on* δ*, such that the inflated part in* $\mathrm{Gr}(X, \lambda)$ *is* D_δ/k*-slim, meaning that any point in the interior of the inflated part, lies at distance at most* D_δ/k *from its boundary.*

Proof. The hyperbolic surface X might not be δ-thick in general, so let us consider its δ-thick-thin decomposition. If it is trivial, that is, X is δ-thick, then there is an upper bound D for its diameter, by compactness of the δ-thick part of the moduli space $\mathcal{M}(S)$. Such D will trivially satisfy our proposition in this case, as the inflated part is obtained by rescaling X by k^{-1} and cutting it along λ, resulting in pieces D/k-slim.

In the general case, assuming without loss of generality δ small enough, the thin components are annuli. Let us then consider a thin component of X, meaning an annulus where every point has injectivity radius less than δ with respect to the hyperbolic metric of X, in particular the shortest closed geodesic γ in such an annulus has length less than 2δ. Then there must be a leaf l of λ crossing γ, otherwise to γ corresponds an even shorter curve in $\mathrm{Gr}(X, \lambda)$, as the grafting does not affect γ and the rescaling is by $k^{-1} = \ell_X(\lambda) < 1$ by assumption. The leaf l, intersecting γ the core of the annulus, by simple hyperbolic geometry, has to cross also both the boundary components of the annulus. From this follows also that every thick component of X intersects at least one leaf of λ, since every thick component is adjacent to at least a thin component.

It is now easy to show the slimness of the inflated part in $\mathrm{Gr}(X, \lambda)$. In X, any point in a thin component, will lie at distance at most δ from the leaf of λ crossing it. On the other side, the δ-thick components have uniformly upper bounded diameter $D = D(\delta, g)$ depending only on δ and on the genus of the surface. We can assume $D > \delta$. Since we showed that every thick component intersects at least one leaf of λ, then any point in a thick component is at distance at most D from λ. In $\mathrm{Gr}(X, \lambda)$ the metric is rescaled by a factor $1/k$ with respect to the one coming from the hyperbolic surface X, so any point will be distant at most D/k from the boundary of the inflated part. □

The lemma above is nevertheless not sufficient to provide a bound for the length of β gained in the inflated parts, as in general β_{infl} will be a union of countably many segments. However, most of the contribution to $\mathrm{Len}(\beta_{\text{infl}})$ is given by few relatively long segments, while most of the segments will not contribute much. The following lemma shows this, and its argument is the analogue of Lemma 5.2.

The key idea is that in each inflated region, the distance between the segments of β_{infl} will be lower-bounded by the systole of the surface $\mathcal{O}(X, \lambda)$, and since each connected component of the inflated region is either compact or exponentially thin, then the total length of β_{infl} has to be bounded. We will now turn these heuristics into a more quantitative estimate.

Let us decompose each inflated region as follows. Consider the spine Sp in $\mathcal{O}(X, \lambda)$ as the diagram of singular horizontal leaves. For some $d > 0$, take V the d-neighbourhood along Sp of the vertices. Then we call $\mathcal{D}^{-1}(V)$ the *thick part* of the inflated region, and $\mathcal{D}^{-1}(\mathrm{Sp} \setminus V)$ the *thin part*. We use this decomposition to partition also $\beta_{\text{infl}} = \beta^T_{\text{infl}} \cup \beta^t_{\text{infl}}$, where the first term β^T_{infl} is the union of all the segments of β_{infl} lying in the thick part of the inflated region, while β^T_{infl} the union of the ones in the thin part of the inflated region.

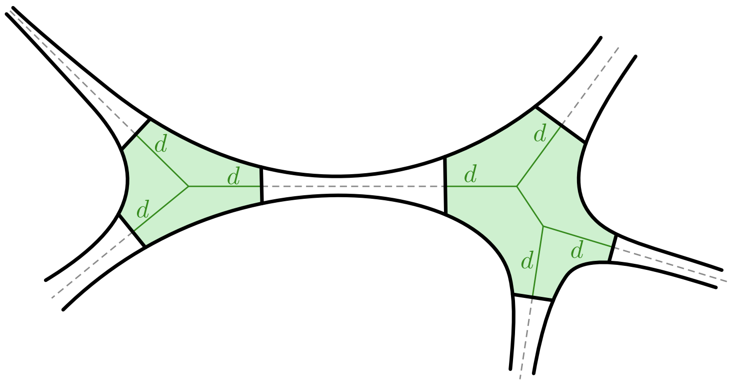

Figure 7.3: In green the thick part of the inflated region, containing all the singularities for $\mathcal{O}(X, \lambda)$.

Lemma 7.8. *Consider $\mathcal{D}\colon \mathrm{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$ as above, and assume $\mathcal{O}(X,\lambda)$ has systole length at least ϵ. Take the parameter $d = \epsilon/2$ for the thick-thin decomposition of the inflated part. Then*

$$\mathrm{Len}(\beta^t_{\mathrm{infl}}) \leq C_\epsilon / k^2$$

where the constant C_ϵ depends only on ϵ.

Proof. As we anticipated, the idea for the argument is the same as in Lemma 5.2: each segment of β^t_{infl} has a fairly sized neighbourhood in the inflated part, but the latter has finite total area $2\pi|\chi(S)|/k^2$.

Every component of β^t_{infl} is mapped via $\mathcal{D}$ to a point $\alpha(t)$ in $\mathrm{Sp}\setminus V$. By checking our construction of pull-back arc, we see that the point $\alpha(t)$ must lie in a non-horizontal subsegment of α.

Let us consider, for every such a t, the horizontal segment U_t centred at $\alpha(t)$, and of length $\epsilon/2$. Note that each U_t does not intersect itself, meaning it is not a closed curve as we assumed that $\mathcal{O}(X,\lambda)$ is ϵ-thick. We also notice that the segments U_t are disjoint from the singularities, as we are using $d = \epsilon/2$, and by definition of the thin part of the inflated region. Moreover, all the segments U_t will be disjoint from each other. Suppose by contradiction that U_t intersects $U_{t'}$, with $t \neq t'$. Then we can connect $\alpha(t)$ and $\alpha(t')$ with a horizontal segment of length less than $\epsilon/2$. Being the systole length less than ϵ, the injectivity radius of $\mathcal{O}(X,\lambda)$ is less than $\epsilon/2$, so such a horizontal segment is length minimizing between $\alpha(t)$ and $\alpha(t')$. But being α length minimizing, it implies that that segment lies in α, which contradicts our observation about α being in $\alpha(t)$ transverse to the horizontal foliation.

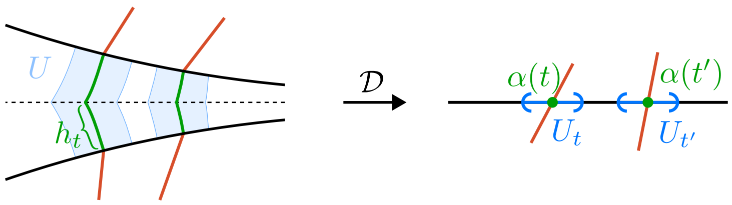

Figure 7.4: We call h_t half the length of $\mathcal{D}^{-1}(\alpha(t))$.

Consider now U the subset of λ formed by all the (disjoint) segments U_t we described

$$U := \bigcup_t U_t.$$

Similarly to what we observed at the end of Chapter 5, we see that $\mathcal{D}^{-1}(U)$, being contained in the inflated part of $\mathrm{Gr}(X,\lambda)$, has area bounded above by the total area of the inflated part.

$$\mathrm{Area}(U) \le 2\pi|\chi(S)|k^{-2}$$

On the other hand, we also have a lower bound for the area of U. Indeed, U is the disjoint union of $\mathcal{D}^{-1}(U_t)$ for countably many values of t. Each region $\mathcal{D}^{-1}(U_t)$ is made up of two isometric pieces, lying on the two sides of the spine, whose area we can lower bound thanks to Lemma 7.9.

$$\mathrm{Area}(U) = \sum_t \mathrm{Area}(\mathcal{D}^{-1}(U_t)) \ge \sum_t 2h_t\epsilon/2 = \epsilon \sum_t h_t = \epsilon\,\mathrm{Len}(\beta^t_{\text{infl}}).$$

Combining the two area bounds we obtain

$$\mathrm{Len}(\beta^t_{\text{infl}}) \le \frac{2\pi|\chi(S)|}{\epsilon}k^{-2}.$$

□

Here is the hyperbolic geometry lemma about area estimate we used in the argument above. Essentially it estimates the area of a quadrilateral in $\mathbb{H}^2$ with two adjacent right angles using the comparison with an analogue Euclidean counterpart.

Lemma 7.9. *Let r be a line in $\mathbb{H}^2$ parametrized by arc-length and η be the foliation whose leaves $(\eta_t)_{t\in\mathbb{R}}$ are such that η_t is the geodesic orthogonal to r in $r(t)$. Let l be any other line disjoint from r. Denote $r(t_0)$ a point on r, and U be the region delimited by $r, l, \eta_{t_0+\delta}, \eta_{t_0-\delta}$ as in Figure 7.5. Call $h(t)$ the length of the segment of η_t between r and l, then*

$$\mathrm{Area}(U) \ge 2\delta h(t_0).$$

Proof. Consider the coordinates $\varphi\colon \mathbb{R}^2 \to \mathbb{H}^2$ that maps the x axis isometrically to r, so $\varphi(t,0) = r(t)$ and sends each horizontal line $\{x = t\}$ isometrically to η_t. Assume also that $l \subset \varphi(\{y > 0\})$.

In this setting l is the image under φ of the graph of the function h and U is the image under φ of the region T under the graph of h restricted to $(t_0 - \delta, t_0 + \delta)$. Because of the negative curvature of $\mathbb{H}^2$, the map φ is a dilation from the Euclidean metric on $\mathbb{R}^2$ to the hyperbolic metric on $\mathbb{H}^2$, so $\mathrm{Area}(U) \ge \mathrm{Area}(T)$. Always due to negative curvature, we know that the function h is convex (it is indeed a hyperbolic cosine or an exponential), so $\mathrm{Area}(T) \ge \mathrm{Area}(T')$ where T' is the trapezium under the tangent in t_0 to the graph of h, hence

$$\mathrm{Area}(U) \ge \mathrm{Area}(T) \ge \mathrm{Area}(T') = 2\delta h(t_0).$$

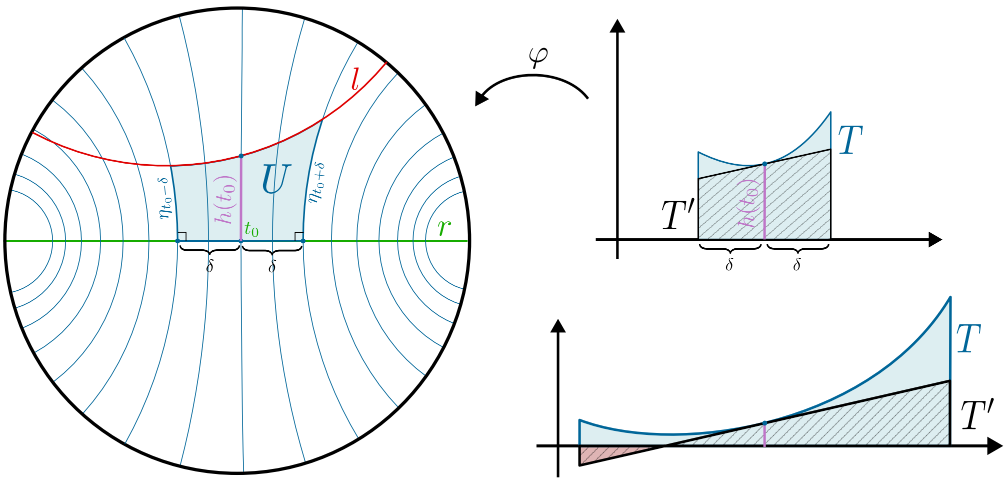

Figure 7.5: Left: the hyperbolic quadrilateral U. Top right: the Euclidean chart. Bottom right: the case when T' is not simple.

Note that even in the case when the tangent line delimiting T' intersects the x axis inside the interval $(t_0 - \delta, t_0 + \delta)$, then one can just consider only the part above the x-axis of T', whose area will be even larger than $2\delta h(t_0)$, as that is the signed area of T', where the part below the x-axis counts as negative. □

Remark . Note that the previous lemma holds true even if we replace $\mathbb{H}^2$ with a rescaled version of it, meaning with constant curvature $-k^2$. Indeed, we just used that in negative curvature the parametrization φ is a dilation. Alternatively, one can just rescale by k the initial data of the problem, and reach the same conclusion, as the inequality would be homogeneous in the rescaling factor k.

7.4 Proof of the main result

Finally, we can combine all the previous lemmas to compose a proof of Theorem 7.1.

Proof of Theorem 7.1. Let us consider the deflation map $\mathcal{D}\colon \mathrm{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$. We know that $\mathcal{D}$ is 1-Lipschitz. This implies that $\delta = \epsilon$ is also a lower bound for the systole length of $\mathrm{Gr}(X,\lambda)$. We want to show that $\mathcal{D}$ also does not shrink distances by more than an additive error proportional to k^{-1}, and then conclude by Lemma 3.3.

For any two points x, y in $\mathrm{Gr}(X,\lambda)$ we want to show that

$$d(\mathcal{D}(x), \mathcal{D}(y)) \geq d(x,y) - Ck^{-1}. \tag{7.1}$$

Let us consider the geodesic arc α joining $\mathcal{D}(x), \mathcal{D}(y)$ and realizing their distance in $\mathcal{O}(X, \lambda)$. Let us consider the pull-back arc $\beta = \mathcal{D}^*\alpha$. The endpoints of β are not necessarily x, y. They lie in $\mathcal{D}^{-1}(\mathcal{D}(x))$ and in $\mathcal{D}^{-1}(\mathcal{D}(y))$, which, as we said above, can be either a point, a segment of a leaf, or a star-shaped union of segments on a singular leaves. In any case, by Proposition 7.7, the length of such segments is at most $D_\delta k^{-1}$, so we can extend β so that it joins x to y. This contributes to $\mathrm{Len}(\beta_{\mathrm{infl}})$.

Now the length of β gives an upper bound for the distance between x and y. By Lemma 7.5 we have indeed

$$d(\mathcal{D}(x), \mathcal{D}(y)) = \mathrm{Len}(\alpha) = \mathrm{Len}(\beta) - \mathrm{Len}(\beta_{\mathrm{infl}}) \geq d(x, y) - \mathrm{Len}(\beta_{\mathrm{infl}})$$

so all we need is an upper bound for $\mathrm{Len}(\beta_{\mathrm{infl}})$. We have separate upper bounds for $\mathrm{Len}(\beta^T_{\mathrm{infl}})$ and $\mathrm{Len}(\beta^t_{\mathrm{infl}})$, where, as before, we choose $d = \epsilon/2$ for the thick-thin decomposition of the inflated region.

In every component of the thick part of the inflated region there can be at most one segment of β_{infl} per prong. Otherwise, we would have two of them that can be connected with a horizontal segment of length less than $\epsilon/2$ which as above gives a contradiction with the assumption on the lower bound of the injectivity radius. The total number of prongs on the surface is at most $6|\chi(S)|$, so using Proposition 7.7 we have

$$\mathrm{Len}(\beta^T_{\mathrm{infl}}) \leq 6|\chi(S)| D_\delta / k.$$

Let us combine this with the bound $\mathrm{Len}(\beta^t_{\mathrm{infl}}) \leq C_\epsilon k^{-2}$ given by Lemma 7.8, and, using the assumption $k > 1$, obtain the following.

$$\begin{aligned}
\mathrm{Len}(\beta_{\mathrm{infl}}) &= \mathrm{Len}(\beta^T_{\mathrm{infl}}) + \mathrm{Len}(\beta^t_{\mathrm{infl}}) \\
&\leq 6|\chi(S)| D_\delta k^{-1} + C_\epsilon k^{-2} \\
&\leq \left(6|\chi(S)| D_\delta + C_\epsilon k^{-1}\right) k^{-1} \\
&\leq \left(6|\chi(S)| D_\delta + C_\epsilon\right) k^{-1}
\end{aligned}$$

This concludes the proof of Theorem 7.1 with constant

$$C = 6|\chi(S)| D_\delta + C_\epsilon = 6|\chi(S)| D_\delta + \frac{2\pi|\chi(S)|}{\epsilon}$$

where we remind that we took $\delta = \epsilon$, so C depends only on ϵ. Then the deflation map $\mathcal{D}\colon \operatorname{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$ is a Ck^{-1}-isometry, where we remind that $k^2 = \ell_X(\lambda)$ so $k^{-1} = \ell_X(\lambda)^{-1/2}$. □

We now have all the tools and the language to show how the main theorem can be deduced from Theorem 7.1.

Proof of Theorem 4.15. Let $q \in \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \subset \mathbb{P}\,\mathrm{MMet}(S)$ be a unit area half-translation surface. Then, by Theorem 6.3, $\mathcal{O}$ is a bijection so we can find a sequence of grafted surfaces $\mathcal{O}^{-1}(nq)$ that, after renormalization, deflate to q and for which $\ell_X(\lambda) = n^2$ is going to infinity. Then, by Theorem 7.1, such a sequence converges to q. See Chapter 8.2 for a more detailed discussion of such sequences of grafted surfaces.

We proved that $\mathcal{PT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \subset \overline{\mathcal{PT}}(S)$ in $\mathbb{P}\,\mathrm{MMet}(S)$. The converse inclusion is provided by Proposition 7.10, whose proof is rather technical and thus contained in the next separated section. □

7.5 Degeneration control

In this section, as anticipated, we will prove an improvement of Theorem 7.1 changing the dependency of the constant C_ϵ from the systole of $\mathcal{O}(X,\lambda)$ to the systole of $\operatorname{Gr}(X,\lambda)$. Indeed, this will allow us to prove the following.

Proposition 7.10. *If* $\operatorname{Gr}(X_n,\lambda_n)$ *is a sequence of area-normalised grafted surfaces with* $\ell_{X_n}(\lambda_n)$ *going to infinity, converging in* $\mathrm{MMet}(S)$ *to a surface* Y*, then* Y *lies in the image of* $\mathcal{QT}(S)$*.*

In order to prove this, we need to show that a lower bound δ for the systole of $\operatorname{Gr}(X,\lambda)$ implies, provided that $\ell_X(\lambda)$ is large enough, that $\mathcal{O}(X,\lambda)$ is also thick. For this scope, let us first show a stronger version of Lemma 7.8.

