

A compactification of the moduli space of grafted surfaces via geodesic currents

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Abstract

Let S be a closed orientable surface of genus at least two. We construct a proper, mapping class group equivariant compactification of the moduli space $\mathcal{PT}(S)$ of complex projective structures on S by embedding it into the space of projective geodesic currents. The boundary consists of projective mixed structures and contains the projective half-translation surfaces as a dense subset. Every complex projective structure induces a grafted metric, in the sense of Thurston, obtained by grafting a hyperbolic surface along a measured lamination, i.e., by inserting flat pieces in place of the leaves. Our embedding into currents is the one compatible with these grafted metrics.

The main contribution is a convergence result: grafted surfaces, under suitable conditions and up to rescaling, converge in the marked Gromov–Hausdorff sense (which we introduce) to singular Euclidean metrics coming from half-translation structures. The key construction relies on the orthogeodesic foliation introduced by Calderon and Farre. In parallel, we study an l^1 -variant of the grafted metric and its associated current, which we use both to complete the proof of our compactification and to study the limits of generalised stretch rays in Teichmüller space, extending a theorem of Papadopoulos.

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Introduction

Let S be an orientable closed surface of genus g at least 2. Teichmüller space $\mathcal{T}(S)$ parametrises all the complete hyperbolic metrics on S up to isotopy, or equivalently, all the complex structures on S up to isotopy. Another family of well known geometric structures on S is given by *half-translation surfaces*. They are surfaces obtained by taking a collection of polygons in $\mathbb{R}^2$ and identifying their sides in pairs via the compositions of translations and possibly rotations by π . A half-translation structure on S is equivalent to the datum of a complex structure up to isotopy and a holomorphic quadratic differential on it, so the moduli space of such marked structures is described as the bundle $\mathcal{QT}(S)$ of quadratic differentials over $\mathcal{T}(S)$, where every fibre is the space of holomorphic quadratic differential on a fixed complex structure. These classical objects and some of their properties are recalled in Section 1.

Grafting

The main object of study in this article will be a third class of geometric structures that endow the surface S with a piecewise hyperbolic and piecewise flat metric. Such surfaces are the result of an operation called *grafting*, of which we recall the definition and a result of Thurston in Section 2. In the simplest case, grafting consists of the following: given a hyperbolic surface X , cutting it open along a simple closed geodesic and gluing a flat cylinder in its place (see Figure 2). Grafting can be then extended to weighted multicurves, which are collections of disjoint simple closed geodesics, with a positive weight for each component, indicating the height of the cylinder to be inserted in place of each curve of the collection.

Grafting has been studied from various points of view and produced many results in the course of the years. The main result due to Thurston on grafting is its use to parametrise the moduli space of complex projective structures $\mathcal{PT}(S)$. Weighted multicurves lie inside a larger space $\mathcal{ML}(S)$ of measured laminations, that can be seen as a completion of the space of weighted multicurves. Thurston showed that grafting extends also to measured laminations. In particular, as shown in his unpublished work and discussed in [Tan97] and [Dum09], a complex projective structure induces a metric on the surface S , called the Thurston metric, that can be interpreted as the metric obtained by grafting a hyperbolic surface X along a measured lamination λ , establishing a mapping class group equivariant homeomorphism

$$\text{Gr}: \mathcal{T}(S) \times \mathcal{ML}(S) \xrightarrow{\sim} \mathcal{PT}(S).$$

For this reason we can think of $\mathcal{PT}(S)$ as parametrising grafted surfaces and consider its points as surfaces with the grafted Thurston metric. A different parametrization of $\mathcal{PT}(S)$ was also studied by Dumas using the Schwarzian derivative [Dum06], [Dum07].

Many more results over the time have shown the important role of grafting as a deformation of complex structures. Indeed, a complex projective structure also induces a complex structure, allowing to define a map $\text{gr}: \mathcal{T}(S) \times \mathcal{ML}(S) \rightarrow \mathcal{T}(S)$. This map gives rise to an extension of Thurston's earthquakes to a complex twist parameter in [McM98] and was further studied in [SW02] and [DW08] and used in [CG26] to obtain asymptotics of renormalised volume. Moreover, grafting rays, which are paths of the form $t \mapsto \text{gr}(X, t\lambda)$, were shown to be asymptotic to Teichmüller geodesics [DK07], [Hen11], [CDR11], [Gup14], and were also used in [Aba26] to study the coarse geometry of the Weil–Petersson metric on Teichmüller space.

