Khovanov homology from a geometric point of view

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Introduction

In his seminal paper [Kho00] Khovanov introduced his celebrated categorification of the Jones polynomial [Jon85, Kau87] for links. More precisely, to a generic plane projection D of a link L in $\mathbb{R}^3$ Khovanov constructs a bigraded chain complex $\mathcal{CKh}(D)$ whose graded Euler characteristic is the Jones polynomial of L. Given two diagrams D and D' representing the same link L, i.e. D and D' are related by a sequence of Reidemeister moves, the corresponding chain complexes turn out to be homotopy equivalent. In particular, the homology groups of the chain complex $\mathcal{CKh}(D)$ are invariants of the link L.

In [Kho02] this homological link invariant was extended to tangles with an even number of top and bottom endpoints categorifying the Reshetkikhin-Turaev invariant [RT90] associated with the quantum group $U_q(\mathfrak{sl}_2)$. The original construction of this tangle homology theory is of combinatorial nature. All the chain groups are obtained by applying a two-dimensional TQFT to smoothings of a generic plane projection of the tangle, after closing it up by certain matchings on top and bottom to create a collection of circles. Cobordisms between these smoothings induce the differentials.

From an algebraic point of view the extension of Khovanov homology from links to tangles is interesting because the chain groups of the complex can be equipped with a left and right action of a graphically-defined algebra, nowadays often referred to as Khovanov's arc algebra, thereby turning them into bimodules. In the case of links these actions degenerate into the action of the ground ring and therefore the additional structure is lost. Although the basic arc algebra first appeared in the context of tangle homology [Kho02] it has been generalized and studied extensively outside the field of low-dimensional topology from a purely combinatorial and representation theoretic point of view (cf. [BS11, Str09]).

Khovanov himself found a connection to the geometry of flag varieties by proving that the center of the arc algebra is isomorphic to the cohomology ring of a two-block Springer fiber [Kho04]. Given a nilpotent endomorphism x of $\mathbb{C}^n$, the Springer fiber $\mathcal{F}l^x$ associated with this operator is the complex projective variety of full flags

$$\{0\} \subset F_1 \subset F_2 \subset ... \subset F_n = \mathbb{C}^n$$

in $\mathbb{C}^n$ fixed by the endomorphism x, i.e. we demand inclusions $x(F_i) \subset F_{i-1}$ for all $i \in \{1, 2, \ldots, n\}$ (set $F_0 := \{0\}$) (partial flag varieties fixed by a nilpotent operator are referred to as Spaltenstein varieties). In general, these varieties are not smooth and have many irreducible components. The structure of the irreducible components is poorly understood for arbitrary nilpotent operators (cf. also [FM10]). However, if we restrict ourselves to two-block Springer fibers, i.e. flag varieties fixed by a nilpotent operator with two Jordan blocks, then the irreducible components can be written down explicitly using the combinatorics of cup diagrams. This goes back to the work of Spaltenstein [Spa76], Vargas [Var79] and Fung [Fun03].

In their recent paper [SW12] Stroppel and Webster extend Khovanov's result on the center of the algebra by providing a geometric construction of the entire algebra as a convolution algebra using the irreducible components of two-block Springer fibers. In fact, their results are much stronger than that since they also provide geometric constructions of interesting generalized Khovanov algebras (cf. [BS11, Str09]) and their quasi-hereditary covers. Based on the work of Stroppel and Webster we want to ask the following question:

Question. Is it possible to extend the geometric construction of the arc algebra to a new geometric construction of Khovanov's chain complex (in which the TQFT does not occur anymore) using Springer fibers or the more general Spaltenstein varieties?

In this thesis we give an affirmative answer to this question. More precisely, we construct subvarieties inside a finite product of Spaltenstein varieties such that after taking the cohomology of these varieties we are able to recover the chain groups of Khovanov's complex. These subvarieties sit inside each other in an interesting way. In particular, we prove that the differentials of the complex can be given a natural geometric meaning by realizing them via pullback or pushforward maps in cohomology induced by the inclusions of these subvarieties. This main result is documented in Theorem 3.34.

One possible motivation for the question above might be the goal of finding a precise connection between Khovanov homology (respectively its Lie-theoretic version [Str05] which is known to agree with Khovanov homology) and the link homology theories by Cautis and Kamnitzer [CK08] via coherent sheaves and the ones by Seidel and Smith [SS06] and Manolescu [Man07] via symplectic geometry. An explicit categorical connection between all these theories is not yet established. Since Springer fibers play a crucial role in the work of Cautis and Kamnitzer as well as Seidel and Smith, our geometric construction of Khovanov homology might provide another small step in comparing all three categorifications in the future.

Structure of the thesis: In the following we outline the contents of our work. This thesis is subdivided into three major parts.

The main agenda of the first part is to identify and describe the topological space underlying the two-block Spaltenstein variety equipped with the analytic topology. A topological description of the equal-block Springer fiber $\mathcal{F}l^{\frac{n}{2},\frac{n}{2}}$ as a certain subspace of a n-fold product of spheres was conjectured by Khovanov [Kho04] and proven by Russell and Tymoczko [RT11], and independently by Wehrli [Weh09]. This was later generalized by Russell to the general two-block-case [Rus11]. It turns out that the topological model from [Rus11] can also be used to describe the topology of two-block Spaltenstein varieties. This is Theorem 1.15.

In the second part we explain how to assign a subvariety inside a finite product of Spaltenstein varieties to a given tangle diagram. We also provide a simple topological model (cf. Proposition 2.26). If two tangle diagrams are related by a local surgery then the associated varieties are related by an inclusion map, i.e. one of them sits inside the other one as a subvariety (cf. Proposition 2.35). In Theorem 2.51 (Theorem 2.50 is the topological equivalent) we explicitly compute the pullback and pushforward of these inclusions in cohomology and relate the result to the maps obtained by applying a TQFT to certain surgery cobordisms between the tangle diagrams. This requires genuine work and is therefore considered as one of the central results.