Lemma 7.11. *Consider* $\mathcal{D}\colon \operatorname{Gr}(X,\lambda) \to \mathcal{O}(X,\lambda)$ *as above and assume that the systole of* $\operatorname{Gr}(X,\lambda)$ *is at least* δ*. Let* α *be a geodesic arc in* $\mathcal{O}(X,\lambda)$*, and* $\beta = \mathcal{D}^*(\alpha)$ *its pull-back in* $\operatorname{Gr}(X,\lambda)$ *and* β_{infl} *the intersection of* β *with the interior of the inflated part. Assume there exist* $N \geq 1, d > 0$ *such that for every segment of* λ *of length at most d,* β *intersects it at most* N *times. Then there is a constant* $C_{d,\delta}$ *such that*

$$\operatorname{Len}(\beta_{\text{infl}}) \leq C_{d,\delta} N/k$$

Proof. Let us consider the thick-thin decomposition of the inflated part, with respect to the parameter $d = \delta$. Let us follow along the proof of Lemma 7.8 to show separate upper bounds for $\mathrm{Len}(\beta^T_{\mathrm{infl}})$ and $\mathrm{Len}(\beta^t_{\mathrm{infl}})$.

In every component of the thick part of the inflated region there can be at most N segments of β_{infl} per prong. Otherwise, we would have N of them that intersect the same leaf of λ more than N times within a segment of length d, which violates our hypothesis. The total number of prongs on the surface is at most $6|\chi(S)|$, so using Proposition 7.7 we have

$$\mathrm{Len}(\beta^T_{\mathrm{infl}}) \leq 6|\chi(S)|D_\delta N/k.$$

We can get an upper bound for $\mathrm{Len}(\beta^t_{\mathrm{infl}})$ by adapting the argument with the area estimates we used in Lemma 7.8. In this case too, for every point of $\alpha(t)$ that is the image of a segment of β^t_{infl} we take a horizontal neighbourhood U_t, of length d along the spine Sp. Such segments U_t are once again away from singularities, by definition of the thin part of the inflated region, but might overlap with each other. However, our hypothesis implies that every point on the spine Sp lies in at most N neighbourhoods U_t's. Summing all the areas of the $\mathcal{D}^{-1}(U_t)$ will yield then at most N times the total area of the inflated region.

$$2N\pi|\chi(S)|k^{-2} \geq \sum_t \mathrm{Area}(\mathcal{D}^{-1}(U_t)) \geq \sum_t 2h(t)d = \mathrm{Len}(\beta^t_{\mathrm{infl}})d$$

We used again the same lower bound for the areas of $\mathcal{D}^{-1}(U_t)$ (see Figure 7.4 and Lemma 7.9). Which, rearranging, yields

$$\mathrm{Len}(\beta^t_{\mathrm{infl}}) \leq 2\pi|\chi(S)|\frac{N}{dk^2}.$$

Combining this with the upper bound of the thick part we obtain

$$\begin{aligned}\mathrm{Len}(\beta_{\mathrm{infl}}) &= \mathrm{Len}(\beta^T_{\mathrm{infl}}) + \mathrm{Len}(\beta^t_{\mathrm{infl}}) \\ &\leq 6|\chi(S)|D_\delta N/k + \frac{2\pi|\chi(S)|}{d}\frac{N}{k^2} \\ &\leq 2|\chi(S)|\,(3D_\delta + \pi/d)\,N/k\end{aligned}$$

which is the thesis, where the constant $C_{d,\delta} = 2|\chi(S)|\,(3D_\delta + \pi/d)$ depends only on d and δ. □

Now we are ready to prove that when grafting goes to infinity, we have topological degeneration in the rescaled grafted surface if and only if we have it in the associated unit area half-translation surface. Let us remind that the normalization we are considering is always the rescaling by $k^{-1} = \ell_X(\lambda)^{-1/2}$, both for $\mathrm{Gr}(X,\lambda)$ and $\mathcal{O}(X,\lambda)$.

Lemma 7.12. *Given $\delta > 0$ there exist L such that if $\mathrm{Gr}(X,\lambda)$ is a rescaled grafted surface with $\ell_X(\lambda) > L$, and $\mathrm{sys}(\mathrm{Gr}(X,\lambda)) > \delta$, then $\mathrm{sys}(\mathcal{O}(X,\lambda)) > \delta/2$.*

Proof. Let us fix $\delta > 0$ small, say $\delta < 1/4$. Take $L = \left(4C_{\frac{\delta}{4},\delta}/\delta\right)^2$. Let $Z = \mathrm{Gr}(X,\lambda)$ be a rescaled grafted surface with $\ell_X(\lambda) > L$ and systole length at least δ. We will show that $q = \mathcal{O}(X,\lambda)$ has $\mathrm{sys}(q) > \delta/2$.

Let us take α the systole for the surface q and assume by contradiction that $\ell_q(\alpha) = \mathrm{sys}(q) < \delta/2$. Heuristically, the idea is that α is a very short curve, but in order for it to correspond to a curve of moderate length in the grafted surface Z, it needs to have a conspicuous contribution in the length by the inflated part of Z. With L large, the inflated part is very thin, so in order to receive such a contribution as inflated length, α needs to intersect "frequently" λ. But this will imply that there is another short curve transverse to α and almost parallel to λ, that will then receive very little inflated length. Let us now carry out a proper proof.

Let us consider the pull-back arc $\beta = \mathcal{D}^*\alpha$ as in Definition 7.4. Using that Z is δ-thick, so $\ell_Z(\alpha) \geq \delta$, by Lemma 7.5 we have that

$$\mathrm{Len}_Z(\beta_{\mathrm{infl}}) = \mathrm{Len}_Z(\beta) - \ell_q(\alpha) \geq \ell_Z(\alpha) - \ell_q(\alpha) \geq \delta - \delta/2 = \delta/2 \tag{7.2}$$

We now want to reach a contradiction by obtaining an incompatible upper bound for $\mathrm{Len}(\beta_{\mathrm{infl}})$ by using the previous lemma. Let us now take N, d as follows

$$d = \delta/4 \quad \text{and} \quad N = 4\,\mathrm{Len}(\beta_{\mathrm{infl}})/\delta$$

and show that the hypothesis of the previous lemma are satisfied. We observe that from (7.2), it follows that $N \geq 1$. Suppose now by contradiction that there exists a segment η of a leaf of λ, of length d, intersecting β transversely N times. Such N intersection points partition β in N arcs, in particular there is one of these, say γ', such that

$$\mathrm{Len}_Z(\beta_{\mathrm{infl}} \cap \gamma') < \mathrm{Len}_Z(\beta_{\mathrm{infl}})/N = \delta/4$$

Then we can just take γ the closed curve obtained by concatenating γ' and the portion of

η between the endpoints of γ'. By construction the length in Z of γ is as follows

$$\begin{aligned}
\mathrm{Len}_Z(\gamma) &\leq \mathrm{Len}(\gamma') + d \\
&= \mathrm{Len}(\gamma' \cap \beta_{\mathrm{infl}}) + \mathrm{Len}(\gamma' \setminus \beta_{\mathrm{infl}}) + d \\
&< \mathrm{Len}_Z(\beta_{\mathrm{infl}})/N + \mathrm{Len}(\beta \setminus \beta_{\mathrm{infl}}) + d \\
&= \delta/4 + \ell_q(\alpha) + d \\
&\leq \delta/4 + \delta/2 + \delta/4 \\
&= \delta
\end{aligned}$$

Which contradicts our assumption of Z being δ-thick, so the hypothesis of the previous lemma must hold. Then by applying it, and inserting the chosen value for $N = 4\,\mathrm{Len}(\beta_{\mathrm{infl}})/\delta$ and $d = \delta/4$ we obtain

$$\begin{aligned}
\mathrm{Len}(\beta_{\mathrm{infl}}) &\leq C_{d,\delta} N k^{-1} \\
k &\leq C_{\frac{\delta}{4},\delta} N / \mathrm{Len}(\beta_{\mathrm{infl}}) \\
k &\leq 4C_{\frac{\delta}{4},\delta}/\delta
\end{aligned}$$

This is precisely a contradiction with the assumption $k^2 = \ell_X(\lambda) > L = \left(4C_{\frac{\delta}{4},\delta}/\delta\right)^2$. So the assumption $\mathrm{sys}(q) < \delta/2$ must have been wrong.

□

Finally, we can complete the proof of Proposition 7.10.

Proof of Proposition 7.10. If the sequence $\mathrm{Gr}(X_n, \lambda_n)$ converges to Y in $\mathbb{P}\,\mathrm{MMet}(S)$, then by Lemma 3.13 there is a uniform lower bound δ for their systole lengths. Then for all n large enough such that $\ell_{X_n}(\lambda_n) > L$, where L is the constant provided by Lemma 7.12, the systole length of $\mathcal{O}(X_n, \lambda_n)$ is at least $\delta/2$. Then Theorem 7.1 shows that the deflation map $\mathcal{D}\colon \mathrm{Gr}(X_n, \lambda_n) \to \mathcal{O}(X_n, \lambda_n)$ is a $(C_{\delta/2} \cdot \ell_{X_n}(\lambda_n)^{-1/2})$-isometry. So, since $\ell_{X_n}(\lambda_n)$ goes to infinity, the sequence $\mathcal{O}(X_n, \lambda_n)$ also converges to Y. But we showed in Proposition 4.10 that the embedding of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \hookrightarrow \mathbb{P}\,\mathrm{MMet}(S)$ is closed, then Y lies in its image and is then a metric surface arising from a half-translation structure. □

8 Inflation

We introduced a bordification $\overline{\mathcal{PT}(S)}$ of the space of grafted surfaces $\mathcal{PT}(S)$. Such a bordification encompasses all the hyperbolic surfaces, as it contains $\mathcal{T}(S)$, the unit-area half-translation surfaces as it contains $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$, and in between those, grafted surfaces that are partly negatively curved and partly flat. Grafting allows us to see this third family of surfaces as deformation of hyperbolic ones. In this chapter we will describe a complementary point of view in which they are seen as deformation of the flat half-translation surfaces. We will call such a deformation *inflation*, as it goes in the opposite direction of the deflation map. The ingredients for this description are all already discussed in the previous chapters; this chapter introduces only a different perspective on the same phenomenon.

8.1 Inflation parametrization

In this section we will define the inflation map, and rewrite the main result in terms of this. As we said, the inflation is meant to go in the opposite direction of the deflation, so given a half-translation surface q, it produces a surface whose deflation is q.

Let us start then defining the inflation as the map $\mathcal{O}^{-1}$

$$\mathrm{Infl} = \mathcal{O}^{-1} \colon \mathcal{QT}(S) \to \mathcal{PT}(S).$$

Given $q \in \mathcal{QT}(S)$ we think of it as the datum of a marked unit area half-translation surface $q' = \frac{q}{\sqrt{|q|}}$ together with a direction given by the horizontal foliation $\mathrm{Im}(q)$, and a positive real parameter k given by $k^2 = |q|$. So we have that $q = kq'$ with q' unit area. While in the case of grafting, the grafting datum is encoded by a pair made of the hyperbolic surface and the grafting lamination, here the initial unit area half-translation surface q', the positive real parameter k and the direction $\mathrm{Im}(q)$, can all be encoded in a single quadratic differential $q = kq'$.

So $\mathrm{Infl}(q) = \mathrm{Infl}(kq')$ is a grafted surface, say $\mathrm{Gr}(X, \lambda)$, then this means that

$$\mathcal{O}_\lambda(X) = \mathrm{Re}(q) = \mathrm{Re}(kq') = k\,\mathrm{Re}(q') \qquad \text{and} \qquad \lambda = \mathrm{Im}(q) = \mathrm{Im}(kq') = k\,\mathrm{Im}(q').$$

Moreover, we remind that the transverse measure to the foliation $\mathcal{O}_\lambda(X)$ is given by the hyperbolic length with respect to X along leaves of λ. This also implies that $\ell_X(\lambda) = i(\mathcal{O}_\lambda(X), \lambda)$ (see [CF24b, Lemma 5.7]). So, by using for this computation the intersection form we introduce in Chapter 9, we have that the following holds.

$$\ell_X(\lambda) = i(\mathcal{O}_\lambda(X), \lambda) = i(\mathrm{Re}(q), \mathrm{Im}(q)) = |q| = k^2 \tag{8.1}$$

where we used that the intersection form valued on vertical and horizontal foliations gives the Euclidean area $|q|$ of the half-translation surface. As a consequence, Theorem 7.1 rewrites in terms of inflation as follows. We remind that in this context $\mathrm{Infl}(kq)$ is a grafted surface considered rescaled such that the area of the grafted part is 1, and by q we mean the associated unit area half-translation surface.

Theorem 8.1. *Let q be a unit quadratic differential, $k > 1$ and let us consider ϵ smaller than the injectivity radius of q. Then the deflation map $\mathcal{D}\colon \mathrm{Infl}(kq) \to q$ is a C_ϵ/k-quasi isometry, where the constant C_ϵ depends only on ϵ. In particular, we have that*

$$d_{GH}(\mathrm{Infl}(kq), q) \leq C_\epsilon/k.$$

Let us remind that $\mathcal{QT}(S)$ is a complex vector bundle over $\mathcal{T}(S)$, where every fibre $Q(X)$ with $X \in \mathcal{T}(S)$ is the vector space, without the origin, of holomorphic quadratic differentials on the Riemann surface X. Each fibre has a projective compactification given by $\overline{Q(X)} = Q(X) \cup \mathbb{P}_{\mathbb{C}} Q(X)$ where $\mathbb{P}_{\mathbb{C}} Q(X) = Q(X)/\mathbb{C}^*$ is the complex projectification of $Q(X)$. Applying this compactification fibre-wise we obtain the bordification $\overline{\mathcal{QT}(S)} = \mathcal{QT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$. We now want to extend the inflation $\mathrm{Infl}\colon \mathcal{QT}(S) \to \mathcal{PT}(S)$ to the bordifications introduced above and obtain a mapping

$$\mathrm{Infl}\colon \overline{\mathcal{QT}(S)} \to \overline{\mathcal{PT}(S)}.$$

In order to do that, we notice that by definition the boundaries of both spaces are identified with $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$. We define then the inflation map Infl to be the identity on the boundary. As already pointed out, Infl is not continuous as the map $\mathcal{O}^{-1}$ is not globally continuous. Our bordification result implies the continuity of Infl at the boundary. We remind that the topology on $\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is the one inherited as subspace of $\mathbb{P}\,\mathrm{MMet}(S)$, so the one of projective marked Gromov-Hausdorff convergence. In this section, we will indicate with $\mathcal{T}(S) \subset \overline{\mathcal{PT}(S)}$ the subset corresponding via grafting to $\mathcal{T}(S) \times \{0\} \subset \mathcal{T}(S) \times \mathcal{ML}_0(S)$.

Corollary 8.2. *The inflation map* $\mathrm{Infl}\colon \overline{\mathcal{QT}(S)} \to \overline{\mathcal{PT}(S)}$ *is a bijection between the domain and the complement of* $\mathcal{T}(S)$ *in* $\overline{\mathcal{PT}(S)}$*, and it is continuous on the boundary* $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$*.*

Proof. It is easy to see that it is a bijection because we defined the inflation on $\mathcal{QT}(S)$ as $\mathcal{O}^{-1}$, which is a bijection between $\mathcal{QT}(S)$ and $\mathcal{PT}(S) \setminus \mathcal{T}(S)$. Then we extended it as the identity between the boundaries, which is again clearly bijective. So also the whole map Infl is bijective.

Let us now check the continuity at the boundary, meaning that for every sequence $(q_n)_n$ converging projectively to $[q] \in \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$, where q is the unit quadratic differential representing the class $[q]$, then the surfaces $\mathrm{Infl}(q_n)$ converge in $\mathbb{P}\,\mathrm{MMet}(S)$ to $\mathrm{Infl}([q]) = q$.

The convergence of q_n to $[q]$ in $\overline{\mathcal{QT}(S)}$ implies that $|q_n|$ goes to infinity and that q_n converges projectively to q. Since $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ embeds in $\overline{\mathcal{PT}(S)}$, then $[q_n] \to [q]$ in $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \subset \mathbb{P}\,\mathrm{MMet}(S)$ as unit area half-translation surfaces. In particular the systole length of q_n, considered area normalized, also converges to the systole length of q, and is then uniformly lower bounded by a positive constant $\epsilon > 0$. Then Theorem 8.1 tells us that the deflation maps $\mathcal{D}_n\colon \mathrm{Infl}(q_n) \to q_n$ are ϵ_n-isometries with $\epsilon_n = C_\epsilon |q_n|^{-1/2}$ going to zero as n tends to infinity, because $|q_n|$ is going to infinity. But then also $\mathrm{Infl}(q_n)$, as $[q_n]$, converge to the unit area half-translation surface q. □

Remark . Note that we only showed that $\mathrm{Infl}\colon \overline{\mathcal{QT}(S)} \to \overline{\mathcal{PT}(S)}$ is continuous at the boundary, meaning that any sequence converging to a point in the boundary is mapped to a sequence converging to the image of the limit. However, Infl is not locally continuous on the boundary.

To sum up we have that the space $\overline{\mathcal{PT}(S)}$ has two partial parametrization, given by grafting and by inflation. For the first one $\mathrm{Gr}\colon \mathcal{T}(S) \times \mathcal{ML}_0(S) \to \overline{\mathcal{PT}(S)}$ we have that

- Gr is a homeomorphism onto the complement in $\overline{\mathcal{PT}(S)}$ of the subspace $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ corresponding to half-translation surfaces;
- to points of $\mathcal{T}(S) \times \{0\}$ correspond hyperbolic surfaces;
- to points of $\mathcal{T}(S) \times \mathcal{ML}(S)$ correspond properly grafted surfaces.

