Configuration Spaces

The n-th *ordered configuration space* $\tilde{C}^n(M)$ of a space M is the space of all n-tupels $(\zeta_1, \ldots, \zeta_n)$ of distinct points in M; and the quotient $C^n(M) = \tilde{C}^n(M)/\mathfrak{S}_n$ by the free action of the symmetric group $\mathfrak{S}_n$ is called the *unordered configuration space.*

These spaces are well-studied for manifolds M. Their fundamental group $\pi_1(C^n(M)) = \mathrm{Br}_n(M)$ is called the *braid group* of M; in particular, $\mathrm{Br}_n(\mathbb{R}^2)$ is the classical braid group on n strings, defined by E.Artin in 1925.

The symmetric group occurs, wherever things are ordered or reordered. Likewise the braid groups occur in all geometric situations where points (in M) are ordered or reordered. The braid group is thus an extension of the symmetric group :

$$1 \to \pi_1(\tilde{C}^n(M)) \to \pi_1(C^n(M)) = \mathrm{Br}_n(M) \to \mathfrak{S}_n \to 1$$

This is particularly important for two- or three-dimensional manifolds. For example, sending $\zeta = (\zeta_1, \ldots, \zeta_n)$ to $p(z) = (z - \zeta_1) \cdots (z - \zeta_n)$ is a homeomorphism from $C^n(\mathbb{C})$ to the space of complex polynomials of degree n, with leading term equal to 1 and distinct roots.

The importance of braids for 3-manifolds, knots and links lies in two facts. First, closing the strings of a braid gives a link in $\mathbb{R}^3$, and any link is the closure of some braid. Secondly, every compact 3-manifold can be obtained from the 3-sphere by surgery along some link. Furthermore, braid groups have many properties in common with mapping class groups.

In homotopy theory, the configuration spaces play an important role as building blocks of mapping spaces. For example, for the m-fold loop space of an m-fold suspension of some connected space X one has a homotopy equivalence

$$\Omega^m \Sigma^m X \quad \simeq \quad (\bigsqcup_{n \geq 0} \tilde{C}^n(\mathbb{R}^m) \times_{\mathfrak{S}_n} X^n)/\sim$$

with $(\zeta_1, \ldots, \zeta_n; x_1, \ldots, x_n) \sim (\zeta_1, \ldots, \zeta_{n-1}; x_1, \ldots, x_{n-1})$ if x_n is the basepoint in X. Such models for mapping spaces are the basis for the Snaith splitting of their suspension spectrum into a wedge of more simple spectra.

C.-F. Bödigheimer (7.12.2004)