In the final part we recall the definition of Khovanov's arc algebra as defined in [Kho02] and sketch the construction of the convolution algebras of Stroppel and Webster [SW12] in the equal-block-case. Then we use the machinery developed in the second part of this thesis to construct some important bimodules and bimodule homomorphisms geometrically via Spaltenstein varieties and pullback and pushforward maps. In Theorem 3.37 this culminates in the promised geometric construction of Khovanov's chain complex associated with a tangle diagram (cf. also Theorem 3.34 for the topological equivalent).

Notation and conventions: Once and for all we fix the following notation and conventions valid throughout this thesis:

- Let X be a topological space. Then we denote by $H^*(X)$ its singular cohomology with $\mathbb{F}_2$-coefficients, i.e. we use the notation $H^*(X)$ for $H^*(X;\mathbb{F}_2)$ (which among topologists is usually reserved for cohomology with $\mathbb{Z}$-coefficients). Similarly, we write $H_*(X)$ for the homology of X with $\mathbb{F}_2$-coefficients.

- A vector space or an algebra will always be a $\mathbb{F}_2$-vector space or a $\mathbb{F}_2$-algebra, respectively. Moreover, all tensor products are defined over $\mathbb{F}_2$, i.e. we use the notation $\otimes$ for $\otimes_{\mathbb{F}_2}$, unless stated otherwise.
- If V is a graded vector space ("graded" always means "$\mathbb{Z}$-graded") we write $V\{j\}$ for the graded vector space whose i-th component is given by $(V\{j\})_i = V\{i-j\}$.

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1 Topology of two-block Spaltenstein varieties

1.1 Two-block Spaltenstein varieties: Structure of irreducible components and a topological model

We begin this section by providing the basic definitions. Then we recall some structural results concerning the irreducible components of two-block Spaltenstein varieties including the involved combinatorics. Moreover, a simple topological model for two-block Spaltenstein varieties is introduced at the end of this subsection.

Let $n > 0$ be a natural number and let $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ be a family of integers such that $0 < i_1 < i_2 < ... < i_m = n$. Define $\mathcal{F}l_I$ to be the set consisting of all sequences

$$F_{i_1} \subset F_{i_2} \subset ... \subset F_{i_m} = \mathbb{C}^n,$$

where $F_{i_l} \subset \mathbb{C}^n$ is a subspace, $\dim F_{i_l} = i_l$, for all $l \in \{1, \dots, m\}$. Such a sequence of subspaces is called a *partial flag of type I* and we will write $(F_{i_1}, \dots, F_{i_m})$ to denote such a flag. Let $\mathrm{Gr}(k, n)$ be the Grassmannian of k-planes inside $\mathbb{C}^n$. Then there is an obvious embedding

$$\mathcal{F}l_I \hookrightarrow \mathrm{Gr}(i_1, n) \times \mathrm{Gr}(i_2, n) \times ... \times \mathrm{Gr}(i_m, n),$$

which can be used to prove that $\mathcal{F}l_I$ is a smooth complex projective variety called the *partial flag manifold of type I*. Alternatively, one could realize $\mathcal{F}l_I$ as the homogeneous space $\mathrm{GL}(n, \mathbb{C})/P$, where P is the parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$ given by all matrices with all entries zero below the block diagonal where the blocks are squares of size $i_1, i_2, \dots, i_m$.

Let $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$ be the Lie algebra of traceless endomorphisms of $\mathbb{C}^n$ and let $\mathcal{N} \subset \mathfrak{g}$ be the *nilcone* consisting of all elements which act nilpotently in all representations of $\mathfrak{g}$. By the general theory of Lie algebras these elements coincide with the elements which are nilpotent as linear endomorphisms in the sense of linear algebra.

For everything that follows it will be convenient to set $i_0 := 0$ and $F_0 := \{0\} \subset \mathbb{C}^n$.

Definition 1.1. The *Spaltenstein variety of type I* associated with some fixed nilpotent operator $x \in \mathcal{N}$ is the variety

$$\mathcal{F}l_I^x := \big\{(F_{i_1}, ..., F_{i_m}) \in \mathcal{F}l_I \mid xF_{i_l} \subset F_{i_{l-1}} \text{ for all } l \in \{1, ..., m\}\big\},$$

consisting of all flags in $\mathcal{F}l_I$ fixed under x.

Remark 1.2. In the case of full flags, i.e. if I consists of all integers between 1 and n, the index I is dropped from the notation and we simply write $\mathcal{F}l^x$. In this case we refer to the Spaltenstein variety as *Springer fiber*.

Spaltenstein varieties are rich geometric objects and they arise naturally as the fibers of a resolution of singularities of the nilcone; see e.g. [CG97, Sections 3.2, 3.5, 3.7, 4.1 and 4.4] for details. In general they are not smooth and have many irreducible components.

In this work we will only study *two-block Spaltenstein varieties*, i.e. the case where x is of Jordan type $(n-k, k)$, where $0 \leq 2k \leq n$. More explicitly, this means that there exists an ordered basis $e_1, ..., e_{n-k}, f_1, ..., f_k$ of $\mathbb{C}^n$ such that

$$x(e_i) = e_{i-1} \qquad \text{and} \qquad x(f_i) = f_{i-1},$$

where we set $e_0 = f_0 = 0$. One can easily show that the Spaltenstein variety does not depend (up to isomorphism) on the particular choice of x, but only on the Jordan type of x. This allows us to speak of the $(n-k, k)$-Spaltenstein variety of type I, which we denote by $\mathcal{F}l_I^{n-k,k}$, without further specifying the nilpotent operator x.