For the second one $\mathrm{Infl}\colon \overline{\mathcal{QT}(S)} \to \overline{\mathcal{PT}(S)}$ we have that

- Infl is a bijection between $\overline{\mathcal{QT}(S)}$ and the complement in $\overline{\mathcal{PT}(S)}$ of the space of hyperbolic surfaces $\mathcal{T}(S)$; it is not globally continuous, but it is continuous at the boundary and on the continuity regions of $\mathcal{O}^{-1}$ (see Theorem 6.4);
- to points of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ correspond unit area half-translation surfaces;
- to points of $\mathcal{QT}(S)$ correspond properly grafted surfaces.

8.2 Inflation rays

In this section we want to summarize the geometric intuitive interpretation of inflation and describe inflation rays. As we introduced grafting with its simplest example, consisting of the insertion of a single cylinder, let us try to describe more concretely the easiest example of inflation.

Example . Let us consider a unit quadratic differential q where the diagram of horizontal separatrices is a (finite) connected graph G. From its embedding in q, G also inherits the structure of a metric ribbon graph. Then from the previous section and the explicit construction of the deflation map, follows that $\mathrm{Infl}(kq)$ can be obtained by cutting open the half-translation surface q along G and gluing in its place a negatively curved surface A_k with geodesic boundary, obtained as follows.

Let us consider kG the ribbon graph G with the metric rescaled by k. Then to it corresponds a hyperbolic surface B_k such that the spine $\mathrm{Sp}(B_k)$ with the metric induced by the orthogeodesic foliation with respect to the border ∂B_k is kG. The surface B_k exists and is unique, as discussed in Chapter 6.3. Then we can define the surface A_k as B_k rescaled by the constant k^{-1}. Note that the curvature of A_k is then $-k^2$.

We observe that the injectivity radius of $\mathrm{Infl}(kq)$ is larger than the one of q, so Proposition 7.7 tells us that A_k is D/k-slim for a constant D depending only on the systole length of q. This implies that for k going to infinity, A_k will shrink and deformation retract onto its spine, converging then to the metric graph G. The convergence of such sequences of rescaled hyperbolic surfaces with geodesic boundary to a metric ribbon graph is also studied in [Xu19].

To sum up, we can picture inflation as taking the flat half-translation surface q and inflating the graph G of horizontal separatrices to a fat graph endowed with a metric with curvature $-k^2$ and geodesic boundary.

Definition 8.3. Similarly to grafting rays $t \mapsto \mathrm{Gr}(X, t\lambda)$, we call *inflation rays* the paths defined as $t \mapsto \mathrm{Infl}(t^{-1}q)$ where q is a unit area quadratic differential.

Note that we used the inverse of t in the definition in order to have the following.

Proposition 8.4. *The inflation ray* $t \mapsto \mathrm{Infl}(t^{-1}q)$ *is continuous, in particular* $\mathrm{Infl}(t^{-1}q)$ *converges to* q *in* $\mathbb{P}\mathrm{MMet}(S)$ *as* t *tends to 0.*

Proof. The inflation ray is continuous for $t \in (0, +\infty)$ since $\mathcal{O}^{-1}$ is continuous restricted to $\{t^{-1}q\}_{t\in(0,+\infty)}$ by Theorem 6.4 as the horizontal foliation of $t^{-1}q$ is only rescaled upon change of t, so its support does not change with t. The geometric convergence for t approaching 0 follows from Theorem 8.1. □

From the definition of the map $\mathcal{O}$, it follows immediately that a grafting ray $t \mapsto \mathrm{Gr}(X, t\lambda)$ is mapped to $q(\mathcal{O}_\lambda(X), t\lambda)$, which, by considering its conformal class, projects to a Teichmüller geodesic. We now want to see, instead, how an inflation ray is described in terms of grafting coordinates. Let us call $X(t), \lambda(t)$ the grafting data such that $\mathrm{Infl}(t^{-1}q) = \mathrm{Gr}(X(t), \lambda(t))$. Let us call $\eta = \mathrm{Re}(q)$ and $\lambda = \lambda(1) = \mathrm{Im}(q)$, then $\lambda(t) = t^{-1}\lambda$. Then we also remind that the map $\mathcal{O}_\lambda$ depends only on the support of λ, so $\mathcal{O}_{\lambda(t)} = \mathcal{O}_\lambda$, thus we have $t^{-1}\eta = \mathcal{O}_{\lambda(t)}(X(t)) = \mathcal{O}_\lambda(X(t))$.

Corollary 8.5. *The inflation ray* $t \mapsto \mathrm{Infl}(t^{-1}q)$ *coincides for* $t > 0$ *with the path* $\mathrm{Gr}(X(t), t^{-1}\lambda)$ *where* $X(t)$ *is the generalized stretch ray defined by* $X(s) = \mathcal{O}_\lambda^{-1}(s^{-1}\eta)$.

As discussed in [CF24b, Section 15], these *generalized stretch rays*, when λ has complementary regions that are regular ideal polygons, or are all ideal triangles or quadrilaterals, then they coincide with Thurston stretch rays, that are directed geodesics in $\mathcal{T}(S)$ with respect to Thurston asymmetric metric. It is not yet understood if this is the case in general. In Section 10.5 we will study the limit at infinity of these generalized stretch rays.

9 Geodesic currents

Geodesic currents are a unifying concept introduced by Bonahon in [Bon88] that encompasses different geometric objects related to a surface, such as closed geodesics, measured laminations and hyperbolic metrics to name a few. Indeed, Bonahon used geodesic currents to reinterpret Thurston's compactification of Teichmüller space, giving in this way a more geometric point of view on the result. Later, further works showed that more structures on a surface can be represented as geodesic currents, like strictly negatively curved Riemannian metrics [Ota90], negatively curved metrics with cone singularities [HP97], singular flat metrics [DLR10], Hitchin representations [Lab07].

A self-contained introduction to geodesic currents can be found in [ESP22, Chapter 3] and in [AL17, Sections 3.4]. We will summarize here the definition and results that we will need in order to realize an embedding of $\mathcal{PT}(S)$ into the space of geodesic currents.

Introduction Let us consider a reference hyperbolic metric $X \in \mathcal{T}(S)$. This induces an identification of the universal cover $\tilde{S}$ with $\mathbb{H}^2$. Let us call $\mathscr{G}$ the space of unoriented geodesics in $\mathbb{H}^2$. Since every geodesic is determined by its endpoints at infinity, we can easily identify $\mathscr{G} \cong (\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta)/\sim$ where Δ is the diagonal and $(p, q) \sim (q, p)$. The action of the deck transformations group $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$ induces an action on $\partial_\infty \mathbb{H}^2$ and so on $\mathscr{G}$. A *geodesic current* is a Γ-invariant Radon measure on $\mathscr{G}$.

Bonahon showed that the space $\mathscr{G}$ together with the group action of $\pi_1(S) \cong \Gamma$ does not depend on the chosen metric X, up to $\pi_1(S)$-equivariant homeomorphisms. Let us denote with $\mathscr{C}(S)$ the space of geodesic currents endowed with the weak-* topology. Note that as a space of measures, it supports the addition operation and the rescaling by positive real numbers. In particular, it is a convex cone inside an infinite dimensional vector space (of signed measures). It is then possible to consider the projectivisation of the space of currents $\mathbb{P}\mathscr{C}(S)$ just by taking the quotient by the rescaling $\mathbb{R}^+$-action. Bonahon showed the crucial fact that $\mathbb{P}\mathscr{C}(S)$ is compact. This is one of the reasons why the space of currents has shown to be very useful as ambient space where to embed various structures on surfaces: if the embedding passes to the projectivisation, taking the closure of its image yields a compactification of the moduli space of such embedded structures.

Curves and laminations Any closed curve is represented in $\mathscr{C}(S)$ as the sum of Dirac deltas on all its lifts in $\mathbb{H}^2$. Similarly, a measured lamination λ is represented by a measure supported on the lines that are lifts in $\mathbb{H}^2$ of leaves of λ, and with mass given by the transverse measure. In particular, the space of measured laminations $\mathcal{ML}(S)$ properly embeds in the space of currents $\mathscr{C}(S)$.

Intersection form Bonahon showed that $\mathscr{C}(S)$ is equipped with a continuous non-degenerate positive bilinear form $i(\cdot,\cdot)$ called *intersection form.*

Call $I \subset \mathscr{G} \times \mathscr{G}$ the subset of the pairs of intersecting lines. It is naturally Γ-invariant. Note that every pair of geodesics $(r, l) \in I \subset \mathscr{G} \times \mathscr{G}$ is described equivalently by their intersection point $p \in \mathbb{H}^2$ and two tangent vectors to l and r respectively. From this we can see that Γ acts freely and properly discontinuously on I, as it does on $\mathbb{H}^2$, so the map $I \to I/\Gamma$ is a covering map. By taking the quotient by the Γ-action we also have the homeomorphism

$$I/\Gamma \leftrightarrow \{(v_1, v_2) \in \mathbb{P}TS \times \mathbb{P}TS \mid v_1 \neq v_2 \text{ have the same basepoint}\},$$

where $\mathbb{P}TS$ is the projectivisation of the tangent bundle, so the fibre over every point p is the space of directions tangent to p.

Definition 9.1. The intersection form is then defined as follows

$$i(\alpha, \beta) = \int_U d\alpha \, d\beta = (\alpha \times \beta)(I/\Gamma),$$

where U is a fundamental domain for the covering $I \to I/\gamma$ and $\alpha \times \beta$ is interpreted as the measure pushed-down to I/Γ.

Remark . The intersection form can be extended by linearity to a bilinear form on the whole vector space of signed currents. However, it is not continuous there, so we will always restrict ourselves to only consider intersection between (positive) geodesic currents.

When interpreting closed geodesics as atomic geodesic currents, the intersection form extends what is known as the geometric intersection number between curves. More precisely, if γ, η are two closed geodesics, then $i(\gamma, \eta)$ is the number of their transverse intersections, counted with multiplicity. Furthermore, Otal proved in [Ota90] that the intersection form of a geodesic current with all the closed curves completely determines the current. Indeed, we have the following result, where we remind that we call $\mathcal{C}$ the set of free homotopy classes of closed curves on S.

Theorem 9.2 ([AL17, Theorem 3.4.12]). *The following map is a proper embedding.*

$$i_*\colon \mathscr{C}(S) \to \mathbb{R}^{\mathcal{C}}, \quad \alpha \mapsto i(\alpha, \cdot)$$

The intersection of a current α with closed curves can then be seen as a length function $\gamma \to i(\alpha, \gamma)$. We will see, when we introduce Liouville currents in the next subsection, that when α is a Liouville current associated to a metric, this actually coincides with the length function with respect to the metric on the surface.

Filling currents and systole We notice that in general for a geodesic current α, the length function induced by the intersection $i(\alpha, \cdot)$ is not necessarily positive for every curve. For example if α is a simple closed curve in S, we can find another closed curve γ that does not intersect it, and so $i(\alpha, \gamma) = 0$. In general, in analogy with the length functions for metrics on surfaces, we can then define the *systole* of a geodesic current as follows.

$$\operatorname{sys}(\alpha) = \inf_{\gamma \in \mathcal{C}} i(\alpha, \gamma)$$

The geodesic currents that have positive systole are called *filling*. We denote by $\mathscr{C}_{\text{fill}}(S)$ the subspace of filling currents. Note that $\mathscr{C}_{\text{fill}}(S)$ is open and dense in $\mathscr{C}(S)$, as for every current α we can take any filling current β and have that the current $\alpha + t\beta$ is filling for any $t > 0$, and converges to α for t going to zero.

9.1 Liouville currents for grafted surfaces

A Liouville current has been constructed for hyperbolic metrics in many equivalent ways: using cross-ratios, as volume induced by a two-form in local coordinates for $\mathscr{G}$, or as geodesic flow invariant measure on the unit tangent bundle over the surface. These constructions adapt to our case of grafted surfaces too, but some extra care for details is needed to take into account of some phenomena that do not appear in the more regular hyperbolic case. We report here some of the constructions and discussion on Liouville currents for Riemannian metrics that can be found in [AL17, Section 3.5] and in [CFF92].

Transverse measures and geodesic flow invariant measures We introduce now other objects closely related to geodesic currents, that in the case of hyperbolic metrics are equivalent to geodesic currents, while in slightly more general settings, like ours of grafted surfaces, present subtle differences from currents.

Consider X a surface diffeomorphic to S equipped with a Riemannian metric. Let us assume that the metric is non-positively curved, or possibly only C^1 regular, but nevertheless locally CAT(0). Call T^1X the unit tangent bundle of X, that is the circle bundle over X obtained by taking only the tangent vectors of unit norm. The geodesic flow on the tangent bundle restricts to a geodesic flow φ^t on T^1X. The orbits form a one-dimensional foliation $\mathcal{F}'$ of T^1X. We call *flip* the homeomorphism that swaps every tangent vector v with $-v$. We notice that $T^1X/\text{flip} \cong \mathbb{P}TX$, so the foliation $\mathcal{F}'$ descends to a foliation $\mathcal{F}$ on $\mathbb{P}TX$.

Definition 9.3. A *transverse measure* for $\mathcal{F}$ is the association of a non-negative number to any subsurface of $\mathbb{P}TX$ which is transverse to $\mathcal{F}$, and that is invariant under homotopies preserving the transversality. Moreover, we require this to be additive on the disjoint union of transverse surfaces.

Proposition 9.4. *There is a natural correspondence between the geodesic flow invariant measures on T^1X, that are also flip invariant, and the transverse measures for $\mathcal{F}$.*

Sketch of the proof: We will identify a transverse measure for $\mathcal{F}$ in $\mathbb{P}TX$ with its lift in T^1X as a flip invariant transverse measure for $\mathcal{F}'$. Given a surface σ transverse for $\mathcal{F}$, there exists ϵ such that the flow-box $\sigma \times [0,\epsilon] \to \bigcup_{t\in[0,\epsilon]} \varphi^t(\sigma)$ is an embedding. So, given a transverse measure to $\mathcal{F}$, we can define a measure on T^1S by defining its value on flow boxes as the product of the transverse measure and the height ϵ of the flow box. One can check that this definition is flip invariant. Vice versa, given the flip and flow invariant measure, we can define a transverse measure on any subsurface σ as the mass given to an embedded ϵ-flowbox, divided by ϵ. □

Consider $\tilde{X}$ the universal cover of X and Γ its deck transformations group. By our assumption on the metric, $\tilde{X}$ is a CAT(0) space, in particular it has a well-defined visual boundary that we denote by $\partial_\infty \tilde{X}$. Let $\tilde{\mathcal{F}}$ be the lift of the foliation $\mathcal{F}$ to $\mathbb{P}T\tilde{X}$.

Definition 9.5. Let us call $\mathscr{G}(X)$ the space of the leaves of $\tilde{\mathcal{F}}$, that is the space of (unoriented) unit-speed geodesics in $\tilde{X}$.

It has naturally the topology of the uniform convergence on compact subsets. Any surface σ homeomorphic to a subset of $\mathbb{R}^2$ and transverse to $\mathcal{F}$ and intersecting each leaf at most once, provides then a chart for $\mathscr{G}(X)$, associating to every point of σ the leaf of $\mathcal{F}$ through it. So $\mathscr{G}(X)$ is a manifold of dimension two. By definition of the visual boundary, every geodesic in $\tilde{X}$ has two endpoints at infinity in $\partial_\infty \tilde{X}$, which are distinct since $\tilde{X}$ is CAT(0). Given a marking $S \to X$, there is a $\pi_1(S)$-equivariant homeomorphism between

$\partial_\infty \tilde{S}$ and $\partial_\infty \tilde{X}$ induced by the lift of the marking. Here we interpret S as endowed with a reference hyperbolic metric, so $\tilde{S} \cong \mathbb{H}^2$.

Remark . There is natural map $\xi_X : \mathscr{G}(X) \to \mathscr{G}$, also called *Morse map*, that associates to a leaf the (unordered) pair of its two endpoints at infinity, as we remind that $\partial_\infty \tilde{X}$ is naturally identified to $\partial_\infty \tilde{S}$.

Theorem 9.6 (Morse). *The Morse map ξ_X is continuous and proper, and if X is negatively curved, it is a homeomorphism.*

In this case, a transverse measure for $\mathcal{F}$ lifts to a Γ-invariant transverse measure for $\tilde{\mathcal{F}}$. Since a transverse measure is by definition invariant under homotopies that preserve transversality, it is equivalent to a measure on $\mathscr{G}(X)$. For the Morse theorem above, when X is negatively curved, this corresponds via ξ_X to a Γ invariant measure on $\mathscr{G}$, i.e. a geodesic current.

In the case of X a grafted surface instead, the Morse map is not always a homeomorphism. If there are flat cylinders in X, so flat strips in $\tilde{X}$, there are multiple geodesics that foliate the strip and then lie at constant distance from each other, and so have the same endpoints at infinity in the visual boundary $\partial_\infty \tilde{X}$. This same phenomenon arises when considering singular flat metrics in [Fra12, Section 4.2.2], and their strategy to deal with this issue while defining a Liouville current applies verbatim in our case too. By [CFF92, Proposition 1.3], the only case where two geodesics in $\tilde{X}$ remain at bounded distance for all times is the one we just described: when they bound a flat strip, which projects to a flat cylinder in X. Let us call such geodesics *strip geodesics*, and the elements of $\mathbb{P}T\tilde{X}$ tangent to strip geodesics, *strip directions*.

Remark . Note that the subset of $\mathscr{G}(X)$ of the strip directions is 1-dimensional: a one-parameter-family of parallel geodesics for every flat strip. In total there are only countably many flat strips, as there are countably many lifts for every flat cylinder in X, and the cylinders are finitely many.

In particular, if we avoid the strip directions, the same argument working in negative curvature, shows us the following.

Proposition 9.7. *The restriction of $\xi_X : \mathscr{G}(X) \to \mathscr{G}$ to the non-strip geodesics is a homeomorphism onto the image.*

Meanwhile, for every strip, which is a 1-dimensional collection of parallel strip geodesics, every geodesic is mapped to the same point in $\mathscr{G}$. With these tools, we are now ready to construct the embedding of $\mathcal{PT}(S)$ into the space of currents. To achieve this, we now follow the construction of Liouville currents reported in [AL17, Section 3.5.1-3.5.2].