Recently, grafting has also appeared in higher Teichmüller theory. In [BHM25], [BHMM25], a notion of grafting along decorated multicurves has been introduced as a deformation of Fuchsian representations into the Hitchin component, and has proved useful for studying the geometry of the moduli space of Hitchin representations. These developments suggest that grafted surfaces are not merely a way of parametrising complex projective structures, but form a class of geometric structures of independent interest. In particular, the geometric degenerations of

grafted surfaces have not been studied, in contrast with the well-understood cases of hyperbolic and half-translation structures.

Compactification via geodesic currents

The main goal of this article is to describe a compactification of the moduli space $\mathcal{PT}(S)$ that captures the geometry of degenerating structures and gives a meaningful interpretation of boundary points. The way this is achieved, in Section 8, is via embedding $\mathcal{PT}(S)$ in the space of geodesic currents, and prove that this interpolates between previously known embeddings of two families of geometric structures: hyperbolic metrics and singular flat metrics induced by half-translation structures.

Geodesic currents were introduced by Bonahon (see Section 8 for the definition) and extend multiple families of objects on a surface S , such as curve, multicurves and measured laminations. The space of geodesic currents $\mathcal{C}(S)$ is equipped with a continuous bilinear form called *intersection form*. Bonahon showed in [Bon88] that each hyperbolic metric X determines a geodesic current L_X , the Liouville current, which induces via the intersection form the marked length spectrum of X . That is, for every free homotopy class γ of closed curves in S , the intersection $i(L_X, \gamma)$ gives the length of the representative of the class of γ that is geodesic with respect to the metric X . Moreover, $\mathcal{C}(S)$ admits a natural free $\mathbb{R}^+$ -action by rescaling. Taking the quotient yields the space of projective geodesic currents $\mathbb{P}\mathcal{C}(S)$, which Bonahon showed to be compact. This allowed him to prove the following renowned result.

Theorem ([Bon88]). *The map $\mathcal{T}(S) \rightarrow \mathcal{C}(S)$ given by $X \mapsto L_X$ is a proper $\text{Mod}(S)$ -equivariant embedding. Projectivising yields a map $\mathcal{T}(S) \rightarrow \mathbb{P}\mathcal{C}(S)$, which is also an embedding. The closure of $\mathcal{T}(S)$ in $\mathbb{P}\mathcal{C}(S)$ is a disc, with boundary consisting of projective currents corresponding to measured laminations.*

Building on Bonahon's approach, Duchin, Leininger and Rafi in [DLR10] constructed Liouville currents for the singular Euclidean metrics associated to any half-translation surface. Since these metrics are invariant under rotation of the half-translation structure, the map $q \mapsto L_q$ descends to a well-defined map $\mathcal{QT}(S)/S^1 \rightarrow \mathcal{C}(S)$, for which they showed the following:

Theorem ([DLR10]). *The map $\mathcal{QT}(S)/S^1 \rightarrow \mathcal{C}(S)$ is a proper $\text{Mod}(S)$ -equivariant embedding. Projectivising it induces the embedding $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S) \rightarrow \mathbb{P}\mathcal{C}(S)$, where the closure of the image is $\mathbb{P}\text{Mix}(S)$, the space of projective currents associated to mixed structures.*

Here, a mixed structure consists of a half-translation structure on a subsurface of S together with a measured lamination on its complement (see Definition 8.11). Similar mixed structures also appear as boundary points in the compactification of the moduli spaces of different structures on surfaces [BIPP21], [OT21], [Ouy23].

The construction of a Liouville current representing uniquely any hyperbolic metric was extended to all negatively curved Riemannian metrics [Ota90], later to non-positively curved smooth Riemannian metrics [CFF92], and more recently to smooth Riemannian metrics with Anosov geodesic flow [GLP25]. Our grafted surfaces, even if retaining non-positive curvature properties, do not have a smooth metric. For this reason we need to carefully extend their result to our setting, obtaining then marked length spectrum rigidity for grafted surfaces. In conclusion, we prove the following.