Notice that for a flag $(F_{i_1}, ..., F_{i_m}) \in \mathcal{F}l_I^x$, where x is of Jordan type $(n-k, k)$, the rank-nullity theorem yields inequalities

$$\dim F_{i_l} = \dim \ker x|_{F_{i_l}} + \dim xF_{i_l} \leq 2 + \dim F_{i_{l-1}}$$

for all $l \in \{1, ..., m\}$. Thus $\mathcal{F}l_I^x$ will be empty if $i_l - i_{l-1} > 2$ for some index in the family I. In order to exclude these trivial cases from the beginning we will always restrict ourselves to the following special type of integer-sequence.

Definition 1.3. A family of increasing positive integers $(i_1, ..., i_m)$ is called *admissible* if the conditions $0 < i_l - i_{l-1} \leq 2$ are satisfied for all $l \in \{1, ..., m\}$.

Given an admissible family I, one can associate to it a word of length n, consisting of letters from the alphabet $\{\bullet, \times\}$, by putting a dot at position i_l of the word, if $i_l - i_{l-1} = 1$, and a cross at positions i_l and $i_l - 1$ if $i_l - i_{l-1} = 2$. Such a word is called a *dot-cross sequence*. Any word obtained in this way consists of $2(n-m)$ crosses and $2m - n$ dots.

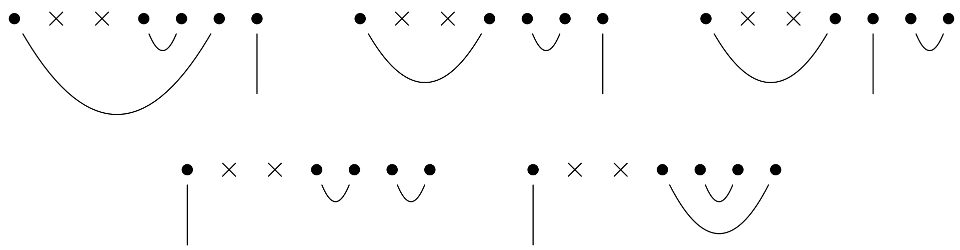
Example 1.4. The dot-cross sequence associated with $(1, 3, 4, 5, 6, 7)$ is given by $\bullet \times \times \bullet\bullet\bullet\bullet$.

Schäfer proved in [Sch12] that the irreducible components of the $(n-k, k)$-Spaltenstein variety of type I are in one-to-one correspondence with cup diagrams of type $(n-k, k)$ on the dot-cross sequence corresponding to I (cf. Proposition 1.11 below).

Definition 1.5. A *cup diagram of type* $(n-k, k)$ on the dot-cross sequence corresponding to some admissible $I = (i_1, ..., i_m)$ is a planar diagram which is obtained by attaching $n - 2k$ downward-pointing rays and $k - (n-m)$ arcs to the dots of the dot-cross sequence. Arcs only pass below the symbols of the dot-cross sequence and we require that every dot is connected to precisely one cup or ray. Moreover, we think of the rays as being "infinitely long", so arcs do not pass below rays. The set of all cup diagrams of type $(n-k, k)$ is denoted by $B_I^{n-k,k}$.

Remark 1.6. In order to tie in with the notation from Remark 1.2 we omit the index I and simply write $B^{n-k,k}$ if the family I consists of all integers between 1 and n.

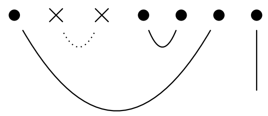
Example 1.7. Here is a complete list of cup diagrams of type $(4, 3)$ on the dot-cross sequence considered in Example 1.4:



Given $a \in B_I^{n-k,k}$ the notation $(i, j) \in a$, where $i < j$, means that the symbols at positions i and j in the dot-cross sequence corresponding to I are connected by an arc in the diagram a. We write $(i) \in a$ if there is a ray in a connected to the dot at position i.

If we slightly modify the cup diagram then it contains all the necessary information to write down explicit relations defining all the flags which lie in the corresponding irreducible component. In order to make these modifications precise let $a \in B_I^{n-k,k}$ be a cup diagram. For certain arguments it is convenient to think of the crosses at positions i_l and $i_l - 1$ as being connected by a dashed arc if $i_l - i_{l-1} = 2$. Such a dashed arc is called *invisible* because it does not appear in the usual cup diagrams from Definition 1.5. If we want to include invisible arcs in the cup diagram a, we will denote this by $\tilde{a}$. Notice that invisible arcs only connect neighboring crosses. In particular, they are never nested inside each other.

Example 1.8. This is a cup diagram from Example 1.7 including invisible arcs:

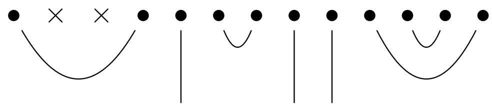


Definition 1.9. Given a cup diagram $a \in B_I^{n-k,k}$ we obtain a map

$$\rho_a \colon \{1, 2, \ldots, n\} \to \mathbb{Z}_{\geq 0}$$

by defining $\rho_a(i)$ to be the number of rays in a which are left of the i-th symbol in the dot-cross sequence corresponding to I. We use the convention that a ray is always considered as being left of itself.

Example 1.10. Consider the following cup diagram:

$a =$ 

Then we have $\rho_a(3) = 0$ and $\rho_a(12) = \rho_a(9) = 3$.

The following proposition is essentially [Sch12, Theorem 6.11].