Liouville currents On $\mathbb{P}TX$ there is a canonical measure ν_X associated to the metric X, called *Liouville measure*. It is the one induced by the volume form on $\mathbb{P}TX$ obtained as the wedge product of the Riemannian volume form on X and $d\theta$ where θ is the angle coordinate of each fibre of $\mathbb{P}TX$. One can check that ν_X is invariant under the geodesic flow (see again [AL17]). Using the correspondence we introduced before, between flow and flip invariant measures and transverse measures, ν_X induces a transverse measure L_X for the foliation $\mathcal{F}$, that lifts to a transverse measure for $\tilde{\mathcal{F}}$, that as we argued before, can be seen as a measure on $\mathscr{G}(X)$, and that for convenience we also call L_X.

We now show that L_X encodes length functions with respect to the metric X. For this reason consider a geodesic segment α in $\tilde{X}$ and call σ_α the surface in $\mathbb{P}TS$ formed by all the tangent directions transverse to α, with basepoint on α. We can parametrize σ_α as

$$\sigma_\alpha \colon [0,T] \times (0,\pi) \to \mathbb{P}T\tilde{X}, \quad \sigma_\alpha(t,\theta) = (\alpha(t),\theta),$$

where θ indicates the angle between the tangent direction and α.

Proposition 9.8. *The transverse measure L_X deposited on σ_α equals to the length of α.*

$$L_X(\sigma_\alpha) = \mathrm{Len}_X(\alpha)$$

Sketch of the proof: As discussed previously, σ_α is a chart for $\tilde{\mathcal{F}}$. There is a simple formula that expresses L_X as a volume form in this chart [AL17, Proposition 3.5.6].

$$L_X(v) = \frac{1}{2}\sin(\theta)\, d\theta\, dt$$

From this formula a simple integration shows the thesis.

$$L_X(\sigma_\alpha) = \frac{1}{2}\int_{[0,T]\times(0,\pi)} \sin(\theta) d\theta\, dt = \frac{1}{2}\left(\int_0^\pi \sin(\theta) d\theta\right)\left(\int_0^T dt\right) = \frac{1}{2}\cdot 2 \cdot T = T$$

□

We define the *Liouville current* as the push-forward though ξ_X of the transverse measure L_X, and still call it L_X. Notice that the subset of $\mathscr{G}(X)$ of strip directions, since it has dimension 1, is negligible with respect to the transverse measure L_X, being it induced by a 2-dimensional volume form. So the push-forward via ξ_X of L_X to $\mathscr{G}$ is the same as the push forward via the restriction of ξ_X to the subset of non-strip directions. In other words, the strip directions can be ignored, as they are negligible with respect to the Liouville current L_X.

Corollary 9.9. *If X is a negatively curved metric, or a grafted metric, then the Liouville current L_X induces, via the intersection form, the length function for X. So for any closed curve $\gamma \in \mathcal{C}$*

$$i(L_X, \gamma) = \ell_X(\gamma).$$

Proof. Let us remind that curves are interpreted as sum of Dirac delta measures on $\mathscr{G}$ on the lifts of γ. Consider the geodesic segment $\gamma'\colon [0,T] \to \tilde{X}$ which is a lift of γ. Using the definition of $i(L_X, \gamma)$, since γ is a Dirac delta, by integrating it, the integral reduces immediately to $L_X(\sigma_{\gamma'})$, which we showed already equals to $\mathrm{Len}_{\tilde{X}}(\gamma') = \ell_X(\gamma)$. □

We can now associate to a grafted metric X on S a geodesic current L_X that induces via the intersection form the length functions with respect to X. Thanks to Theorem 9.2, since the map $i_*\colon \mathscr{C}(S) \to \mathbb{R}^{\mathcal{C}}$ is an embedding, we can study the properties of the mapping into currents, simply by studying the length functions of the metric.

9.2 Compactification via geodesic currents

The constructions of the previous section have been carried out first in the more specific case of hyperbolic metrics by Bonahon with the goal of reinterpreting, with geodesic currents, Thurston's compactification of Teichmüller space.

Theorem 9.10 (Thurston). *The length functions induce a mapping $\mathcal{T}(S) \to \mathbb{R}^{\mathcal{C}}$ which is a proper* $\mathrm{Mod}(S)$*-equivariant embedding. Projectivising yields again an embedding $\mathcal{T}(S) \to \mathbb{P}\mathbb{R}^{\mathcal{C}}$, where the closure of the image of $\mathcal{T}(S)$ is homeomorphic to a closed disc.*

Bonahon, in the celebrated work [Bon88], replicated essentially the same result, but embedding $\mathcal{T}(S)$ into the space of geodesic currents.

Theorem 9.11 ([Bon88]). *The map $\mathcal{T}(S) \to \mathscr{C}(S)$ given by $X \mapsto L_X$ is a proper* $\mathrm{Mod}(S)$*-equivariant embedding. Projectivising yields a map $\mathcal{T}(S) \to \mathbb{P}\mathscr{C}(S)$ which is also a* $\mathrm{Mod}(S)$*-equivariant embedding. Taking the closure of the image of $\mathcal{T}(S)$ in $\mathbb{P}\mathscr{C}(S)$ yields Thurston's compactification of Teichmüller space, and the boundary consists of the projective currents coming from measured laminations.*

As we mentioned before, other studies have been carried out following Bonahon's idea, embedding other moduli spaces of geometric structures into the space of geodesic currents: negatively curved metrics [Ota90], possibly with conical singularities [HP97], half-translation surfaces [DLR10]. Our case of grafted surfaces does not follow from these works. However, the only difficulty in producing a Liouville current in our case, compared to the case of negative curvature, was the non-smoothness of the metric and the possible presence of flat strips. Problems that have been dealt with in other cases, for example for half-translation surfaces. We now observe that the Liouville metrics we produced in the previous section induce a nice map into the space of currents. But let us first prove the following lemma.

Lemma 9.12. *For any grafted surface* $\mathrm{Gr}(X,\lambda)$ *and closed curve* γ

$$\ell_{\mathrm{Gr}(X,\lambda)}(\gamma) \geq i(\lambda,\gamma). \tag{9.1}$$

Proof. Let us first consider the case of λ multicurve, say $\lambda = \sum_i a_i\gamma_i$. If γ crosses transversely n_i times γ_i in X, then the geodesic representative of γ in $\mathrm{Gr}(X,\lambda)$ has n_i subsegments contained in the cylinder corresponding to γ_i, connecting the two opposite boundary components, and so of length at least a_i. Then the length of γ in $\mathrm{Gr}(X,\lambda)$ is at least $\sum_i n_i a_i = i(\gamma,\lambda)$. If λ is any measured lamination, we can approximate it with a sequence of weighted multicurves μ_n, such that $\mu_n \to \lambda$ in $\mathscr{C}(S)$ and by continuity of the intersection form $i(\lambda_n,\gamma) \to i(\lambda,\gamma)$. On the left-hand side instead we have continuity of grafting telling us that $\mathrm{Gr}(X,\lambda_n) \to \mathrm{Gr}(X,\lambda)$, and by Corollary 3.15, also convergence of the length functions $\ell_{\mathrm{Gr}(X,\lambda_n)}(\gamma) \to \ell_{\mathrm{Gr}(X,\lambda)}(\gamma)$, which allows us to conclude. □

Theorem 9.13. *The map* $\mathcal{PT}(S) \to \mathscr{C}(S)$ *is continuous and proper.*

Proof. We showed that the Liouville currents of grafted surfaces represent their length function via the intersection form, that is the following diagram commutes.

$$\begin{array}{ccc} \mathcal{PT}(S) & \longrightarrow & \mathrm{MMet}(S) \\ \big\downarrow L. & & \big\downarrow \ell.(-) \\ \mathscr{C}(S) & \xrightarrow{\;i(\cdot,-)\;} & \mathbb{R}^{\mathcal{C}} \end{array} \tag{9.2}$$

We showed in Proposition 4.9 that the map $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is an embedding and in Proposition 3.14 that the map $\mathrm{MMet}(S) \to \mathbb{R}^{\mathcal{C}}$ is continuous. The bottom horizontal map is an embedding thanks to Theorem 9.2, in particular it has continuous inverse from its image. By commutativity of the diagram, then we have that $L.\colon \mathcal{PT}(S) \to \mathscr{C}(S)$ is also continuous.

To show properness it is enough to show that for a diverging sequence $\mathrm{Gr}(X_n,\lambda_n)$ of grafted surfaces, there is a closed curve whose length diverges. If X_n diverges, then up to subsequences it converges to a projective measured lamination η, meaning that there is a sequence a_n going to zero, such that $a_n L_{X_n} \to \eta$ in $\mathscr{C}(S)$. If we take any curve γ

transverse to η, then thanks to Corollary 2.10 we have that

$$\ell_{\mathrm{Gr}(X_n,\lambda_n)}(\gamma) \geq \ell_{X_n}(\gamma) = i(L_{X_n},\gamma).$$

But we also have that $a_n i(L_{X_n},\gamma) \to i(\lambda,\gamma) > 0$, so $i(L_{X_n},\gamma)$ must go to infinity. On the other side, if we have that X_n is bounded and λ_n diverges, meaning $\ell_{X_n}(\lambda_n)$ goes to infinity, then, up to subsequence, $\lambda_n \to [\lambda]$ projectively, meaning that there is a_n going to zero such that $a_n\lambda_n \to \lambda$ in $\mathscr{C}(S)$. Using Lemma 9.12 then we get

$$\ell_{\mathrm{Gr}(X_n,\lambda_n)}(\gamma) \geq i(\lambda_n,\gamma)$$

which must go to infinity as $a_n i(\lambda_n,\gamma) = i(a_n\lambda_n,\gamma) \to i(\lambda,\gamma) > 0$.

□

In order to understand if the map is actually an embedding, injectivity is left to be studied. Since a current is completely determined by the length function it induces via the intersection form, the injectivity is equivalent to the marked length spectrum rigidity for grafted surfaces.

Conjecture 9.14. Grafted surfaces are spectrally rigid, that is, two grafted surfaces have the same marked length spectrum if and only if they are isometric and the isometry preserves the markings.

Marked length spectrum rigidity has been studied extensively and been proved for various classes of metrics, but none including grafted surfaces. One closely related result can be found in [CFF92], where they prove the rigidity for smooth non-positively curved Riemannian metrics. More specifically for grafted surfaces, in [Kim99], it is shown that if two grafted surfaces $\mathrm{Gr}(X_1,\lambda_1)$ and $\mathrm{Gr}(X_2,\lambda_2)$ have the same length spectrum and $i(\lambda_1,\lambda_2)=0$, then they are isometric with an isometry compatible with the markings. Another related result on the marked length spectrum rigidity of grafted surfaces can also be found in [Sun16], comparing length spectra of grafted surfaces to the ones of hyperbolic surfaces.

We believe that the conjecture holds true, although we are not able to provide a proof yet.[1] In place of this, we were able to prove the injectivity for a more explicit mapping of $\mathcal{PT}(S)$ into currents that arises when considering the l^1-metric variant of grafting, that we will introduce in the next chapter. Meanwhile, the analogous rigidity result has been proved for half-translation surfaces.

[1]Post-submission update: Conjecture 9.14 holds true. A short argument is provided in Appendix A.3, adapting the proof from [CFF92].

Theorem 9.15 ([DLR10, Theorem 4]). *Given any half-translation surface q there exists a Liouville current L_q that induces the length function of q. The induced map $\mathcal{QT}(S)/S^1 \to \mathscr{C}(S)$, $q \mapsto L_q$ is a proper* Mod(S)*-equivariant embedding, and also after projectivising, $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \to \mathbb{P}\mathscr{C}(S)$ is also an embedding.*

Remark . Notice that the Liouville current depends only on the metric induced by q. For this reason the embeddings are considered from the quotients of $\mathcal{QT}(S)$ by S^1 and by $\mathbb{C}^*$, exactly as we did when considering the embeddings into MMet(S) in Proposition 4.10.

The advantage of considering the map into currents, as we mentioned before, is that the convergence can be deduced just by the convergence of the induced length functions. For this reason our main result of projective convergence in MMet(S) of grafted surfaces to half-translation ones is stronger and implies the convergence also in the setting of currents.

Proposition 9.16. *The image of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ in $\mathbb{P}\mathscr{C}(S)$ is contained in the closure of the image of $\mathcal{PT}(S)$.*

Proof. We showed the analogue of this in Theorem 4.15 for the embeddings into MMet(S). We proved in Proposition 3.14 that the map MMet$(S) \to \mathbb{R}^{\mathcal{C}}$ is continuous, so we also have that the marked length spectrum of half-translation surfaces is limit, up to rescaling, of the marked length spectra of grafted surfaces. Since Liouville currents are constructed for grafted surfaces and for half-translation surfaces to induce the length functions, and the mapping $\mathscr{C}(S) \to \mathbb{R}^{\mathcal{C}}$ induced by the intersection form is an embedding, then the convergence happens also at the level of Liouville currents. □

The space of projective currents is compact, so if Conjecture 9.14 holds true, taking the closure of the image of $\mathcal{PT}(S)$ in $\mathbb{P}\mathscr{C}(S)$ would yield a compactification of $\mathcal{PT}(S)$. Such a compactification will then strictly contain the bordification we obtained via MMet(S), as any sequence of grafted surfaces, even the degenerating ones, must admit accumulation points, as the closure in $\mathbb{P}\mathscr{C}(S)$ will be compact. Meanwhile, we can recover our bordification by restricting to filling currents, which is the analogue of avoiding the topological degeneration of the surface due to the pinching of a curve.

Proposition 9.17. *The boundary of the image of $\mathcal{PT}(S)$ in $\mathbb{P}\mathscr{C}_{\text{fill}}(S)$ is exactly the image of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$.*

Proof. Let us consider a diverging sequence Z_n in $\mathcal{PT}(S)$ with converging Liouville currents $L_{Z_n} \to \mu \in \mathbb{P}\mathscr{C}_{\text{fill}}(S)$. Since the mapping into currents is proper and Z_n diverge, also L_{Z_n} must diverge, meaning that there exists a sequence of scaling parameters a_n such that $a_n L_{Z_n} \to \mu$ in $\mathscr{C}(S)$, with $a_n \to 0$ for n going to infinity. Since $a_n \operatorname{sys}(Z_n) \to \operatorname{sys}(\mu)$ and filling currents have positive systole, then Z_n cannot degenerate, meaning that the area-normalized grafted surfaces associated to Z_n have uniformly lower bounded systole. Then, by Lemma 7.12 together with Theorem 7.1, Z_n is converging in $\mathbb{P}\,\mathrm{MMet}(S)$ to a half-translation surface q. But then by convergence of marked length spectrum, it must be that $\mu = L_q$. □

What we showed is that both taking the closure $\overline{\mathcal{PT}(S)}^{\mathscr{C}_{\text{fill}}(S)}$ of the image of the map $\mathcal{PT}(S) \to \mathbb{P}\mathscr{C}_{\text{fill}}$, and taking the closure $\overline{\mathcal{PT}(S)}^{\mathrm{MMet}(S)}$ of the image of the embedding $\mathcal{PT}(S) \to \mathbb{P}\,\mathrm{MMet}(S)$, has the effect of adding a boundary at infinity for $\mathcal{PT}(S)$ made of points that represent half-translation surfaces. We now show that the topology of the bordifications obtained are also essentially the same.

Theorem 9.18. *For sequences of grafted surfaces, the convergence in $\mathbb{P}\mathscr{C}_{\text{fill}}(S)$ is equivalent to the convergence in $\mathbb{P}\,\mathrm{MMet}(S)$. In particular, if Conjecture* 9.14 *holds true, the bordifications $\overline{\mathcal{PT}(S)}^{\mathscr{C}_{\text{fill}}(S)}$ and $\overline{\mathcal{PT}(S)}^{\mathrm{MMet}(S)}$ are equivalent.*

Proof. Since we have already that both $\mathcal{PT}(S)$ and $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ are homeomorphic to their images in both $\mathbb{P}\mathscr{C}_{\text{fill}}(S)$ and in $\mathbb{P}\,\mathrm{MMet}(S)$, then what is left to check is that the convergence to the boundary (that is convergence of grafted surface to half-translation surfaces) is equivalent when looking at the associated currents, or when considered with respect to marked Gromov-Hausdorff convergence.

We know that the embeddings above induce the same length spectrum, in other words Diagram (9.2) commutes. We also showed in Corollary 3.15 that marked Gromov-Hausdorff convergence implies convergence of the marked length spectrum, which in turns implies convergence of the currents inducing such lengths, since $i_* \colon \mathscr{C}(S) \to \mathbb{R}^{\mathcal{C}}$ is an embedding. So the convergence in $\mathbb{P}\,\mathrm{MMet}(S)$ implies convergence in $\mathbb{P}\mathscr{C}_{\text{fill}}(S)$.

Vice versa, if we have a sequence of area-renormalized grafted surfaces $\mathrm{Gr}(X_n, \lambda_n)$ whose associated Liouville currents converge projectively to the Liouville current representing q, then we have also convergence of the systole length for $\mathrm{Gr}(X_n, \lambda_n)$ as they are considered area normalized. In particular there is a uniform lower bound $\delta > 0$ for the systole lengths. Then by the previous Lemma 7.12 we also have a uniform lower bound $\delta/4$ for the systole lengths of the area-normalized half-translation surfaces $\mathcal{O}(X_n, \lambda_n)$.

Let us now show that the sequence $\mathcal{O}(X_n, \lambda_n)$ converges in $\mathbb{P}\,\mathrm{MMet}(S)$ to q. Since $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is homeomorphic to its image both in $\mathbb{P}\mathscr{C}(S)$ and in $\mathbb{P}\,\mathrm{MMet}(S)$, it is enough to show convergence of their currents, that is convergence of their marked length spectrum.

Observe first that because of the assumption on the projective convergence of the currents representing $\mathrm{Gr}(X_n, \lambda_n)$ to the one representing q, the marked length spectrum of $\mathrm{Gr}(X_n, \lambda_n)$ converges projectively pointwise to the one q. Moreover, we must have that $\ell_{X_n}(\lambda_n)$ goes to infinity. Then, by Theorem 7.1, we have that $\mathrm{Gr}(X_n, \lambda_n)$ is ϵ_n-isometric to $\mathcal{O}(X_n, \lambda_n)$ with $\epsilon_n = C_{\delta/4}\ell_{X_n}(\lambda_n)^{-1/2}$ going to zero. Since we have a uniform lower bound for the systole, thanks to Corollary 3.15, we have that the marked length spectrum of $\mathcal{O}(X_n, \lambda_n)$ is quantitatively close to the one of $\mathrm{Gr}(X_n, \lambda_n)$, which we showed is converging pointwise to the one of q. Therefore, the marked length spectrum of $\mathcal{O}(X_n, \lambda_n)$ also pointwise converges to the marked length spectrum of q.