Theorem A. *The map $\mathcal{PT}(S) \rightarrow \mathcal{C}(S)$ induced by considering the Liouville current is a proper $\text{Mod}(S)$ -equivariant embedding. The projectivisation is also an embedding and the closure of the image of $\mathcal{PT}(S)$ in $\mathbb{P}\mathcal{C}(S)$ is*

$$\overline{\mathcal{PT}(S)}^{\mathcal{C}} := \overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}\text{Mix}(S).$$

Where we remind that, by [DLR10], $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ is dense in $\mathbb{P}\text{Mix}(S)$ the boundary of $\mathcal{PT}(S)$, and it corresponds exactly to the boundary of $\mathcal{PT}(S)$ taken in the space of projective filling currents (Theorem 8.10).

The proof of Theorem A consists of two parts: showing that half-translation surfaces, and then automatically also mixed structures, are contained in the closure of $\mathcal{PT}(S)$, and then showing that to every point in the boundary correspond a mixed structure. For the first part we in fact prove a stronger geometric convergence in Theorem B, for which we first introduce the space of marked metric surfaces, and endow it with what we call marked Gromov–Hausdorff convergence.

The space of marked metric surfaces

In Section 3 we introduce the space $\text{MMet}(S)$ of marked metrics on S . Elements of $\text{MMet}(S)$ are isotopy classes of length metrics inducing the topology of S . We adapt the notion of Gromov–Hausdorff convergence to this setting of marked metric spaces, and so define a topology on $\text{MMet}(S)$, and develop some results on continuous approximate isometries between surfaces.

The space $\text{MMet}(S)$ contains a much wider variety of metrics on S than those we focus on in this article, and is then an object of independent interest. With this in mind, we prove a few results that hold in the full generality for metrics in $\text{MMet}(S)$. While Gromov–Hausdorff convergence is generally a rather weak notion, we show that in our restricted setting, where all spaces are homeomorphic to S , it becomes rather strong. For instance, we show that convergence in $\text{MMet}(S)$ implies a Lipschitz-type convergence of the marked length spectrum. And by restricting to a subset of locally CAT(0) metrics, the convergence is even stronger (see Section 3.4).

The space $\text{MMet}(S)$ also carries a natural $\mathbb{R}^+$ -action by rescaling the metric, so also in this case we can pass to the projectivization $\mathbb{P}\text{MMet}(S)$, which, unlike $\mathbb{P}\mathcal{C}(S)$, is not compact. For this reason $\text{MMet}(S)$ is not suited to play the role of an ambient space where to realise a compactification, but rather just a bordification. Indeed, in Section 4 we have the following.

Theorem B. *Both the inclusions of $\mathcal{PT}(S)$ and $\mathcal{QT}(S)/S^1$ into $\text{MMet}(S)$ are proper $\text{Mod}(S)$ -equivariant embeddings. Projectivising yields embeddings of $\mathcal{PT}(S)$ and $\mathbb{P}_{\mathbb{C}}\mathcal{QT}(S)$ into $\mathbb{P}\text{MMet}(S)$, and identifying them with their images, we obtain*

$$\overline{\mathcal{PT}(S)}^{\text{MMet}} := \overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}_{\mathbb{C}}\mathcal{QT}(S).$$

Note that for its nature, convergence in $\text{MMet}(S)$ does not allow for topological degenerations, as all its points represent surfaces homeomorphic to S . It nevertheless can capture geometric degeneration of our grafted metrics to singular flat ones. Moreover, we prove (Proposition 3.19) that the mapping class group acts properly on it, allowing us to easily switch between the marked setting, and the unmarked setting.