Proposition 1.11. *Let I be an admissible family and x a nilpotent operator of Jordan type $(n-k, k)$. Then the irreducible component $K_a \subset \mathcal{F}l_I^x$ corresponding to $a \in B_I^{n-k,k}$ consists of precisely those flags $(F_{i_1}, ..., F_{i_m}) \in \mathcal{F}l_I$ which satisfy the following conditions imposed by the cup diagram $\tilde{a}$:*

(i) If $(i,j) \in \tilde{a}$ (the arc connecting the symbols i and j might be an invisible arc), then

$$F_j = x^{-\frac{1}{2}(j-i+1)} F_{i-1}.$$

(ii) If $(i) \in \tilde{a}$, then we have

$$F_i = F_{i-1} + \operatorname{span}\left(e_{\frac{1}{2}(i+\rho_a(i))}\right).$$

Proof. The proof is omitted here (see [Sch12, §6] for a detailed argument). The rough idea is to reduce the statement of the proposition to the case of Springer fibers which is treated in the work of Fung [Fun03]. □

Remark 1.12. The original version of Proposition 1.11 in [Sch12] is formulated using the combinatorics of *dependence graphs*. In order to avoid confusion we remark that these graphs differ slightly from the cup diagrams used here. Our cup diagrams correspond to what Schäfer calls *extended cup diagrams* if we replace the rays by *green arcs*.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the two-dimensional standard unit sphere and let $\mathbf{p} = (0,0,1)$ be its north pole. Given a cup diagram $a \in B^{n-k,k}$, define a smooth submanifold $S_a \subset (\mathbb{S}^2)^n$ of the n-fold cartesian product of the sphere with itself by

$$S_a := \left\{(x_1, ..., x_n) \in \left(\mathbb{S}^2\right)^n \mid x_i = x_j \text{ if } (i,j) \in a \text{ and } x_i = (-1)^i \mathbf{p} \text{ if } (i) \in a\right\}.$$

Following Russell [Rus11, §2] we define the $(n-k,k)$ *topological Springer fiber* as

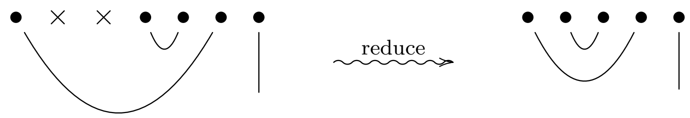
$$\mathcal{S}^{n-k,k} := \bigcup_{a \in B^{n-k,k}} S_a \subset \left(\mathbb{S}^2\right)^n .$$

Definition 1.13. For $a \in B_I^{n-k,k}$ let a_{red} denote the *reduced* extended cup diagram obtained by erasing all crosses. In particular, this assignment induces a map

$$\mathrm{red}\colon B_I^{n-k,k} \to B^{m-k,m+k-n}$$

to which we refer as a *reduction map.*

Example 1.14. The reduced version of the diagram in Example 1.7 is the following:



The main result of the first part is the following theorem:

Theorem 1.15. *The irreducible component K_a of the Spaltenstein variety $\mathcal{F}l_I^{n-k,k}$ corresponding to $a \in B_I^{n-k,k}$ is homeomorphic to $S_{a_{\mathrm{red}}}$ and we have a homeomorphism*

$$\mathcal{F}l_I^{n-k,k} \cong \bigcup_{a \in B_I^{n-k,k}} S_{a_{\mathrm{red}}} \subset (\mathbb{S}^2)^{2m-n}.$$

A proof of this theorem is provided in section 1.4. For the reader's convenience we summarize the main ideas of the argument to motivate the next sections.

Sketch of proof (Theorem 1.15):
Our proof closely follows and generalizes the train of thought of Wehrli's argument in [Weh09] for the $(\frac{n}{2},\frac{n}{2})$-Springer fiber. The general idea is as follows: In section 1.2 we introduce a smooth projective variety Y_I which contains the Spaltenstein variety $\mathcal{F}l_I^{n-k,k}$. We prove that Y_I is homeomorphic to a $(2m-n)$-fold product projective spaces $\left(\mathbb{P}^1\right)^{2m-n}$, cf. Corollary 1.20. Proposition 1.11 above enables us to explicitly compute the images $\phi_I(K_a)$ of the irreducible components K_a under this homeomorphism in section 1.3. The most important result in this context is Proposition 1.35. In the last section we introduce another homeomorphism $\gamma_{2m-n}\colon \left(\mathbb{P}^1\right)^{2m-n} \xrightarrow{\cong} \left(\mathbb{S}^2\right)^{2m-n}$ such that the composition

$$Y_I \xrightarrow{\phi_I} \left(\mathbb{P}^1\right)^{2m-n} \xrightarrow{\gamma_{2m-n}} \left(\mathbb{S}^2\right)^{2m-n}$$

maps the irreducible components $K_a \subset \mathcal{F}l_I^{n-k,k} \subset Y_I$ to the sets $S_{a_{\mathrm{red}}}$. In particular, this homeomorphism maps $\mathcal{F}l_I^{n-k,k}$, which is just the union of its irreducible components, to the subset

$$\bigcup_{a \in B_I^{n-k,k}} S_{a_{\mathrm{red}}} \subset (\mathbb{S}^2)^{2m-n},$$

which will prove the theorem.

1.2 A smooth space containing the Spaltenstein variety

In this section the first step in proving Theorem 1.15 is provided by embedding the Spaltenstein variety into a smooth space homeomorphic to a finite product of projective spaces.

1.2.1 Some linear algebra

Let $N > 0$ be a large[1] integer and let $z : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ be a nilpotent linear endomorphism with two equally-sized Jordan blocks, i.e. there exists a basis $e_1, ..., e_N, f_1, ..., f_N$ of $\mathbb{C}^{2N}$ such that

$$z(e_i) = e_{i-1} \qquad \text{and} \qquad z(f_i) = f_{i-1},$$

for all $i \in \{1, 2, ..., N\}$, where $e_0 = f_0 = 0$. We equip $\mathbb{C}^{2N}$ with the structure of a unitary vector space by defining

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = \delta_{ij}\,, \qquad \langle e_i, f_j \rangle = 0$$

for all $i, j \in \{1, 2, ..., N\}$, where δ_{ij} is the Kronecker delta.