Now that we have convergence of $\mathcal{O}(X_n, \lambda_n)$, once again, being it ϵ_n-isometric to $\mathrm{Gr}(X_n, \lambda_n)$, we get that the latter also converges in $\mathrm{MMet}(S)$ to q. □

In other words, we showed that the mapping $\mathcal{PT}(S) \to \mathscr{C}_{\mathrm{fill}}(S)$ extends continuously to the bordification $\overline{\mathcal{PT}(S)}^{\mathbb{P}\,\mathrm{MMet}(S)}$, and if Conjecture 9.14 holds true, the extension is an embedding. Let us now drop the restriction to filling currents and try to describe geometrically all the points of the closure in $\mathbb{P}\mathscr{C}(S)$.

Mixed structures In [DLR10] they study the closure of the embedding $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \to \mathbb{P}\mathscr{C}(S)$ and they show that all the limiting currents are the ones associated to the so-called *mixed structures*. Mixed structures arise in various similar contexts, and one can find in literature multiple definitions that are similar but not equivalent to each other. We report the one of [DLR10], as it is compatible with the interpretation of mixed structures as geodesic currents. Other definitions, compared to the one we will use, might retain additional data as decoration (for example as in [Bur+21b], [OT21], [Ouy23]).

We briefly recall that for holomorphic quadratic differentials on a surface with boundary we mean the following. When considering a conformal or complex structure on a surface with boundary, it is irrelevant if we replace each boundary component with a puncture. Then the holomorphic quadratic differential on the punctured surface is meant as the restriction of a meromorphic quadratic differential whose poles are simple and contained in the set of punctures. Note that simple poles can be considered as zeroes of order -1, and to them correspond cone singularities of angle π for the associated flat singular metric.

Definition 9.19. A *mixed structure* (Σ, q, λ) on S is the datum of a subsurface $\Sigma \subset S$ with a singular Euclidean metric coming from a quadratic differential q on Σ and a measured lamination supported in the complement $S \setminus \Sigma$.

By sub-surface we mean here a π_1-injective subsurface $\Sigma \subset S$ (possibly not connected, possibly empty, and possibly the whole surface) of negative Euler characteristic. Note that as far as it concerns the mixed structure, on the subsurface Σ we care only about the induced metric by q, so we can freely rotate q by multiplying it with a unitary complex number (possibly a different one for each connected component of Σ). In [DLR10] they show how a mixed structure is naturally described by a geodesic current obtained by summing the Liouville current associated to q, which we remind depends only on the induced metric, and λ seen as current. Let us call $\mathrm{Mix}(S)$ the space of currents associated to mixed structures. Then we have the following.

Theorem 9.20 ([DLR10, Theorem 5]). *In $\mathbb{P}\mathscr{C}(S)$, the closure of the space of half-translation surfaces $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is the space of projective currents associated to mixed structures $\mathbb{P}\,\mathrm{Mix}(S)$.*

Since we showed that in $\mathbb{P}\mathscr{C}(S)$ the image of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is contained in the closure of the image of $\mathcal{PT}(S)$, from the previous theorem follows that $\mathbb{P}\,\mathrm{Mix}(S)$ is also contained in such a closure. We believe that this is all one can get as limit of currents associated to grafted surfaces.

Conjecture 9.21. In $\mathbb{P}\mathscr{C}(S)$ we have that

$$\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}\,\mathrm{Mix}(S).$$

Idea for a proof: The containment $\mathbb{P}\,\mathrm{Mix}(S) \subset \overline{\mathcal{PT}(S)}$ follows from what we just discussed. We only need to show that any diverging sequence of grafted surfaces has Liouville currents projectively converging to a mixed structure.

Considering the grafted surfaces to be area normalized, there are mainly three cases for a diverging sequence. First case: $\ell_{X_n}(\lambda_n)$ goes to infinity and $\mathrm{sys}(\mathrm{Gr}(X_n, \lambda_n))$ stays lower bounded. Proposition 9.17 shows that in this case we have convergence up to subsequence to a half-translation surface.

In the second case $\ell_{X_n}(\lambda_n)$ stays bounded, while $\mathrm{sys}(\mathrm{Gr}(X_n, \lambda_n))$ goes to zero. In this case it is must be that X_n diverges, so up to subsequence, it converges to a projective measured lamination $[\eta]$. One can show that finite grafting cannot change length functions by much, except for curves that are transverse to η. Our claim is that this shows that the limit of the Liouville currents associated to $\mathrm{Gr}(X_n, \lambda_n)$ will converge to a measured lamination that might be different from η, but whose support will be contained in the support of η.

In the third case $\ell_{X_n}(\lambda_n)$ diverges and $\mathrm{sys}(\mathrm{Gr}(X_n, \lambda_n))$ goes to zero. In this case we need to show the convergence to a mixed structure. Our marked Gromov-Hausdorff topology by design do not admit convergence for such sequences. However, we believe that our same argument adapts to the context where one considers instead the equivariant Gromov-Hausdorff convergence introduced in [Pau88]. One way to think of this is that if the systole of $\mathrm{Gr}(X_n, \lambda_n)$ goes to zero, there might be still a subsurface Σ with a positive uniform lower bound for the systole: on this subsurface an adaptation of our argument should show that the grafted metric converges to a singular flat metric on Σ; while on the complement, the metric degenerates to an action on a $\mathbb{R}$-tree dual to a measured lamination supported in $S \setminus \Sigma$. This will yield the limit mixed structure. □

We will instead carry out completely this study on a very similar family of metrics, we will call l^1-metrics, where being able to write more explicit formulas for the embedding into the space of currents will allow us to conclude the analogue of the above conjecture in that setting.

10 l^1-grafting and l^1-half-translation surfaces

The goal of this chapter is to realize a compactification of $\mathcal{PT}(S)$ by studying a different embedding into the space of currents. This time we will not use Liouville currents for the Thurston metrics of grafted surfaces, but we will use currents that represent, via the intersection form, length functions for another class of metrics obtained by what we will call l^1-grafting.

For this scope we will start studying maps from $\mathcal{PT}(S)$ and $\mathcal{QT}(S)$ into geodesic currents, independently of the geometric description of l^1-metrics. Such maps have very simple and natural formulae defining them, so we believe that they are quite natural to consider, hence also of independent interest of the metrics whose length functions we will show they represent.

Similarly to what we achieved for the classic grafting, in this case if we restrict ourselves to taking the closure in projective filling currents, we obtain the bordification of $\mathcal{PT}(S)$ by half translation surfaces. The advantage compared to the previous section is that when taking the closure in the whole space of projective currents, it will also yield a compactification, but this time it will be easier to describe explicitly all the points in the boundary.

10.1 Embeddings into geodesic currents

In this section we introduce two very natural mappings from $\mathcal{PT}(S)$ and $\mathcal{QT}(S)$ into the space of geodesic currents and show that they are (or induce) embeddings.

Theorem 10.1. *The following map is a proper embedding.*

$$\Phi\colon \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathscr{C}(S)\,, \quad (X, \lambda) \mapsto L_X + \lambda$$

Proof. The map Φ is obtained by summing two proper embeddings of $\mathcal{T}(S)$ and of $\mathcal{ML}(S)$ into $\mathscr{C}(S)$, so it is automatically continuous and proper. We are left to show that Φ is injective. Let us assume that there exist $X, Y \in \mathcal{T}(S)$ and $\mu, \nu \in \mathcal{ML}(S)$ such that

$$L_X + \mu = L_Y + \nu.$$

As mentioned in [Bon88], when $X \neq Y$, the associated Liouville currents L_X, L_Y are not only different, but they are also singular to each other, meaning that there is a subset $U \subset \mathscr{G}$ that is of full measure for L_X, and negligible for the L_Y, i.e. $L_X(U^c) = L_Y(U) = 0$. As mentioned by Bonahon, this follows from the fact that the corresponding Liouville measure, the geodesic flow invariant measures on the unit tangent bundle of the surface, is ergodic.

Moreover, L_X is a Lebesgue measure when $\mathscr{G}$ is modelled by taking X as reference metric. From [Bon01] we know that the support of measured laminations has Hausdorff dimension zero, so they must have zero mass with respect to L_X. Up to subtracting the support of ν from U, we can then assume that also $\nu(U) = 0$. But then the equality above gives a contradiction as U has zero mass for the right-hand side and positive mass for L_X on the left-hand side, which is a contradiction. So $X = Y$, and then automatically from the equation follows also that $\mu = \nu$.

□

Let us now consider the following mapping of holomorphic quadratic differentials into geodesic currents.

Theorem 10.2. *The following map*

$$\mathcal{QT}(S) \to \mathscr{C}(S), \quad q \mapsto \mathrm{Re}(q) + \mathrm{Im}(q)$$

is a double cover onto its image, and then it induces an embedding $\mathcal{QT}(S)/\{\pm 1\} \to \mathscr{C}(S)$.

Proof. The map is continuous as taking the horizontal and vertical foliations gives an embedding $\mathcal{QT}(S) \to \mathcal{MF}(S) \times \mathcal{MF}(S)$, then $\mathcal{MF}(S)$ embeds in $\mathscr{C}(S)$ and the addition is continuous.

We also know that the image lies in the subspace $\mathscr{C}_{\text{fill}}(S)$ of filling geodesic currents. We now show that if we restrict the codomain to $\mathscr{C}_{\text{fill}}(S)$, then the map is proper. Assume there is a diverging sequence q_n such that $\mathrm{Re}(q_n)+\mathrm{Im}(q_n)$ converges in $\mathscr{C}(S)$. In particular both $\mathrm{Re}(q_n)$ and $\mathrm{Im}(q_n)$ do not diverge, so by compactness of $\mathbb{P}\mathcal{MF} \cong \mathbb{P}\mathcal{ML}(S)$, up to passing to a subsequence, we can assume that they both converge, respectively to η and λ. If η and λ were transverse and binding, then q_n would converge to $q = q(\eta, \lambda)$, which contradicts the assumption. Then $\eta + \lambda$ is not filling, proving properness of the map into the space of filling currents.

The map is not injective as $\mathrm{Re}(-q) = \mathrm{Im}(q)$ and $\mathrm{Im}(-q) = \mathrm{Re}(q)$, so q and $-q$ are mapped to the same current. But we can then take the map induced on the quotient $\mathcal{QT}(S)/\{\pm\,\mathrm{id}\}$ and show that it is injective, which combined with continuity and properness implies that it is an embedding. The injectivity follows from Theorem 10.26, as being $\mathrm{Re}(q) + \mathrm{Im}(q)$ filling, its decomposition is trivial, so its intersection graph is connected, and so it admits only one bipartition.

□

Theorem 10.3. *By passing to the projectivisation, the embeddings of Theorem* 10.1 *and Theorem* 10.2 *induce the embeddings*

$$\mathcal{PT}(S) \to \mathbb{P}\mathscr{C}(S) \quad \text{and} \quad \mathcal{QT}(S)/\mathbb{R}^* \to \mathbb{P}\mathscr{C}(S).$$

Proof. For the first embedding, we start by extending linearly the embedding Φ as follows.

$$\overline{\Phi}\colon \mathbb{R}^+ \times \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathscr{C}(S), \quad \varphi(a, X, \lambda) = a\,(L_X + \lambda)$$

The injectivity of $\overline{\Phi}$ follows from the same proof of the injectivity of Φ in Theorem 10.1: the argument does not change if we add a constant in front of the Liouville current. To show that $\overline{\Phi}$ is an embedding we just need to show the continuity of the inverse, meaning that if $a_n L_{X_n} + a_n\lambda_n$ converges to $aL_X + a\lambda$, then $(a_n, X_n, \lambda_n) \to (a, X, \lambda)$. Since $a_n\lambda_n$ cannot diverge, otherwise also $a_n L_{X_n} + a_n\lambda_n$ would diverge, then by compactness of $\mathbb{P}\mathcal{ML}(S)$ it converges up to subsequence to a measured lamination λ'. So we have that

$$\lim_{n\to\infty} a_n L_{X_n} + \lambda' = \lim_{n\to\infty} (a_n L_{X_n} + a_n\lambda_n) = aL_X + a\lambda.$$

But again λ' is negligible with respect to L_X, this means that $a_n L_{X_n}$ must converge to a measure which is non-singular to aL_X. By Theorem 9.11, the limit in $\mathscr{C}(S)$ of a sequence of rescaled Liouville currents can only be a rescaled Liouville current or a measured lamination. So the only possible option, in order to have that the previous equality holds, is for $a_n L_{X_n}$ to converge to aL_X, which in particular implies that a_n converges to a and X_n to X. From the equality above then it follows also that $\lambda' = a\lambda$, in particular λ_n converges to λ.

Now it is enough to notice that the embedding $\overline{\Phi}$ commutes with the free action by $\mathbb{R}^+$ given by scalar multiplication on the factor $\mathbb{R}^+$, so it induces an embedding of the quotient into projective currents $\mathcal{T}(S) \times \mathcal{ML}(S) \to \mathbb{P}\mathscr{C}(S)$.

Similarly, the embedding $\mathcal{QT}(S)/\{\pm 1\} \to \mathscr{C}(S)$ commutes with the free action by $\mathbb{R}^+$ given by scalar multiplication, so it induces an embedding of the quotient into the space of projective currents $\mathcal{QT}(S)/\mathbb{R}^+/\{\pm \mathrm{id}\} = \mathcal{QT}(S)/\mathbb{R}^* \to \mathbb{P}\mathscr{C}(S)$.

□

Let us observe that we can easily deduce that the image of $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is contained in the closure of the image of $\mathcal{PT}(S)$. Indeed, given $q = q(\eta, \lambda)$, by Theorem 9.11 we can find a sequence of metrics $X_n \in \mathcal{T}(S)$ converging in $\mathbb{P}\mathscr{C}(S)$ to $[\eta]$. Then it is sufficient to take a sequence of weights a_n diverging with the correct rate, such that $[X_n + a_n\lambda] \to [\eta + \lambda]$. In the next section we will introduce the l^1 metrics, so that we can interpret this convergence also in a more geometric sense. A more complete discussion on the whole closure of the image of the map $\mathcal{PT}(S) \to \mathbb{P}\mathscr{C}(S)$ is in Section 10.6.

10.2 l^1-grafting

In this section we will introduce a new class of geometric objects that we will call l^1-grafted surfaces. In the multicurve case, the difference from regular grafting consists in equipping the inserted cylinders not with a Euclidean metric, but with a metric of type l^1. Let us then first spend some words explaining more precisely what we mean by l^1-metric.

Consider $\mathbb{R}^2$ with the standard l^1-norm $\|(x,y)\|_{l^1} = |x| + |y|$. The norm induces a metric on the space, which we refer to as *l^1-metric*.

Remark . Alternatively, the l^1-metric can be seen also as the length metric induced by integrating the measures $|dx|$ and $|dy|$ transverse to the vertical and horizontal foliations of $\mathbb{R}^2$, and taking their sum. We notice in particular that the l^1-metric is not invariant under rotations of any angle, but it is for rotations of angle $\pi/2$.

If we consider a rectangle $[0,l] \times [0,h]$ in $\mathbb{R}^2$ equipped with the l^1-metric, and take the quotient of it by the horizontal translation of l, we obtain a cylinder of height h and circumference l, where the vertical and horizontal foliations associated to the metric are the ones parallel or orthogonal to the boundary. Below, when we refer to cylinders with l^1-metric, we will always imply that the metric is as we just described. We now define the operation of l^1-grafting.

Example . In the simple case of multicurves, as anticipated, we proceed as follows. Given $X \in \mathcal{T}(S)$ and a weighted multicurve λ, the l^1-grafted surface is obtained by inserting cylinders equipped with the l^1-metric, with heights determined by the weights of λ.

Let us now give the definition of l^1-grafting for the general case when λ is any measured lamination. We first recall that we call $\kappa\colon \operatorname{Gr}(X,\lambda) \to X$ the Thurston collapsing map (see Theorem 2.9), and that when we refer to λ as measured lamination on the grafted surface, we mean the special representative on $\operatorname{Gr}(X,\lambda)$, geodesic with respect to the metric of $\operatorname{Gr}(X,\lambda)$, where closed leaves with atomic transverse measure are replaced by a family of geodesics foliating a cylinder, with a uniform transverse measure (see Chapter 6.5).

Definition 10.4 (l^1-grafting). Given $(X,\lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$, the *$l^1$-grafted surface* $\operatorname{Gr}^{l^1}(X,\lambda)$ is defined as the grafted surface $\operatorname{Gr}(X,\lambda)$ but equipped with the following path metric. Given a smooth path $c\colon [0,1] \to \operatorname{Gr}(X,\lambda)$, we define its length as follows.

$$\operatorname{Len}_{\operatorname{Gr}^{l^1}(X,\lambda)}(c) = \operatorname{Len}_X(\kappa \circ c) + \int_c d\lambda \tag{10.1}$$

The induced distance between any two points is then the one defined as the infimum of the length of paths connecting them.

Remark . Observe that the definition above induces a well-defined length metric. Indeed, the representative of λ we consider on $\operatorname{Gr}(X,\lambda)$ has no atomic leaf, so the above defined length functional $\operatorname{Len}_{\operatorname{Gr}^{l^1}}$ is continuous. So, being S compact, the infimum of the lengths of all the arcs connecting two points is always attained. And if a path c has length zero, it must be constant. Indeed, $\operatorname{Len}_X(\kappa \circ c) = 0$ implies that the path is contained in a fibre of κ which is either a point, or a segment transverse to λ. In the latter case, $\int_c d\lambda = 0$ implies that c is constant.

