Seminar on Quantum Groups & Quantum Cohomology of Symplectic Varieties

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Let X be a complex variety with an action by a torus $T = (\mathbb{C}^*)^n$. Since the topological Euler characteristic of $\mathbb{C}^*$ vanishes, the Euler characteristic of X is the Euler characteristic of the fixed locus X^T under the torus action:

$$e(X) = e(X^T).$$

More generally, the cohomology of X is determined by the cohomology of X^T together with the restriction maps on cohomology, given that the cohomology is taken *equivariantly* with respect to the torus T. This yields a practical and powerful tool to understand the cohomology of algebraic varieties with torus actions.

The seminar discusses the cohomology of symplectic varieties with torus actions and their relation to quantum groups. The key is to understand the equivariant restriction morphisms

$$H^*_T(X) \to H^*_T(X^T)$$

and to construct suitable maps in the other direction called *stable envelopes*. Understanding this in detail for Nakajima quiver varieties will lead to intresting R-matrices and from there to quantum groups. The quantum groups act on the cohomology of these varieties by multiplication by Chern classes of tautological bundles.

A further enhancement of the theory is obtained by considering the quantum cohomology of the symplectic varieties. Quantum cohomology is a commutative and associate deformation of the cohomology of a smooth projective variety whose structure constants encode counts of rational curves on the underlying variety. Formally it is defined in terms of the Gromov–Witten theory of the variety. For Nakajima quiver varieties we will see that the operators of quantum multiplication with tautological classes are expressed through the action of a certain (generalized) Yangian.

The main reference for the course is the book [**?**] by Maulik and Okounkov.

The theory presented here is best understood through concrete examples. Already the seemingly simple case of $T^*\mathbb{P}^1$, the cotangent bundle of $\mathbb{P}^1$, shares many of the properties of the general theory. Hence the $T^*\mathbb{P}^1$ case should be discussed in every talk. More general examples include:

- Cotangent bundle of projective space: $T^*\mathbb{P}^n$
- Cotangent bundle of the Grassmannian: $T^*\mathrm{Gr}(k,n)$ or more generally of flag varieties.
- Hilbert scheme of points on the affine plane, $\mathsf{Hilb}(\mathbb{C}^2)$, or on ADE surfaces
- Springer resolution $T^*(G/B)$
- Moduli space of framed rank r sheaves on the plane $\mathcal{M}(r,n)$

References

[1] D. Maulik, A. Okounkov, *Quantum groups and quantum cohomology*, Astrisque No. 408 (2019), ix+209 pp.