Let e, f denote the canonical basis of $\mathbb{C}^2$. Define a linear map $C\colon \mathbb{C}^{2N} \to \mathbb{C}^2$ by

$$C(e_i) = e \qquad \text{and} \qquad C(f_i) = f$$

for all $i \in \{1, ..., N\}$. Notice that $\mathbb{C}^2$ has the structure of a unitary vector space coming from the standard Hermitian inner product.

In general it does not make sense to ask whether the map C is unitary, i.e. whether the equality $\langle C(v), C(w)\rangle = \langle v, w\rangle$ holds for all $v, w \in \mathbb{C}^{2N}$, because this would imply that C is injective, which cannot be the case if $N > 1$. However, it is meaningful to ask for two-dimensional subspaces $U \subset \mathbb{C}^{2N}$ such that the restriction of C to U yields a unitary isomorphism. The following lemma contained in [CK08, Lemma 2.2] (cf. also [Weh09, Lemma 2.1]) constructs important examples of such spaces.

Lemma 1.16. *Let $U \subset \mathbb{C}^{2N}$ be a z-stable subspace, i.e. $zU \subset U$, such that $U \subset \operatorname{im}(z)$. Then C restricts to a unitary isomorphism*

$$C : z^{-1}U \cap U^{\perp} \xrightarrow{\cong} \mathbb{C}^2.$$

Before going into the proof of the lemma, let us record the following general fact:

Lemma 1.17. *If $U \subset \mathbb{C}^{2N}$ is a subspace contained in $\operatorname{im}(z)$, then $\dim(z^{-1}U) - \dim(U) = 2$. Similarly one obtains $\dim(U) - \dim(zU) = 2$ if a subspace $U \subset \mathbb{C}^{2N}$ contains $\ker(z)$.*

Proof. The inclusion $U \subset \operatorname{im}(z)$ implies that z restricts to a surjection $z\colon z^{-1}U \twoheadrightarrow U$. Applying the rank-nullity theorem to this map yields

$$\dim(z^{-1}U) - \dim(U) = \dim\left(\ker(z|_{z^{-1}U})\right) = 2,$$

where the last equality follows from the inclusion $\ker(z) = z^{-1}(0) \subset z^{-1}U$. An analogous argument for the map $z\colon U \twoheadrightarrow zU$ proves the second claim. □

Proof (Lemma 1.16). There is an isomorphism $z^{-1}U \cap U^{\perp} \cong z^{-1}U/U$ by elementary linear algebra. Thus one obtains

$$\dim(z^{-1}U \cap U^{\perp}) = \dim(z^{-1}U/U) = \dim(z^{-1}U) - \dim(U) \overset{(1.17)}{=} 2.$$

Hence the domain and target space of $C\colon z^{-1}U \cap U^{\perp} \to \mathbb{C}^2$ are equidimensional and it only remains to show unitarity because unitary maps are always injective. Let $v, w \in z^{-1}U \cap U^{\perp}$ and write

$$v = v_1 + ... + v_N, \qquad w = w_1 + ... + w_N$$

[1] See also Remark 1.18 for a more accurate description of what is meant by the word “large”.

with $v_j, w_j \in \mathrm{span}(e_j, f_j)$. Since the restriction $C : \mathrm{span}(e_j, f_j) \to \mathbb{C}^2$ is easily seen to be unitary, we deduce

$$\langle v, w\rangle = \sum_{i=1}^{N} \langle v_i, w_i\rangle = \sum_{i=1}^{N} \langle C(v_i), C(w_i)\rangle.$$

Comparing this to the equality

$$\langle C(v), C(w)\rangle = \sum_{i,j} \langle C(v_i), C(w_j)\rangle,$$

we see that it suffices to show

$$\sum_{i \neq j} \langle C(v_i), C(w_j)\rangle = 0 \tag{1}$$

to complete the proof.

For $w \in z^{-1}U$ and $v \in U^{\perp}$ we have $\langle v, z^l w\rangle = 0$ for all $l \geq 1$ because U is z-stable by assumption. A calculation shows

$$\langle v, z^l w\rangle = \langle C(v_1), C(w_{l+1})\rangle + ... + \langle C(v_{N-l}), C(w_N)\rangle$$

for all $l \geq 1$. Adding up all these equations for every l gives

$$0 = \sum_{i<j} \langle C(v_i), C(w_j)\rangle. \tag{2}$$

Interchanging the roles of v and w (i.e. interpreting v as an element of $z^{-1}U$ and w as an element of $U^{\perp}$), we obtain

$$0 = \left(\sum_{i<j} \langle C(w_i), C(v_j)\rangle\right)^{\dagger} = \sum_{i<j} \langle C(w_i), C(v_j)\rangle^{\dagger} = \sum_{i<j} \langle C(v_j), C(w_i)\rangle, \tag{3}$$

where the dagger $\dagger$ denotes complex conjugation. Adding up (2) and (3) yields (1) and therefore the second claim. $\square$

1.2.2 The variety Y_I

Fix an admissible family $I = (i_1, ..., i_m)$ and let $N \geq m$. For every $q \in \{1, 2, ..., m\}$ define a complex projective variety

$$Y_{i_1 i_2 \dots i_q} := \left\{(F_{i_1}, ..., F_{i_q}) \mid F_{i_l} \subset \mathbb{C}^{2N} \text{has dimension } i_l, F_{i_1} \subset ... \subset F_{i_q}, zF_{i_l} \subset F_{i_{l-1}}\right\}$$

and set Y_{i_0} to be the one-point space. We equip all these varieties with the analytic topology. In particular, they are compact Hausdorff spaces.