Proposition 10.5. *In the case of λ multicurve, the length metric restricts to an l^1-metric on cylinders and a hyperbolic one on the complementary regions.*

Proof. Consider a path c contained in the cylinders $\kappa^{-1}(\lambda)$. The map κ restricted to the cylinders collapses each of them in the direction orthogonal to λ. If we call dl the length measure in X along λ, we can write

$$\operatorname{Len}_X(\kappa \circ c) = \int_{\kappa \circ c} dl = \int_c \kappa^* dl,$$

where $\kappa^* dl$ is the transverse measure to the foliation of the cylinders orthogonal to their boundaries. As observed before $\int_c (dl + d\lambda)$ is exactly the l^1-metric on the cylinders. On the other hand, if a path c is contained in a complementary region, then the second term in (10.1) is zero, and we know from Theorem 2.9 that κ is a local homeomorphism there. So the defined metric coincides with the hyperbolic one X on the complementary regions. □

Remark . Note that the l^1-metric on the cylinders is a Finsler metric, where the unit ball in the tangent space is a square rotated by $\pi/4$ with respect to the axes. We observe then that when the grafting lamination is a multicurve, the l^1-grafted metric is a piecewise Finsler metric. Usually in its definition, a Finsler metric is required to be smooth, or at least have some degree of continuity regularity, which we do not have in our case. In the general case of any measured lamination, it is even more subtle to talk about piecewise continuity as the grafted region can have empty interior. For these reason we renounce trying to inherit known properties of Finsler metrics in our case, and we will just use instead our definition of the l^1-grafted metric as a length metric.

As we did for the classic grafting, we can consider the map $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ induced by taking the l^1-grafted metrics. We believe that this map is a proper embedding as in the case of classical grafting. In the case of classic grafting we could just deduce the continuity of the map from Thurston's work, and then use that together with Theorem 9.2 to deduce the continuity of the mapping into currents. This time, for l^1-grafting, we will study directly the map into currents, as it is easier. We still believe that it is possible to show the continuity of l^1-grafting by constructing rather explicitly ϵ-isometries between l^1-grafted metrics, although we will not do it in this dissertation and focus instead on the embedding into currents.

Conjecture 10.6. The space $\mathrm{Gr}^{l^1}(X, \lambda)$ depends continuously on $(X, \lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$ with respect to the marked Gromov-Hausdorff convergence. In other words, the l^1-grafted metric induces a continuous map $\mathcal{PT}(S) \to \mathrm{MMet}(S)$.

We want to study instead directly geodesic currents associated to such metrics, that is currents that induce the same length functions.

Remark . Notice that the l^1-metric on $\mathbb{R}^2$ is not locally CAT(0). Indeed, quite on the opposite, it is not CAT(k) for any k. For example one can verify that the perimeter of an arbitrarily small square determines a closed geodesic in $\mathbb{R}^2$ with respect to the l^1-metric.

As a consequence, even if length functions are still well-defined, there is not a bijection between homotopy classes and closed geodesics as there are geodesics that are not length minimizing in their homotopy class. However, we will show that in every free homotopy class of closed curves in $\mathrm{Gr}^{l^1}(X, \lambda)$ there is essentially only one representative that minimizes the length. This representative is not unique only in the case when the closed curve is a closed leaf of λ.

Lemma 10.7. *In an l^1-grafted surface, the closed curves that are length minimizers in their homotopy class are either contained in a cylinder and parallel to its boundary, or preimages via the Thurston collapsing map $\kappa\colon \mathrm{Gr}(X,\lambda) \to X$ of closed geodesics on X.*

Proof. From the definition of the metric, it follows that given a closed curve γ in $\mathrm{Gr}^{l^1}(X,\lambda)$, its length is at least $\ell_X(\gamma) + i(\gamma,\lambda)$ as we can lower bound each of the terms in (10.1) as follows.

$$\mathrm{Len}_X(\kappa \circ \gamma) \geq \ell_X(\gamma) \quad \text{and} \quad \int_\gamma d\lambda \geq i(\gamma,\lambda)$$

The first inequality coming from the definition of length function and the second one from the interpretation of the intersection number as the minimal geometric intersection between curves. This same lower bound holds for every curve in the same homotopy class of γ. We show that in every homotopy class of closed curves such a minimum is always realized. Assume γ realizes the minimum, then both the inequalities above must hold and be the equalities. The first one implies that $\kappa(\gamma)$ is geodesic in X.

Let us distinguish two cases. If $\kappa(\gamma)$ is a closed leaf of λ, then $i(\gamma,\lambda) = 0$, so the second equality holds only when $\int_\gamma \lambda = 0$, that is when γ is a curve in the cylinder $\kappa^{-1}(\kappa(\gamma))$ and is parallel to its boundary. In the second case $\kappa(\gamma)$ is a geodesic transverse to λ. Then γ is forced to coincide with the preimage $\kappa^{-1}(\kappa(\gamma))$, which we saw in Chapter 5 being a closed curve. It is known that geodesics realize the minimum intersection number in their homotopy class, so also the second equality $\int_\gamma d\lambda = i(\gamma,\lambda)$ holds. □

As we mentioned before, for the Liouville embedding of $\mathcal{T}(S)$ into the space of geodesic currents, the intersection form induces the length functions for the hyperbolic metric. We can then easily deduce that the embedding of Theorem 10.1 induces, via the intersection form, the length functions for the l^1-grafted surfaces.

Corollary 10.8. *For any $(X,\lambda) \in \mathcal{T}(S) \times \mathcal{ML}(S)$, the intersection pairing with $L_X + \lambda$ gives the length function with respect to the l^1-grafted metric. So for any simple closed curve γ,*

$$i(L_X + \lambda, \gamma) = \ell_{\mathrm{Gr}^{l^1}(X,\lambda)}(\gamma).$$

Proof. It follows from the previous lemma observing that by linearity of the intersection form we have that

$$i(L_X + \lambda, \gamma) = i(L_X, \gamma) + i(\lambda, \gamma) = \ell_X(\gamma) + i(\lambda, \gamma).$$

□

Moreover, observe that the l^1-grafting is not that different from the regular grafting, in the following sense.

Proposition 10.9. *The marked length spectrum of* $\mathrm{Gr}(X,\lambda)$ *and of* $\mathrm{Gr}^{l^1}(X,\lambda)$ *are* $\sqrt{2}$*-bilipschitz related, that is for every* γ *closed curve*

$$\frac{1}{\sqrt{2}}\,\ell_{\mathrm{Gr}(X,\lambda)}(\gamma) \leq \ell_{\mathrm{Gr}^{l^1}(X,\lambda)}(\gamma) \leq \sqrt{2}\,\ell_{\mathrm{Gr}(X,\lambda)}(\gamma)$$

Proof. For λ multicurve it is easy to see: the identity is a local isometry in the hyperbolic pieces, while on the cylinders, the l^1 and the Euclidean metrics are $\sqrt{2}$-bilipschitz. So the identity is globally $\sqrt{2}$-bilipschit between the two metrics.

For the general case we can approximate any measured lamination λ by a sequence of multicurves μ_n. Then we have that the length functions with respect to the classic grafting converge because in Proposition 4.9 we showed $\mathcal{PT}(S) \to \mathrm{MMet}(S)$ is an embedding; and the length functions with respect to the l^1-grafting also converge because in Theorem 10.1 we showed that $\mathcal{PT}(S) \to \mathscr{C}(S)$ is an embedding. Then the inequalities pass to the limit. □

10.3 l^1 half-translation surfaces

We now introduce an l^1-type metric we can associate to half-translation surfaces.

Definition 10.10 (l^1-half-translation surfaces). Given $q \in \mathcal{QT}(S)$ we define q^{l^1} as the associated half-translation surface, but instead of having the usual Euclidean metric, we endow the surface with the pull-back of the standard l^1-metric on $\mathbb{R}^2$ via the charts given by the half-translation structure.

Remark . As observed before, differently from the Euclidean case, the l^1-metric is not invariant under rotations of arbitrary angles, but only under rotations of $\pi/2$, corresponding to multiplying q by -1.

Proposition 10.11. *The embedding of Theorem* 10.2 *induces the length functions with respect to the* l^1*-metric on half-translation surfaces, i.e., for any* $q \in \mathcal{QT}(S)$ *and* γ *closed curve,*

$$\ell_{q^{l^1}}(\gamma) = i(\mathrm{Re}(q) + \mathrm{Im}(q), \gamma).$$

Proof. It follows immediately from the fact that the l^1-length of a path in $\mathbb{R}^2$ equals the sum of the integrals of the transverse measures to the vertical and horizontal foliations along the path. □

Lemma 10.12. *The metric space q^{l^1} depends continuously on $q \in \mathcal{QT}(S)$ with respect to the marked Gromov-Hausdorff convergence.*

Proof. Charts for $\mathcal{QT}(S)$ are given by the set of coordinates of the vertices in a representation of q as polygon with glued sides. It is easy to see that under small variations of such coordinates, the polygon with the l^1-metric will be ϵ-isometric to the original one. □

10.4 Metric convergence in the l^1-metric case

In this subsection we will show that, similarly to what we showed for the standard grafting, also in the case of l^1-metrics, we have convergence of sequences of grafted surfaces to half-translation ones. More precisely the following analogue of Theorem 7.1 holds. We remind that also in this case the surfaces are meant to be considered renormalized, meaning that $\mathrm{Gr}^{l^1}(X, \lambda)$ is considered rescaled by the factor $\ell_X(\lambda)^{-1/2}$ and half-translation surfaces $q^{l^1} = \mathcal{O}^{l^1}(X, \lambda)$ are rescaled to have $i(\mathrm{Re}(q), \mathrm{Im}(q)) = 1$. Note that these are the same renormalization we used in the case of classical grafting.

Theorem 10.13. *Let $\mathrm{Gr}^{l^1}(X, \lambda)$ be an l^1-grafted surface. Assume $\mathcal{O}^{l^1}(X, \lambda)$ has injectivity radius larger than ϵ and that $\ell_X(\lambda) > 1$. Then there exists C depending only on ϵ such that the deflation map $\mathcal{D}\colon \mathrm{Gr}^{l^1}(X, \lambda) \to \mathcal{O}^{l^1}(X, \lambda)$ is a $C \cdot (\ell_X(\lambda))^{-1/2}$-isometry compatible with the markings. In particular*

$$d_{GH}(\mathrm{Gr}^{l^1}(X, \lambda), \mathcal{O}^{l^1}(X, \lambda)) \leq C \cdot (\ell_X(\lambda))^{-1/2} .$$

In order to prove this we first show a couple of lemmas, that we also proved for regular grafting, and that in this case we can prove more directly as we have an explicit expression for the length of paths.

Lemma 10.14. *The deflation map $\mathcal{D}\colon \mathrm{Gr}^{l^1}(X, \lambda) \to \mathcal{O}^{l^1}(X, \lambda)$ is 1-Lipschitz.*

Proof. Consider the orthogeodesic foliation $\mathcal{O}_\lambda(X)$ on X. By definition, its transverse measure along the lamination λ coincides with the hyperbolic length along λ. We remind that the closest point projection from $X \setminus \mathrm{Sp}$ to λ is locally 1-Lipschitz. So we can observe that for any path c in X, its hyperbolic length is always greater than its transverse measure given by $\mathcal{O}_\lambda(X)$. Then given two points x, y in $\mathrm{Gr}^{l^1}(X, \lambda)$, consider α the shortest geodesic segment connecting them. We have that

$$\begin{aligned}
\mathrm{Len}_{\mathrm{Gr}^{l^1}(X,\lambda)}(\alpha) &= \mathrm{Len}_X(\kappa\circ\alpha) + \int_c d\lambda \\
&\geq \int_c d\mathcal{O}_\lambda(X) + \int_c d\lambda \\
&= \mathrm{Len}_{\mathcal{O}^{l^1}(x,\lambda)}(\mathcal{D}\circ\alpha) \\
&\geq d_{\mathcal{O}^{l^1}(x,\lambda)}(\mathcal{D}(x),\mathcal{D}(y))
\end{aligned}$$

□

Let us remind that for a geodesic arc α in $\mathcal{O}^{l^1}(x,\lambda)$, we defined in Chapter 7.1 a way to construct an arc $\beta = \mathcal{D}^*\alpha$ in $\mathrm{Gr}^{l^1}(X,\lambda)$, which we called pull-back of α, such that $\mathcal{D}\circ\beta = \alpha$. The main point in the argument to show that $\mathcal{D}$ is an ϵ-isometry in the case of regular grafting was that we could compare easily the length of β to the length of α. The exact same strategy works in the case of l^1-metrics as well. Let us call once again β_{infl} the portion of β contained in the interior of the negatively curved regions. Then we have the following analogue of Lemma 7.5.

Lemma 10.15. *Given α a geodesic arc in $\mathcal{O}^{l^1}(X,\lambda)$ and $\beta = \mathcal{D}^*\alpha$ its pull-back in $\mathrm{Gr}^{l^1}(X,\lambda)$, then*

$$\mathrm{Len}(\beta) = \mathrm{Len}(\alpha) + \mathrm{Len}(\beta_{\mathrm{infl}}).$$

Proof. We interpret the length functional Len as the induced 1-Lebesgue measure along the arc β. Let us partition β in $\beta_{\mathrm{infl}} \cup (\beta\setminus\beta_{\mathrm{infl}})$. Observe that $\beta\setminus\beta_{\mathrm{infl}}$ is by definition mapped by κ inside λ, where the hyperbolic length coincides with the transverse measure to $\mathcal{O}_\lambda(X)$. Then we obtain

$$\begin{aligned}
\mathrm{Len}(\beta\setminus\beta_{\mathrm{infl}}) &= \mathrm{Len}_X(\kappa(\beta\setminus\beta_{\mathrm{infl}})) + \int_{\beta\setminus\beta_{\mathrm{infl}}} d\lambda \\
&= \int_{\kappa(\beta\setminus\beta_{\mathrm{infl}})} d\mathcal{O}_\lambda(X) + \int_{\beta\setminus\beta_{\mathrm{infl}}} d\lambda \\
&= \int_{\beta\setminus\beta_{\mathrm{infl}}} d\mathcal{O}_\lambda(X) + \int_{\beta\setminus\beta_{\mathrm{infl}}} d\lambda \\
&= \int_\beta d\mathcal{O}_\lambda(X) + \int_\beta d\lambda \\
&= \mathrm{Len}_{\mathcal{O}^{l^1}(X,\lambda)}(\alpha),
\end{aligned}$$

where we used first that κ is a contraction along the leaves of $\mathcal{O}_\lambda(X)$ so it preserves the transverse measure with respect to $\mathcal{O}_\lambda(X)$ considered as foliation on $\mathrm{Gr}^{l^1}(X,\lambda)$ and on X; and then that β_{infl} has zero transverse measure both with respect to λ and to $\mathcal{O}_\lambda(X)$.

By the definition of the l^1-grafted metric, we have that κ is a local isometry on the interior of the negatively curved part so we can conclude that

$$\mathrm{Len}(\beta) = \mathrm{Len}(\beta \setminus \beta_{\mathrm{infl}}) + \mathrm{Len}(\beta_{\mathrm{infl}}) = \mathrm{Len}(\alpha) + \mathrm{Len}(\beta_{\mathrm{infl}}).$$

□

Finally, we can prove the marked Gromov-Hausdorff convergence of grafted surfaces to half-translation ones, also in the case of l^1-metrics.

Proof of Theorem 10.13. The proof is identical to the one of the analogue Theorem 7.1.

Let us consider the deflation map $\mathcal{D}\colon \mathrm{Gr}^{l^1}(X,\lambda) \to \mathcal{O}^{l^1}(X,\lambda)$, defined as before, but this time the two spaces are considered with the l^1-metrics. By Lemma 10.14 it is 1-Lipschitz also with respect to the l^1-metrics. We want to show that $\mathcal{D}$ also does not shrink distances by more than an additive error proportional to $k^{-1} = \ell_X(\lambda)^{1/2}$, and then conclude by Lemma 3.3. More precisely, we want to show that for any two points x, y in $\mathrm{Gr}^{l^1}(X,\lambda)$ the following bound holds.

$$d(\mathcal{D}(x), \mathcal{D}(y)) \geq d(x,y) - Ck^{-1} \tag{10.2}$$

Let us consider the geodesic arc α joining $\mathcal{D}(x), \mathcal{D}(y)$ and realizing their distance on $\mathcal{O}^{l^1}(X,\lambda)$. Let us consider the pull-back arc $\beta = \mathcal{D}^*\alpha$ defined as in Definition 7.4.

As before, the length of β gives an upper bound for the distance between x and y. By Lemma 10.15 we have that $\mathrm{Len}(\beta) = \mathrm{Len}(\alpha) + \mathrm{Len}(\beta_{infl})$ and then

$$d(\mathcal{D}(x), \mathcal{D}(y)) = \mathrm{Len}(\alpha) = \mathrm{Len}(\beta) - \mathrm{Len}(\beta_{infl}) \geq d(x,y) - \mathrm{Len}(\beta_{infl}).$$

The upper bound $\mathrm{Len}(\beta_{infl}) \leq Ck^{-1}$ we showed in the proof of Theorem 7.1 with C depending only on S and ϵ, still holds, as β_{infl} lies in the negatively curved part of the grafted surface, which is unchanged between the regular and l^1-grafting. This concludes, showing that $\mathcal{D}$ is a Ck^{-1}-isometry also for the l^1-metrics. □

In particular, as in the case of classic grafting, given q, there is a special one-parameter family of l^1-grafted surfaces that converges to q obtained as preimage via $\mathcal{O}$ of a scaling ray $t \mapsto tq$ in $\mathcal{QT}(S)$. The combination of this corollary and the explicit formula for the embedding $\mathcal{PT}(S) \to \mathscr{C}(S)$ will allow us to extend in the next section a result of Papadopoulos and study the behaviour of the map $\mathcal{O}_\lambda$ at infinity.

Corollary 10.16. *Consider the path of the form $(X_t, t\lambda) = \mathcal{O}^{-1}(tq)$, then $\mathrm{Gr}^{l^1}(X_t, t\lambda)$, considered rescaled by $\ell_X(\lambda)^{-1/2}$, converges according to the marked Gromov-Hausdorff topology to $\mathcal{O}^{l^1}(X_t, t\lambda) = q^{l^1}$ considered rescaled to have $i(\mathrm{Re}(q), \mathrm{Im}(q)) = 1$.*

Notice that Theorem 10.13 assumes the thickness of the renormalized l^1-half-translation surface $\mathcal{O}^{l^1}(X, \lambda)$. As it happened for the regular grafting, we can upgrade the statement to requiring only the thickness of the renormalized l^1-grafted surface $\mathrm{Gr}^{l^1}(X, \lambda)$, assuming $\ell_X(\lambda)$ large enough. The argument we used to show this in Chapter 7.5 is quite complicated, so instead of reproducing it in this context, we import directly the result, up to losing a factor of 2 on the multiplicative constants, by using the bilipschitz comparison of Proposition 10.9 between length functions with respect to regular and l^1-grafted metrics.