Deflation

The main content of the previous theorem is that the metric of a grafted surface with large grafted area, under suitable conditions and up to rescaling, closely approximates that of a half-translation surface. Central to our construction for the proof is the map $\mathcal{O}: \mathcal{PT}(S) \rightarrow \mathcal{QT}(S)$ introduced by Calderon and Farre in [CF24] and [CF25], which we recall in Section 5. Originally devised to extend Mirzakhani’s results [Mir10] and to study dynamical conjugacies between $\mathcal{PT}(S)$ and $\mathcal{QT}(S)$, this map assigns canonically to each grafted surface $\text{Gr}(X, \lambda)$ a half-translation surface $\mathcal{O}(X, \lambda)$. It does so by constructing a measured foliation $\mathcal{O}_{\lambda}(X)$, called *orthogeodesic foliation*, made of piecewise geodesic segments orthogonal to the grafting lamination, and considering the translation surface that has it as horizontal foliation, and λ as vertical foliation. In our setting, this map provides a natural candidate for the limiting geometry of a grafted surface as the grafted

area becomes large. Let us see a simple example of a sequence of grafted surfaces converging, up to rescaling, to a half-translation one.

Example. Let X be a hyperbolic surface and $\mu = \sum_i a_i \gamma_i$ a weighted pants decomposition: a collection of disjoint simple closed geodesics γ_i , equipped with weights a_i , and whose complement consists of pairs of pants. Given $l_i > 0$, for every $n \geq 1$ one can choose a hyperbolic structure X_n on S so that γ_i has length nl_i for every i . The grafted surface $\text{Gr}(X_n, n\mu)$ is obtained by inserting Euclidean cylinders of height na_i and circumference nl_i along each γ_i . With n going to infinity, the pairs of pants with metric X_n have boundary growing longer and so collars becoming thinner, while the cylinders grow proportionally to n . By rescaling by $1/n$ the metric of $\text{Gr}(X_n, n\mu)$, the cylinders become of constant size, while the pair pf pants become of curvature $-n^2$ and shrink until, for n going to infinity, they collapse into graphs as in Figure 1. The geometric projective limit is then a surface obtained by gluing the cylinders along graphs, which has a natural half-translation structure.

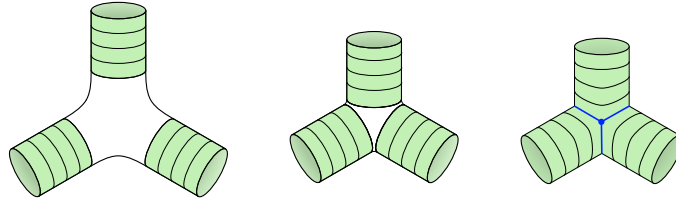


Figure 1: A pair of pants in the grafted surface $\text{Gr}(X_n, n\mu)$ becomes thinner as n increases, eventually collapsing to a graph.

To make a quantitative comparison between $\text{Gr}(X, \lambda)$ and $\mathcal{O}(X, \lambda)$, in Section 5 we explicitly construct a map, called *deflation* $\mathcal{D}$: $\text{Gr}(X, \lambda) \rightarrow \mathcal{O}(X, \lambda)$. In the example above, where μ is a multicurve, the half-translation surface $\mathcal{O}(X_n, n\mu)$ is the one obtained by gluing together only the grafted cylinders of $\text{Gr}(X, \lambda)$; the deflation map $\mathcal{D}$: $\text{Gr}(X_n, n\mu) \rightarrow \mathcal{O}(X_n, n\mu)$ is a local isometry on the cylinders, while it collapses the hyperbolic parts to graphs. As n goes to infinity, the rescaled grafted surface $\text{Gr}(X_n, n\mu)$ converges to the half-translation surface $\mathcal{O}(X, \mu)$. The very core of the article is Section 7, devoted to prove a quantitative version of this approximation in general.