We will be most interested in the variety $Y_{i_1 \dots i_m}$, for which we also use the short notation Y_I, because we can obviously identify $\mathcal{F}l_I^{n-k,k}$ with the subset

$$\{(F_{i_1}, ..., F_{i_m}) \in Y_I \mid F_{i_m} = \mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k)\} \tag{4}$$

and thus obtain an embedding $\mathcal{F}l_I^{n-k,k} \hookrightarrow Y_I$.

Before we prove that Y_I is homeomorphic to a finite product of projective spaces we make the following important remark.

Remark 1.18. Notice that the conditions $zF_{i_l} \subset F_{i_{l-1}}$ imply

$$F_{i_m} \subset z^{-1}F_{i_{m-1}} \subset ... \subset z^{-m}(0) = \operatorname{span}(e_1, ..., e_m, f_1, ..., f_m).$$

In particular, the variety Y_I is independent of the choice of N as long as $N \geq m$ because the spaces $F_{i_l} \subset \mathbb{C}^{2N}$ of the flags in Y_I never "see" the basis vectors e_l and f_l for $l > m$. This turns out to be extremely useful because we can always assume (by increasing N if necessary) that all the subspaces of a flag in Y_I are contained in the image of z.

Proposition 1.19. *Let $q \in \{1, ..., m\}$. If $i_q - i_{q-1} = 1$, then the projection map*

$$\pi_q \colon Y_{i_1 ... i_q} \to Y_{i_1 ... i_{q-1}}\ ,\ (F_{i_1}, ..., F_{i_q}) \mapsto (F_{i_1}, ..., F_{i_{q-1}})$$

defines a trivial fiber bundle with fiber $\mathbb{P}^1$. More precisely, the map

$$\phi_q \colon Y_{i_1 i_2 ... i_q} \to Y_{i_1 i_2 ... i_{q-1}} \times \mathbb{P}^1\ ,\ (F_{i_1}, ..., F_{i_q}) \mapsto \Big(F_{i_1}, ..., F_{i_{q-1}}, C(F_{i_q} \cap F_{i_{q-1}}^{\perp})\Big)$$

defines a homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} Y_{i_1 i_2 ... i_q} \arrow[rr, "\phi_q"] \arrow[dr, "\pi_q"'] & & Y_{i_1 i_2 ... i_{q-1}} \times \mathbb{P}^1 \arrow[dl, "pr_1"] \\ & Y_{i_1 i_2 ... i_{q-1}} & \end{tikzcd}$$

If $i_q - i_{q-1} = 2$, then $\pi_q \colon Y_{i_1 ... i_q} \to Y_{i_1 ... i_{q-1}}$ is a homeomorphism.

Proof. Let $q \in \{1, ..., m\}$ such that $i_q - i_{q-1} = 1$. We clearly have $\pi_q = pr_1 \circ \phi_q$ and ϕ_q is a continuous map between compact Hausdorff spaces. Hence it will be a homeomorphism if it is bijective.

Notice that the vector space $F_{i_{q-1}} \subset \mathbb{C}^{2N}$ is z-stable and by Remark 1.18 we may assume that it is contained in the image of z. Hence Lemma 1.16 yields and isomorphism

$$C : z^{-1}F_{i_{q-1}} \cap F_{i_{q-1}}^{\perp} \xrightarrow{\cong} \mathbb{C}^2.$$

For given $(F_{i_1}, ..., F_{i_{q-1}}, l) \in Y_{i_1 ... i_{q-1}} \times \mathbb{P}^1$ it is easy to check that the flag

$$\left(F_{i_1}, ..., F_{i_{q-1}}, \left(C|_{z^{-1}F_{i_{q-1}} \cap F_{i_{q-1}}^{\perp}}\right)^{-1}(l) \oplus F_{i_{q-1}}\right) \tag{5}$$

is a well-defined preimage under ϕ_q. Let $(F_{i_1}, ..., F_{i_q}) \in Y_{i_1 ... i_q}$ be another preimage, i.e. we have

$$\phi_q(F_{i_1}, ..., F_{i_q}) = \Big(F_{i_1}, ..., F_{i_{q-1}}, C(F_{i_q} \cap F_{i_{q-1}}^{\perp})\Big) = (F_{i_1}, ..., F_{i_{q-1}}, l).$$

Since $F_{i_q} \cap F_{i_{q-1}}^{\perp} \subset z^{-1}F_{i_q-1} \cap F_{i_{q-1}}^{\perp}$ is the unique one-dimensional subspace which is mapped to l under the isomorphism C we have

$$F_{i_q} \cap F_{i_{q-1}}^{\perp} = \left(C|_{z^{-1}F_{i_{q-1}} \cap F_{i_{q-1}}^{\perp}}\right)^{-1}(l),$$

which implies

$$F_{i_q} = \left(C|_{z^{-1}F_{i_{q-1}} \cap F_{i_{q-1}}^{\perp}}\right)^{-1}(l) \oplus F_{i_{q-1}}.$$

Thus the preimage (5) is unique which proves that ϕ_q is indeed bijective.

In the case $i_q - i_{q-1} = 2$, the map $\pi_q : Y_{i_1 \dots i_q} \to Y_{i_1 \dots i_{q-1}}$ is clearly a continuous surjection between compact Hausdorff spaces and it suffices to show injectivity to prove the claim. By definition we have $F_{i_q} \subset z^{-1} F_{i_{q-1}}$ for every flag in $(F_{i_1}, ..., F_{i_q}) \in Y_{i_1 \dots i_q}$. By the assumption $i_q - i_{q-1} = 2$, Lemma 1.17 and Remark 1.18 we obtain

$$\dim(z^{-1} F_{i_{q-1}}) = \dim(F_{i_{q-1}}) + 2 = i_q,$$

showing that this inclusion is in fact an actual equality. Thus the vector space F_{i_q} is completely determined by $F_{i_{q-1}}$. In particular, there is precisely one preimage of $(F_{i_1}, ..., F_{i_{q-1}}) \in Y_{i_1 \dots i_{q-1}}$ under π_q. □

As a corollary we obtain a slight generalization of [CK08, Theorem 2.1] which also covers partial flags inside $\mathbb{C}^{2N}$ corresponding to admissible I.