Lemma 10.17. *Given $\delta > 0$ there exist L such that if $\mathrm{Gr}^{l^1}(X, \lambda)$ is a rescaled l^1-grafted surface with $\ell_X(\lambda) > L$, and $\mathrm{sys}(\mathrm{Gr}^{l^1}(X, \lambda)) > \delta$, then $\mathrm{sys}(\mathcal{O}^{l^1}(X, \lambda)) > \delta/4$.*

Proof. By Proposition 10.9, since $\mathrm{Gr}^{l^1}(X, \lambda)$ has systole at least δ, then the regularly grafted rescaled surface $\mathrm{Gr}(X, \lambda)$ has systole at least $\delta/\sqrt{2}$. Then by Lemma 7.12 there exists L such that for $\ell_X(\lambda) > L$ then $\mathcal{O}(X, \lambda)$ has systole length at least $\delta/2\sqrt{2}$. And again by the bilipschitz comparison of Proposition 10.9, the systole of $\mathcal{O}^{l^1}(X, \lambda)$ must be at least $\delta/4$. □

We now show the analogue of Theorem 9.18.

Proposition 10.18. *If $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ in $\mathcal{PT}(S)$ are such that there is convergence at the level of projective currents to $q \in \mathcal{QT}(S)$, that is $[L_{X_n} + \lambda_n] \to [\mathrm{Re}(q) + \mathrm{Im}(q)]$, then the convergence is also geometric, that is $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ converge to q with respect to the marked Gromov-Hausdorff topology.*

Proof. The argument is the same as in the proof of Theorem 9.18. If the currents $\Phi(X_n, \lambda_n) = L_{X_n} + \lambda_n$ converge projectively to the current $[\mathrm{Re}(q) + \mathrm{Im}(q)]$ representing q^{l^1}, then we have also convergence of the systole length for $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ when considered renormalized. In particular there is a uniform lower bound $\delta > 0$ for the systole lengths. Then by the previous Lemma 10.17 we also have a uniform lower bound $\delta/4$ for the systole lengths of the renormalized half-translation surfaces $\mathcal{O}^{l^1}(X_n, \lambda_n)$.

Let us now show that the sequence $\mathcal{O}^{l^1}(X_n, \lambda_n)$ converges in $\mathbb{P}\,\mathrm{MMet}(S)$ to q^{l^1}. Since $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is homeomorphic to its image both in $\mathbb{P}\mathscr{C}(S)$ and in $\mathbb{P}\,\mathrm{MMet}(S)$, it is enough to show convergence of their currents, that is convergence of their marked length spectrum. Observe first that because of the assumption on the projective convergence of the currents representing $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ to the one representing q^{l^1}, the marked length spectrum of $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ converges projectively pointwise to the one of q^{l^1}. Moreover, we must have that $\ell_{X_n}(\lambda_n)$ goes to infinity. Then, by Theorem 10.13, we have that $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ is ϵ_n-isometric to $\mathcal{O}^{l^1}(X_n, \lambda_n)$ with $\epsilon_n = C_{\delta/4}\ell_{X_n}(\lambda_n)^{-1/2}$ going to zero. Since we have a uniform lower bound for the systole, thanks to Corollary 3.15, we have that the marked length spectrum of $\mathcal{O}^{l^1}(X_n, \lambda_n)$ is quantitatively close to the one of $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$, which we showed being converging pointwise to the one of q^{l^1}, so it also pointwise converges to the one of q^{l^1}.

Now that we have convergence of the sequence $\mathcal{O}^{l^1}(X_n, \lambda_n)$ to q^{l^1} in $\mathbb{P}\,\mathrm{MMet}(S)$, once again, being $\mathcal{O}^{l^1}(X_n, \lambda_n)$ ϵ_n-isometric to $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$, we get that the sequence $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ also converges in $\mathrm{MMet}(S)$ to q^{l^1}.

□

10.5 Limits of generalized stretch rays

In this section we combine the geometric convergence along generalized stretch rays of Corollary 10.16, with the explicit formula of Theorem 10.1 for the embedding $\mathcal{PT}(S) \to \mathscr{C}(S)$ to deduce a generalization of a previous result by Papadopoulos.

Let us remind that as we stated in Theorem 6.3, Calderon and Farre proved that the map $\mathcal{O}_\lambda \colon \mathcal{T}(S) \to \mathcal{MF}(\lambda)$ is a homeomorphism. In the case when λ is maximal, usually called F_λ is obtained by considering the horocycle foliation and had already been studied by many, among which Thurston and later Mirzakhani, with the name of F_λ, and it is obtained by considering the horocycle foliation.

In this case with λ maximal it was shown in [Pap91] that F_λ^{-1} extends to the boundary in the following sense. For $\mathcal{MF}(\lambda)$ we can consider the projective bordification obtained by adding at infinity the limit points of rays in $\mathcal{MF}(\lambda)$, meaning $\overline{\mathcal{MF}(\lambda)} = \mathcal{MF}(\lambda) \cup \mathbb{P}\mathcal{MF}(\lambda)$, while for Teichmüller space we just consider the Thurston compactification $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$. Then we have the following.

Theorem 10.19 ([Pap91]). *For any $\lambda \in \mathcal{ML}(S)$ maximal, the homeomorphism $F_\lambda^{-1} \colon \mathcal{MF}(\lambda) \to \mathcal{T}(S)$ extends by continuity to a map*

$$F_\lambda^{-1} \colon \overline{\mathcal{MF}(\lambda)} \to \overline{\mathcal{T}(S)}$$

which restricts to the inclusion $\mathbb{P}\mathcal{MF}(\lambda) \hookrightarrow \mathbb{P}\mathcal{ML}(S)$ on the boundary.

We now show that this holds in general for the map $\mathcal{O}_\lambda$ for any $\lambda \in \mathcal{ML}(S)$.

Theorem 10.20. *The homeomorphism* $\mathcal{O}_\lambda^{-1} \colon \mathcal{MF}(\lambda) \to \mathcal{T}(S)$ *extends by continuity to a map*

$$\mathcal{O}_\lambda^{-1} \colon \overline{\mathcal{MF}(\lambda)} \to \overline{\mathcal{T}(S)}$$

which restricts to the inclusion $\mathbb{P}\mathcal{MF}(\lambda) \hookrightarrow \mathbb{P}\mathcal{ML}(S)$ *on the boundary.*

Proof. Consider $\eta \in \mathcal{MF}(\lambda)$ a measured foliation transverse to λ and call $q = q(\eta, \lambda)$. By definition, the ray $t \mapsto t\eta$ converges projectively to the point $[\eta] \in \mathbb{P}\mathcal{MF}(\lambda)$. Consider now the ray of quadratic differentials $q_t = q(t\eta, t\lambda) = tq$, then

$$\mathcal{O}^{-1}(q_t) = \mathcal{O}^{-1}(t\eta, t\lambda) = (\mathcal{O}_{t\lambda}^{-1}(t\eta), t\lambda) = (\mathcal{O}_\lambda^{-1}(t\eta), t\lambda).$$

Call $X_t = \mathcal{O}_\lambda^{-1}(t\eta)$. Such paths, as we mentioned at the end of Chapter 8, are the so called generalized stretch rays. We want to show that X_t converges to $[\eta]$ in the Thurston boundary of Teichmüller space. By Corollary 10.16 we have marked Gromov-Hausdorff convergence of $\mathrm{Gr}^{l^1}(X_t, t\lambda)$ to q^{l^1}, so by Proposition 3.14 we have projective convergence of their marked length spectrum, then also the convergence of the projective currents $[L_{X_t} + t\lambda] \to [\eta + \lambda]$, meaning that there is a sequence $a_t > 0$ such that $a_t(L_{X_t} + t\lambda) \to \eta + \lambda$ in $\mathscr{C}(S)$.

We can take for example a_t such that the self intersection number of $a_t(L_{X_t} + t\lambda)$ is constantly equal to the one of $\eta + \lambda$.

$$i(\eta + \lambda, \eta + \lambda) = i(\lambda, \lambda) + i(\eta, \eta) + 2i(\eta, \lambda) = 2i(\eta, \lambda)$$

To compute the self intersection of $a_t(L_{X_t} + t\lambda)$ we will use that $i(L_X, L_X) = \pi^2|\chi(S)|$ for any $X \in \mathcal{T}(S)$ and that

$$i(L_X, \lambda) = \ell_X(\lambda) = i(\mathcal{O}_\lambda(X), \lambda),$$

where the second equality is due to [CF24b, Lemma 5.7]. Then we get the following.

$$\begin{aligned} i(a_t(L_{X_t} + t\lambda), a_t(L_{X_t} + t\lambda)) &= a_t^2(\pi^2|\chi(S)| + 2i(L_{X_t}, t\lambda)) \\ &= a_t^2(\pi^2|\chi(S)| + 2i(\mathcal{O}_\lambda(X_t), t\lambda)) \\ &= a_t^2(\pi^2|\chi(S)| + 2i(t\eta, t\lambda)) \\ &= a_t^2(\pi^2|\chi(S)| + 2i(\eta, \lambda)t^2) \end{aligned}$$

And from this we obtain

$$a_t = \sqrt{\frac{2i(\eta,\lambda)}{2i(\eta,\lambda)t^2 + \pi^2|\chi(S)|}} \sim \frac{1}{t}$$

In particular, $a_t t \to 1$ as t goes to infinity. So from

$$a_t L_{X_t} + a_t t\lambda = a_t(L_{X_t} + t\lambda) \to \eta + \lambda$$

we can deduce, since $a_t t\lambda \to \lambda$, that $a_t L_{X_t} \to \eta$ as currents, meaning $X_t \to [\eta]$ in the Thurston boundary, as we wanted.

□

10.6 Compactification and l^1-mixed structures

In the previous sections we showed that the map $\Phi\colon \mathcal{PT}(S) \to \mathscr{C}(S)$ that represents the length functions of l^1-grafted surfaces is a proper embedding, and we showed also that the composition with the projection to the projectivisation $\mathcal{PT}(S) \to \mathbb{P}\mathscr{C}(S)$ is also an embedding. Let us now study the closure in $\mathbb{P}\mathscr{C}(S)$ of the image via Φ of $\mathcal{PT}(S)$. The map Φ is obtained as sum of the two proper embeddings $\mathcal{T}(S) \to \mathscr{C}(S)$ and $\mathcal{ML}(S) \to \mathscr{C}(S)$. Therefore, if a sequence (X_n, λ_n) in $\mathcal{PT}(S)$ diverges, then $L_{X_n} + \lambda_n$ will also diverge in $\mathscr{C}(S)$, so one or both the currents L_{X_n} and λ_n must also diverge. We remind that by Theorem 9.11 if L_{X_n} diverges, then up to passing to a subsequence, being $\mathbb{P}\mathcal{ML}(S)$ compact, it converges projectively to a projective measured lamination $[L_{X_n}] \to [\eta]$. Similarly, up to passing to a subsequence, we can also assume that $[\lambda_n] \to [\lambda]$ in $\mathbb{P}\mathscr{C}(S)$.

If exactly one between L_{X_n} and λ_n diverges, then the projective limit in $\mathbb{P}\mathscr{C}(S)$ will be the projective limit of the diverging one, that is in both cases a projective measured lamination. If they both diverge, then the projective limit of their sum will be a linear combination $[a\eta + b\lambda]$ with $a, b \geq 0$, where the ratio between a, b depends on the speeds of divergence of the two sequences L_{X_n} and λ_n. Let us call $\mathcal{ML}^{(2)}(S)$ the space of currents that can be written as sum of two measured laminations, or in other words the image of the map

$$\mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathscr{C}(S)\,, \qquad (\eta, \lambda) \mapsto \eta + \lambda\,.$$

Then we have just showed the following.

Corollary 10.21. *Let (X_n, λ_n) be a diverging sequence in $\mathcal{PT}(S)$, then, up to subsequence,* *it converges projectively to a current in $\mathcal{ML}^{(2)}(S)$. In other words in $\mathbb{P}\mathscr{C}(S)$ we have that*

$$\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}\mathcal{ML}^{(2)}(S).$$

We have already observed that in the case of a pair of transverse filling currents (η, λ), the current $\eta + \lambda$ represents the half-translation surface $q = q(\eta, \lambda)$ equipped with the l^1-metric. In Proposition 10.18 we proved that a sequence $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ of l^1-grafted surfaces whose currents converge in $\mathbb{P}\mathscr{C}(S)$ to the current $\eta + \lambda$, converges up to rescaling in the marked Gromov-Hausdorff sense to q^{l^1}. We now want to give a geometric interpretation to the convergence to currents in $\mathcal{ML}^{(2)}$ that are not sum of two transverse and binding measured laminations. Let us introduce for this reason l^1-mixed structures.

Definition 10.22. An l^1*-mixed structure* (Σ, q, λ) on S is the datum of a subsurface $\Sigma \subset S$ with an l^1-metric coming from a quadratic differential q on Σ and a measured lamination supported in the complement of the interior of Σ.

Notice that l^1-mixed structures are not equivalent to mixed structures as in Definition 9.19, in the same way there is not a bijection between the l^1-metric for half-translation surfaces and the Euclidean one. Indeed, to the Euclidean metric is associated a holomorphic quadratic differential up to S^1, while for an l^1-metric, only up to a ± 1 factor. In the Euclidean case, as we mentioned in Chapter 9.2, mixed structures can be described equivalently as certain geodesic currents, obtained as sum of a Liouville current for a quadratic differential on a subsurface and a measured lamination in the complement. In this case we will see that l^1-mixed structures correspond exactly to the currents that arise as boundary points for the currents associated to l^1-grafted metrics. Let us call $\mathrm{Mix}^{l^1}(S)$ the space of l^1-mixed structures on S.

Theorem 10.23. *There is a natural correspondence between* $\mathrm{Mix}^{l^1}(S)$ *and* $\mathcal{ML}^{(2)}(S)$.

To show this, we first invoke a structure theorem for geodesic currents proved in [Bur+21a], that for any geodesic current provides a unique decomposition in currents supported on subsurfaces. For this, let us remind that a current μ' is said to be *filling a subsurface* if it has positive systole on the subsurface, that is if the infimum of $i(\mu', \gamma)$ over all the essential closed curves γ in the subsurface, is positive.

Theorem 10.24 ([Bur+21a, Corollary 1.9]). *Given a geodesic current $\mu \in \mathscr{C}(S)$, there is a collection of disjoint connected subsurfaces $(\Sigma_i)_{i=1}^n$ of S, whose boundary components* *are closed geodesics such that μ decomposes as*

$$\mu = \nu + \sum_{i=1}^{n} \mu_i \,, \tag{10.3}$$

where μ_i is a geodesic current whose support is made by geodesics entirely contained in the interior of Σ_i, and ν is a measured lamination whose support is made by geodesics disjoint from the interior of all the subsurfaces Σ_i's.

Remark . Note that in the statement of the cited article, the decomposition is even finer, but for our scope it is enough to group all the disjoint measured laminations and closed curves of the decomposition into a single measured lamination ν.

In our case, when the current μ is the sum of two measured laminations η and λ, we will now show that we can describe explicitly the terms in the decomposition above. Recall that every measured lamination is sum of finitely many components, which are supported on disjoint minimal geodesic laminations. Let us then consider the decompositions of η and λ in minimal components.

$$\eta = \sum_{i=1}^{k} \eta_i \quad \text{and} \quad \lambda = \sum_{i=1}^{l} \lambda_i$$

Any current μ in $\mathcal{ML}^{(2)}(S)$ can be written as sum of two laminations whose supports do not share any common minimal component. Indeed, if $\mu = \eta + \lambda$ and η_i, λ_j have the same minimal geodesic lamination as support, then we can change η and λ by subtracting the component λ_j from λ and adding it to η. So we can assume that any two minimal components, one taken from the decompositions of η and the other from λ are either intersecting transversely or disjoint. Consider then the following graph.

Definition 10.25. Let μ be a current in $\mathcal{ML}^{(2)}(S)$ and $\mu = \lambda + \eta$ as above. Let us call *intersection graph* G_μ, the graph that has one vertex for every minimal lamination in the collection $\{\eta_i\}_{i=1}^{k} \cup \{\lambda_i\}_{i=1}^{l}$ and an edge between two vertices if and only if their associated laminations intersect.

Note that the collection of minimal laminations in μ is independent of the choice of η, λ for the writing of $\mu = \lambda + \eta$. Indeed, such components can be identified as all the possible measured laminations obtained as restriction of the measure μ to the closures of a leaf that lie in the support of μ.

Remark . The intersection graph G_μ of $\mu = \eta + \lambda$ is a bipartite graph. Indeed, if two minimal laminations η_i, η_j or λ_i, λ_j are components of the same lamination, then they cannot intersect each other.

Theorem 10.26. *If $\mu \in \mathcal{ML}^{(2)}(S)$ and $\mu = \nu + \sum_{i=1}^n \mu_i$ is the decomposition given by Theorem* 10.24*, then*

i) ν is the sum of all the measured laminations associated to the isolated vertices of G_μ

ii) G_μ has n non-trivial connected components, and μ_i equals to the sum of the currents represented by the vertices in the i-th component.

Proof. We have two decompositions of μ: one given by the structure theorem above, and one in minimal measured laminations associated to the vertices of G_μ. Let us call Σ_i the subsurface filled by μ_i. Note that the current μ_i have support made of geodesics that do not intersect any other geodesic in the support of another μ_j or of ν. Then each minimal lamination represented by a vertex of G_μ is fully contained either in ν or in a single μ_i; that is its measure cannot be split among multiple μ_i's or ν. Moreover, if two minimal laminations intersect, so the corresponding vertices are connected by an edge in G_μ, then they both also need to be contained in the same μ_i. So the decomposition of μ into μ_i's and ν is obtained simply by grouping together terms of the decomposition in minimal laminations associated to the vertices of G_μ, where laminations associated to vertices in the same connected component of G_μ have to end up in the same μ_i.