Theorem C. *Assume that the systole of $\mathcal{O}(X, \lambda)$ is at least ϵ and $\ell_X(\lambda) > 1$. Consider now both $\text{Gr}(X, \lambda)$ and $\mathcal{O}(X, \lambda)$ with their metric rescaled to have unit area. Then the map $\mathcal{D}$: $\text{Gr}(X, \lambda) \rightarrow \mathcal{O}(X, \lambda)$ is a $C_\epsilon(\ell_X(\lambda))^{-1/2}$ -isometry, i.e. a map that preserves distances up to a small additive error, where the constant C_ϵ depends only on ϵ .*

We observe in Section 6 that the deflation map not only allows us to compare the geometry of grafted surfaces with the one of half-translation surfaces, but also suggests considering another operation analogue to grafting. Namely, if we can view a half-translation surface as obtained by collapsing the negatively curved parts of a grafted surface, then reversing this collapse yields a deformation of the half-translation surface in which one-dimensional subspaces are "inflated" into negatively curved hyperbolic surfaces with geodesic boundary. We call this inverse process *inflation*. From this perspective, inflation acts as a counterpart to Thurston's grafting: while grafting inserts flat cylinders into hyperbolic surfaces, inflation introduces negative curved parts into half-translation surfaces.

Comparison with l^1 -metrics

In order to complete the proof of Theorem A, we are then left to study the possible topological degenerations of grafted surfaces. To achieve this, in Section 9 we compare the Liouville currents

of grafted surfaces and half-translation surfaces with the much more manageable currents given by the following maps of $\mathcal{PT}(S)$ and $\mathcal{QT}(S)$ into $\mathcal{C}(S)$. We show indeed that these currents and the Liouville currents of Theorem A are bilipschitz related, and in particular they degenerate if and only if their counterparts degenerate. They have the following expressions

$$\mathrm{Gr}(X, \lambda) \mapsto L_X + \lambda \quad \text{and} \quad q \mapsto \mathrm{Re}(q) + \mathrm{Im}(q)$$

where L_X is the Liouville current representing the hyperbolic metric X , whose possible limits in $\mathbb{P}\mathcal{C}(S)$ were already studied by Bonahon. Indeed, we can prove the following.

Theorem D. *The two maps above induce embeddings of $\mathcal{PT}(S)$ and $\mathcal{QT}(S)/\{\pm 1\}$ into the space of currents $\mathcal{C}(S)$. Projectivising yields embeddings of $\mathcal{PT}(S)$ and $\mathbb{P}_{\mathbb{R}}\mathcal{QT}(S)$ into $\mathbb{P}\mathcal{C}(S)$, and identifying the spaces with their images in $\mathbb{P}\mathcal{C}(S)$ we have*

$$\overline{\mathcal{PT}(S)}^{l^1} := \overline{\mathcal{PT}(S)} = \mathcal{PT}(S) \cup \mathbb{P}\mathcal{ML}^{(2)}$$

where $\mathcal{ML}^{(2)}$ is the subspace of the currents that are sums of two measured laminations, which contains $\mathbb{P}_{\mathbb{R}}\mathcal{QT}(S)$ as an open dense subset.

We then observe (Corollary 9.8, Proposition 9.11) that the currents we just introduced represent length functions with respect to natural alternative metrics on both grafted and half-translation surfaces. Such metrics are obtained by replacing the Euclidean ones with the Finsler metric induced by the l^1 -norm on $\mathbb{R}^2$, both in half-translation surfaces and in the cylinders of surfaces grafted along multicurves. In particular, Theorem D contains marked length spectrum rigidity for this class of l^1 -metrics on half-translation surfaces. The currents in the boundary are also associated to geometric structures. Indeed, using the decomposition of geodesic currents introduced in [BIPP21], we establish a correspondence between $\mathcal{ML}^{(2)}$ and the space of l^1 -mixed structures: a variant of mixed structures where the subsurface with a half-translation structure is endowed with the l^1 metric.

Finally, by observing that the proof of Theorem C applies directly to this l^1 variant of metrics as well, we draw the connection between $\overline{\mathcal{PT}(S)}^{l^1}$ and $\overline{\mathcal{PT}(S)}^{\mathcal{C}}$, which allows us to complete the proof of Theorem A, classifying the degenerations of grafted surfaces as mixed structures. As by-product we also obtain that the map $\mathcal{O}_{\lambda}: \mathcal{T}(S) \rightarrow \mathcal{ML}(S)$ induced by taking the orthogeodesic foliation extends naturally at infinity, generalising a result of Papadopoulos [Pap91].

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AI statement

No AI has been used, except for improving fluency in the introduction.