Corollary 1.20. *Let $(i_{j_1}, ..., i_{j_{2m-n}})$ be the subfamily of I consisting of all entries where $i_{j_l} - 1 = i_{j_l - 1}$, i.e. $i_{j_1}, ..., i_{j_{2m-n}}$ are precisely the places of the dots in the dot-cross-sequence. Then the map $\phi_I : Y_I \to (\mathbb{P}^1)^{2m-n}$ defined by*

$$(F_{i_1}, ..., F_{i_m}) \mapsto \left(C(F_{i_{j_1}} \cap F^{\perp}_{i_{j_1 - 1}}), C(F_{i_{j_2}} \cap F^{\perp}_{i_{j_2 - 1}}), ..., C(F_{i_{j_{2m-n}}} \cap F^{\perp}_{i_{j_{2m-n}-1}}) \right)$$

is a homeomorphism.

Proof. Fix j_l for some $l \in \{1, ..., 2m-n\}$ and set $q = j_l$ and $p = j_{l-1}$, where $j_0 = 0$, to simplify notation. Let $\pi_{i_p \dots i_q}$ denote the composition:

$$Y_{i_1 \dots i_{q-1}} \xrightarrow{\pi_{i_{q-1}}} Y_{i_1 \dots i_{q-2}} \xrightarrow{\pi_{i_{q-2}}} \dots \xrightarrow{\pi_{i_{p+1}}} Y_{i_1 \dots i_p}.$$

Notice that all the projections in this composition are homeomorphisms by the definition of the sequence $(j_1, ..., j_{2m-n})$ and the second part of Proposition 1.19.

For the particular choice $q = j_1$ and $p = j_0$ we can apply the first part of Proposition 1.19 and obtain a homeomorphism

$$\phi_{i_1 \dots i_q} \colon Y_{i_1 \dots i_q} \xrightarrow{\phi_q} Y_{i_1 \dots i_{q-1}} \times \mathbb{P}^1 \xrightarrow{\pi_{i_p \dots i_q} \times \mathrm{id}} Y_{i_0} \times \mathbb{P}^1 \cong \mathbb{P}^1$$

which is explicitly given by

$$(F_{i_1}, ..., F_{i_q}) \mapsto C\left(F_{i_q} \cap F^{\perp}_{i_q - 1} \right).$$

If $q = j_l$ and $p = j_{l-1}$ and we assume that the homeomorphism $\phi_{i_1 \dots i_p} \colon Y_{i_1 \dots i_p} \to \left(\mathbb{P}^1\right)^{l-1}$ given by

$$(F_{i_1}, ..., F_{i_p}) \mapsto \left(C(F_{i_{j_{l-1}}} \cap F^{\perp}_{i_{j_{l-1}-1}}), ..., C(F_{i_{j_1}} \cap F^{\perp}_{i_{j_1 - 1}}) \right)$$

is already constructed, then we define $\phi_{i_1 \dots i_q}$ as the composition

$$Y_{i_1 \dots i_q} \xrightarrow{\phi_q} Y_{i_1 \dots i_{q-1}} \times \mathbb{P}^1 \xrightarrow{\pi_{i_p \dots i_q} \times \mathrm{id}} Y_{i_1 \dots i_p} \times \mathbb{P}^1 \xrightarrow{\phi_{i_1 \dots i_p} \times \mathrm{id}} \left(\mathbb{P}^1\right)^l.$$

The reader easily verifies that this homeomorphism is given by

$$(F_{i_1}, ..., F_{i_q}) \mapsto \left(C(F_{i_{j_l}} \cap F^{\perp}_{i_{j_l - 1}}), ..., C(F_{i_{j_1}} \cap F^{\perp}_{i_{j_1 - 1}}) \right).$$

Thus the claim of the corollary follows (after reversing the order of the factors). □

Remark 1.21. If the sequence I consists of all integers between 1 and n we will also write Y_n instead of Y_I and similarly ϕ_n instead of ϕ_I.

1.3 Topology of irreducible components

By the results of the last section we have a homeomorphism $\phi_I\colon Y_I \xrightarrow{\cong} (\mathbb{P}^1)^{2m-n}$. Moreover, the $(n-k,k)$-Spaltenstein variety of type I sits inside Y_I via the identification (4). For the rest of this section the notation $\mathcal{F}l_I^{n-k,k}$ will always refer to this embedded Spaltenstein variety. The next goal is to see what the images of the irreducible components look like under ϕ_I.

1.3.1 Technical preliminaries

We begin by proving some statements which will simplify some of the arguments later on.

Notice that we have an explicit description of the irreducible components of the embedded Spaltenstein variety $\mathcal{F}l_I^{n-k,k} \subset Y_I$ by replacing the map x in condition (i) of Proposition 1.11 with the map $z_{n-k,k}$ defined to be the restriction of z to $\mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k)$. A priori it is necessary to work with this restricted map because preimages under z might not be contained in $\mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k)$ anymore. The following lemma shows that this does not happen.