Let us now consider the following claim: all the minimal laminations forming μ_i must be in the same connected component of G_μ. This is enough to conclude the proof, as it shows that every μ_i is exactly the sum of the measured laminations associated to a non-trivial connected component of G_μ. We say non-trivial because a single measured lamination cannot be filling a subsurface, so to all the isolated vertices of G_μ are associated disjoint measured laminations that will form ν.

Let us now prove the claim. Suppose by contradiction that there is a μ_i that is sum of the minimal measured laminations associated to vertices of more than one connected component of G_μ. Then by using its connected components and the bipartition given by η, λ, we can group the minimal laminations and find four measured laminations $\alpha_0, \alpha_1, \beta_0, \beta_1$ supported on geodesics contained in Σ_i, and such that for $i = 0, 1$, α_i intersects β_i and $\alpha_0 \cup \beta_0$ is disjoint from $\alpha_1 \cup \beta_1$.

We observe then that there is a connected component A of $\Sigma_i \setminus (\alpha_0 \cup \alpha_1 \cup \beta_0 \cup \beta_1)$ whose boundary intersects both $(\alpha_0 \cup \beta_0)$ and $(\alpha_1 \cup \beta_1)$. Take for instance the connected component of a point of Σ_i equidistant from the two subsets $(\alpha_0 \cup \beta_0)$ and $(\alpha_1 \cup \beta_1)$. By construction, A is a hyperbolic surface with convex boundary. Each boundary component is piecewise geodesic formed by finitely many segments, half-lines and lines, forming angles smaller than π each with the next one (or converging to an ideal vertex). In particular every leaf intersecting such a boundary component either intersect or is at distance 0 from the two adjacent ones. This implies that each boundary component is either fully contained in $\alpha_0 \cup \beta_0$ or in $\alpha_1 \cup \beta_1$. This implies that A has at least two boundary components.

By the classification of orientable surfaces, having at least two boundary components implies that A has non-trivial fundamental group, and because it has convex boundary, it contains a closed geodesic γ. Note that γ cannot lie on the boundary of Σ_i, because then it would lie on the boundary of A, that by construction is made of geodesics in $\alpha_0 \cup \alpha_1 \cup \beta_0 \cup \beta_1$, which by Theorem 10.24 are contained in the interior of Σ_i.

By construction γ has zero intersection with the current $\mu_i = \alpha_0 + \alpha_1 + \beta_0 + \beta_1$, which violates the hypothesis of μ_i being filling the subsurface Σ_i. □

We can now finally prove the equivalence between $\mathcal{ML}^{(2)}(S)$ and l^1-mixed structures.

Proof of Theorem 10.23. Given an l^1-mixed structure (Σ, q, λ) we can associate to it the current $\mathrm{Re}(q) + \mathrm{Im}(q) + \lambda$. Since λ lies in the complement of the interior of Σ, it is disjoint from both $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$, so $\mathrm{Im}(q) + \lambda$ is still a measured lamination, and the current $\mathrm{Re}(q) + (\mathrm{Im}(q) + \lambda)$ is then sum of two measured laminations.

Vice versa, given a current μ that is sum of two measured laminations, we now show that it determines an l^1-mixed structure, that is there exists a unique mixed structure whose associated current is μ. Consider the intersection graph G_μ, and the decomposition of μ as in (10.3), where each μ_i is filling a subsurface Σ_i. The previous Theorem 10.26 tells us that every μ_i is sum of minimal laminations in a connected component G_{μ_i} of the bipartite graph G_μ. Since G_{μ_i} is connected, it admits only one possible bipartition. Following such a bipartition we can write $\mu_i = \alpha_i + \beta_i$ where α_i, β_i are transverse measured laminations that are together filling Σ_i; and such a decomposition is unique, up to swapping α_i and β_i. This means that μ_i is the current that represents the l^1-metric induced by the half-translation structure $q_i = q(\alpha_i, \beta_i)$ on Σ_i, determined up to sign, as $q(\beta_i, \alpha_i) = -q$. The union of the q_i gives a quadratic differential on the union of the subsurface $\Sigma = \bigcup_i \Sigma_i$. So given μ, using Theorem 10.26 we produced the l^1-mixed structure (Σ, q, ν) with $\mu = \mathrm{Re}(q) + \mathrm{Im}(q) + \nu$ by construction. □

There is a strong connection between the l^1-mixed structures we have just introduced and the universal dual space to the geodesic currents in $\mathcal{ML}^{(2)}(S)$. Universal dual spaces to geodesic currents were introduced in [Rei23], where the author describes explicitly the ones associated to classes of currents arising as limits of Hitchin representations. Such currents have a special restriction on the possible intersection configurations of the geodesics in their support, which implies that their associated space is a finite dimensional cube complex. Currents in $\mathcal{ML}^{(2)}(S)$ fall in the class of currents whose dual space is 2-dimensional. We claim the following.

Conjecture 10.27. If $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ is a sequence of l^1-grafted surfaces whose currents $L_{X_n} + \lambda_n$ are converging projectively to $\eta + \lambda$, then the universal cover of $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ converges up to rescaling, according to the equivariant Gromov-Hausdorff convergence, to the universal dual space of $\eta + \lambda$, which is a cube complex and where each cube is equipped with the natural l^1-metric.

To conclude, Theorem 10.23 allows us to rewrite the compactification of $\mathcal{PT}(S)$ in $\mathbb{P}\mathscr{C}(S)$, via currents associated to l^1-grafting, as

$$\overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}\,\mathrm{Mix}^{l^1}(S).$$

Notice that the topology of this compactification is more complicated than when we limit ourselves to filling currents, as in this case there are substantially different diverging sequences in $\mathcal{PT}(S)$ that in this compactification converge to the same mixed structure.

Remark . Observe, indeed, that as we discussed before, given a current $\mu \in \mathcal{ML}^{(2)}(S)$, its decomposition as sum of two laminations in not unique. For every non-trivial connected component G_{μ_i} of G_μ there are two options to write μ_i either as $\alpha_i + \beta_i$, or $\beta_i + \alpha_i$. For every minimal component α in ν, instead there is a finite dimensional family worth of options, as the space of transverse measures to a minimal lamination is a finite dimensional simplicial cone.

By combining all such choices for every connected component of G_μ, we obtain all the possible decomposition of μ as sum of two measured laminations. For each decomposition $\mu = \eta + \lambda$, we have a substantially different way of approaching μ in $\mathbb{P}\mathscr{C}(S)$ with a sequence of l^1-grafted surfaces $\mathrm{Gr}^{l^1}(X_n, \lambda_n)$ such that $X_n \to [\eta]$ and $\lambda_n \to [\lambda]$.

A Appendices

A.1 Proper actions on non-locally compact spaces

In this appendix we recap the definitions and theorems of the point-set topology we used for the space $\mathrm{MMet}(S)$. In literature there are multiple definitions available for proper actions and proper maps, which are equivalent under mild assumptions. Unfortunately, one of such assumptions is locally compactness, which we do not have for the space $\mathrm{MMet}(S)$. The results we will need are still valid without having locally compactness, up to choosing the appropriate definitions.

Let X, Y be topological spaces, Hausdorff and first countable. This will give us that compact subsets are closed and allow us to check continuity of maps, compactness and closedness of subsets via sequences.

Definition A.1. A map $f\colon X \to Y$ is proper if the preimage of any compact subsets is compact.

We say that a sequence *diverges* if it does not have accumulation points.

Proposition A.2. *A continuous map $f\colon X \to Y$ is proper if the following is true. A sequence $(x_n)_{n\in\mathbb{N}}$ in X diverges if and only if $(f(x_n))_{n\in\mathbb{N}}$ diverges.*

Proof. Suppose x_n diverges. If $f(x_n)$ has accumulation points, then up to passing to a subsequence, it converges to $y \in Y$. Consider the compact subset of Y defined as $K = \{f(x_n)\}_{n\in\mathbb{N}} \cup y$. Since f is proper, $f^{-1}(K)$ is compact too. But by assumption it contains a sequence x_n without any accumulation point in X, which is a contradiction.

The opposite implication is just continuity: if a sequence x_n converges to x, then $f(x_n)$ converges to $f(x)$. □

Lemma A.3. *If $f\colon X \to Y$ is continuous, injective and proper, then it is homeomorphism onto its image.*

Proof. Since the map is injective, there exists an inverse from its image. We just need to show that the inverse is continuous, that is that f is closed.

Let C be a closed subset of X. Let us show that $f(C)$ is closed. It is enough to check that it is sequentially closed. Take $(y_n)_n$ a sequence in $f(C)$ converging to $y \in Y$. The sequence $f^{-1}(y_n)$ is contained in C. It cannot diverge, because otherwise the properness of f would imply that y_n also diverges. Then $f^{-1}(y_n)$, up to passing to a subsequence, converges to a point x, which by closedness of C also lies in C. By continuity of f then $f(x) = y$. In particular $y \in f(C)$. □

Let now G be a discrete group that acts on X by homeomorphisms. The following is a definition of proper action introduced by Palais in [Pal61], that is the correct notion to make the next lemma hold true even for non-locally compact spaces.

Definition A.4. The action $G \curvearrowright X$ is called *proper* if for every two points $x, y \in X$ there exist neighbourhoods U_x and U_y such that

$$\#\{g \in G \mid gU_x \cap U_y \neq \emptyset\} < \infty$$

Lemma A.5. *Let $f\colon X \to Y$ be a continuous map, equivariant with respect to proper actions of G on X and Y. If the map induced at the quotient $\overline{f}\colon X/G \to Y/G$ is proper, then f is also proper.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ a divergent sequence in X. We want to show that $(f(x_n))_n$ also diverges. Assume by contradiction that, up to passing to a subsequence, $f(x_n)$ converges to $y \in Y$. Let us call $\pi_X\colon X \to X/G$ and $\pi_Y\colon Y \to Y/G$ the projections to the quotients. Let us consider two cases. First, if $\pi_X(x_n)$ diverges, then $\overline{f}(\pi_X(x_n))$ also diverges since $\overline{f}$ is proper. But $\overline{f}(\pi_X(x_n)) = \pi_Y(f(x_n))$ and $f(x_n)$ converges by assumption, which gives a contradiction.

Consider the case now where $\pi_X(x_n)$ converges, up to passing to a subsequence. By definition of convergence at the quotient, there is a point $x \in X$ and a sequence g_n in G such that $g_n x_n \to x$. The sequence g_n must diverge, otherwise being G discrete, it means that, up to passing to a subsequence, it is constantly g, and we would get $gx_n \to x$, that means $x_n \to g^{-1}x$, against our assumptions. Then, by G equivariance, $g_n f(x_n) = f(g_n x_n) \to f(x)$. But we also assumed $f(x_n) \to y$. So we have that for any two neighbourhoods $U_{f(x)}$ and U_y respectively of $f(x)$ and of y, for all n large enough $f(x_n) \in U_y$ and $g_n f(x_n) \in U_{f(x)}$, that is $f(x_n) \in g_n^{-1} U_{f(x)}$. In particular for infinitely many n we have that U_y intersects $g_n^{-1} U_{f(x)}$. This contradicts the definition of proper G-action on Y. □

A.2 One-dimensional grafting along a Cantor set

The scope of this appendix section is to give a more explicit description of a one-dimensional model for grafting in the case when the lamination is generic, so without closed leaves, since there is no straight forward cut-and-paste operation to describe the grafted metric. This one-dimensional model can be thought as what happens to a smooth arc on a hyperbolic surface X, transverse to a measured lamination λ, when the surface is grafted along λ. The analogies with this are the ones suggested by the choice of notation.

Let $L \subset (0,1)$ be a Cantor set, meaning closed and with empty interior and Lebesgue measure 0, but of uncountable cardinality, like for example the very well known ternary Cantor set. Its complement L^c is open, and in particular can be written as a countable union of disjoint intervals.

$$U = \bigcup_{n\in\mathbb{N}} (a_n, b_n)$$

Let us now consider any finite measure λ in $(0,1)$ concentrated on L. The function

$$f\colon (0,1) \to \mathbb{R}; \quad f(x) = \lambda((0,x])$$

is non-decreasing, and constant on each interval (a_n, b_n). Let us now consider the set obtained from U by translating each interval (a_n, b_n) according to f as follows.

$$U' = \bigcup_{n\in\mathbb{N}} (a_n + f(a_n), b_n + f(a_n)) \subset (0, 1 + |\lambda|)$$

Notice that the intervals in the union are disjoint because (a_n, b_n) are, and f is non-decreasing, so each pair of intervals can only get further away from each other. By sigma additivity and translation invariance of the Lebesgue measure, we have that $|U'| = 1$. As a consequence, its complement $\hat{L}$ in $(0, 1 + |\lambda|)$ is a closed set with measure $|\lambda|$. If the measure λ has no atoms, then $\hat{L}$ also has empty interior, but, as we showed, positive Lebesgue measure. Such sets are also known as Smith-Volterra-Cantor sets, or fat Cantor sets.

What follows can also be generalized in the case where λ is allowed to have atoms. For sake of brevity of the exposition we stick to the generic case with no atoms, so that f is continuous and makes the exposition easier.

Let us consider the function $(0,1) \to (0, 1+|\lambda|)$ given by $x \mapsto x + f(x)$. It is monotonic increasing and bijective. Let us call its inverse $\kappa\colon (0, 1+|\lambda|) \to (0,1)$.

Proposition A.6. *The function κ maps isometrically the interval $(a_n + f(a_n), b_n + f(a_n))$*

to (a_n, b_n) *for every* $n \in \mathbb{N}$. *This is the analogue of Thurston's collapsing map of Chapter 2.4.*

Let us consider now the composition $\mathcal{D}\colon f \circ \kappa\colon (0, 1 + |\lambda|) \to (0, |\lambda|)$. One can easily verify the following.

Proposition A.7. *The function* $\mathcal{D}$ *is constant on each connected component of* U'*, that is the complement of* $\hat{L}$*, meaning on each interval* $(a_n + f(a_n), b_n + f(a_n))$*, and its restriction to the fat Cantor set* $\hat{L}$ *is surjective onto* $(0, |\lambda|)$ *and Lebesgue measure preserving.*

A.3 Marked length spectrum rigidity for grafted surfaces

This section provides a positive answer to Conjecture 9.14. The argument was found during fruitful discussions with Didac Martinez-Granado, few weeks after the defence of my thesis.

Theorem A.8. *If two grafted surfaces* $\mathrm{Gr}(X_1, \lambda_1), \mathrm{Gr}(X_2, \lambda_2)$ *have the same marked length spectrum, then* $X_1 = X_2$ *and* $\lambda_1 = \lambda_2$.

The proof follows from the one of the following theorem, by replacing the final argument with one ad-hoc for grafted surfaces. Indeed, the whole statement does not apply directly to our case as grafted metrics are not smooth.

Theorem A.9 ([CFF92, Theorem A]). *Let* S *be a closed surface and let* g_1, g_2 *be smooth non-positively curved smooth Riemannian metrics on* S. *If* (S, g_1) *and* (S, g_2) *have the same marked length spectrum, then they are isometric with an isometry homotopic to the identity.*

The proof described by Croke, Fathi, Feldman of the theorem above follows step by step the one of Otal in [Ota90] for strictly negatively curved metrics. They adapt the constructions to non-positive curvature case, making sure that the presence of flat strips does not invalidate the arguments. However, what they obtain with Otal's construction is an isometry only between the negatively curved parts of the surfaces, which only then they extend to the whole surface while relying on the smoothness of the metric to show that the extension is also an isometry.

Proof of Theorem A.8. Let us call S_i the universal cover of $\mathrm{Gr}(X_i, \lambda_i)$ and $U_i \subset S_i$ the hyperbolic part, that is the open subset of all the points that have a neighbourhood locally isometric to $\mathbb{H}^2$.

By [CFF92, Lemma 2.5], there exists a $\pi_1(S)$-equivariant bijection $\varphi\colon U_1 \to U_2$ that is a metric isometry with respect to subspace metric on U_i. By this we mean that φ preserves the distances measured in S_i between points in U_i.

The complement of U_i is a geodesic lamination $\tilde{\lambda}_i$, preimage via the covering map of the grafted locus λ_i in $\mathrm{Gr}(X_i, \lambda_i)$. The complement $\tilde{\lambda}_i$ of U_i is nowhere dense, unless λ_i has closed leaves, that is if $\tilde{\lambda}_i$ has flat strips. So the closure of U_i in S_i is the whole surface S_i, minus the interior of the flat strips, if S_i has any.

In the case of λ_1 without closed leaves, U_1 is dense in S_1. Then S_1 is the metric completion of U_1. Similarly, being S_2 metrically complete, the closure $\overline{U_2}$ of U_2 in S_2 will be the metric completion of U_2. Since $\varphi : U_1 \to U_2$ is an isometry, by uniqueness of the metric completion, φ extends to an isometry between S_1 and $\overline{U_2}$, in particular $\overline{U_2}$ is connected, so S_2 has no flat strips and $\overline{U_2} = S_2$. By density, the extension is also $\pi_1(S)$-equivariant.

If λ_1 has closed leaves, then S_1 has flat strips, and $\overline{U_1}$ is the complement of the interior of the flat strips. As before, extending $\varphi\colon U_1 \to U_2$ to the metric completion, yields an isometry between $\overline{U_1}$ and $\overline{U_2}$. By construction, S_i is obtained by gluing the connected components of $\overline{U_i}$ with countably many flat strips. The width of the strips and the gluing maps can be easily deduced by the distances between pairs of points on the boundary components of $\overline{U_i}$. In other words the metric on S_i is uniquely determined by the metric (in the sense of distance function) on $\overline{U_i}$. So the $\pi_1(S)$-equivariant isometry between $\overline{U_1}$ and $\overline{U_2}$ extends to a $\pi_1(S)$-equivariant isometry between S_1 and S_2.

An equivariant isometry induces an isometry that preserves the markings between $\mathrm{Gr}(X_1, \lambda_1)$ and $\mathrm{Gr}(X_2, \lambda_2)$ and by Proposition 4.9 the metric determines the grafting data, so $X_1 = X_2$ and $\lambda_1 = \lambda_2$. □

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