Lemma 1.22. *The irreducible component $K_a \subset \mathcal{F}l_I^{n-k,k} \subset Y_I$ corresponding to $a \in B_I^{n-k,k}$ consists of precisely those flags $(F_{i_1}, ..., F_{i_m}) \in Y_I$ which satisfy the following conditions imposed by the cup diagram $\tilde{a}$:*

(i') If $(i,j) \in \tilde{a}$ (the arc connecting the symbols i and j might be an invisible arc), then

$$F_j = z^{-\frac{1}{2}(j-i+1)} F_{i-1}.$$

(ii') If $(i) \in \tilde{a}$, then we have

$$F_i = F_{i-1} + \mathrm{span}\left(e_{\frac{1}{2}(i+\rho_a(i))}\right).$$

Before we prove the above lemma, it is useful to note the following (trivial) fact:

Lemma 1.23. *Let $U, U' \subset \mathbb{C}^{2N}$ be two subspaces. Then we have*

$$z(U+U') = zU + zU'.$$

If we additionally assume $U, U' \subset \mathrm{im}(z)$, then we also get

$$z^{-1}(U+U') = z^{-1}U + z^{-1}U'. \tag{6}$$

The reader is invited to make up examples showing that the assumption about the containment of U, U' in the image of z is indeed necessary for equation (6) to be true.

Proof (Lemma 1.22). Assume $(F_{i_1}, ..., F_{i_m}) \in K_a \subset \mathcal{F}l_I^{n-k,k} \subset Y_I$. By condition (i) of Proposition 1.11 we have an inclusion

$$F_j = z_{n-k,k}^{-\frac{1}{2}(j-i+1)} F_{i-1} \subseteq z^{-\frac{1}{2}(j-i+1)} F_{i-1}, \tag{7}$$

whenever $(i,j) \in \tilde{a}$. We obtain a chain of equation

$$\dim\left(z^{-\frac{1}{2}(j-i+1)} F_{i-1}\right) = j - i + 1 + \dim(F_{i-1}) = \dim(F_j),$$

where the first equation follows from Lemma 1.17 and Remark 1.18. Hence the inclusion (7) is in fact an equality and we obtain (i'). Moreover, condition (ii) clearly implies (ii').

For the converse take a flag $(F_{i_1}, ..., F_{i_m}) \in Y_I$ satisfying conditions (i') and (ii') of the lemma. Again, condition (ii) trivially holds. In order to prove condition (i) for this flag, it suffices to show that

$$F_{i_m} = \mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k), \tag{8}$$

because this equality implies

$$\begin{aligned} F_j &= F_j \cap \mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k) \\ &= z^{-\frac{1}{2}(j-i+1)} F_i \cap \mathrm{span}(e_1, ..., e_{n-k}, f_1, ..., f_k) \\ &= z_{n-k,k}^{-\frac{1}{2}(j-i+1)} F_i, \end{aligned}$$

whenever $(i, j) \in \tilde{a}$.

Let $r_1, ..., r_{n-2k}$ be the positions of the dots in $\tilde{a}$ to which the rays are connected, numbered from left to right. We additionally set $r_0 := 0$. So for fixed $s \in \{0, 1, ..., n-2k-1\}$, the diagram consists of arcs only in between nodes r_s and r_{s+1}. In particular, there is a sequence of outermost arcs (some of which might be invisible)

$$(p_1, q_1), ..., (p_t, q_t)$$

such that $p_1 = r_s + 1$, $q_t = r_{s+1} - 1$ and $q_l + 1 = p_{l+1}$ for all $l \in \{1, ..., t-1\}$. By (i') we have relations

$$F_{q_l} = z^{-\frac{1}{2}(q_l - p_l + 1)} F_{p_l - 1}$$

for all $l \in \{1, ..., t\}$. We claim that

$$F_{q_l} = z^{-\frac{1}{2}(q_l - p_1 + 1)} F_{p_1 - 1} \tag{9}$$

holds for all $l \in \{1, ..., t\}$. This is obviously true for $l = 1$. So suppose (9) holds for $l - 1$, $l > 1$. Since $q_{l-1} = p_l - 1$ we calculate

$$\begin{aligned} F_{q_l} &= z^{-\frac{1}{2}(q_l - p_l + 1)} F_{p_l - 1} \\ &= z^{-\frac{1}{2}(q_l - p_l + 1)} z^{-\frac{1}{2}(q_{l-1} - p_1 + 1)} F_{p_1 - 1} \\ &= z^{-\frac{1}{2}(q_l - p_1 + 1)} F_{p_1 - 1}, \end{aligned}$$

which proves (9) by induction. Setting $l = t$ one obtains the equation

$$F_{q_t} = z^{-\frac{1}{2}(q_t - p_1 + 1)} F_{p_1 - 1} = z^{-\frac{1}{2}(q_t - p_1 + 1)} F_{r_s} \tag{10}$$

which we will use below.

After these local considerations the next step is to relate the vector spaces of a flag beyond the confines of two subsequent rays, too. From now on set $q_s := r_{s+1} - 1$ for $s \in \{0, 1, ..., n-2k-1\}$ and $q_{n-2k} := n$. The next claim will be the following:

$$F_{q_s} = z^{-\frac{1}{2}(q_s - s)}(0) + \sum_{\alpha=1}^{s} \mathrm{span}\left(e_{\frac{1}{2}(q_s - s) + \alpha}\right) \tag{11}$$

For $s = 0$ this claim is obviously true, because equation (11) reduces to equation (10). Now assume that the claim holds for all $0 \leq l \leq s$. Then we have

$$\begin{aligned} F_{q_{s+1}} &= z^{-\frac{1}{2}(q_{s+1} - r_{s+1})} F_{r_{s+1}} \\ &= z^{-\frac{1}{2}(q_{s+1} - r_{s+1})} \left(F_{q_s} + \mathrm{span}(e_{\frac{1}{2}(r_{s+1} + s + 1)})\right) \\ &= z^{-\frac{1}{2}(q_{s+1} - r_{s+1})} F_{q_s} + z^{-\frac{1}{2}(q_{s+1} - r_{s+1})} \mathrm{span}(e_{\frac{1}{2}(r_{s+1} + s + 1)}), \end{aligned}$$

and inserting the induction hypothesis into the first summand yields

$$
\begin{aligned}
z^{-\frac{1}{2}(q_{s+1}-r_{s+1})}F_{q_s} &= z^{-\frac{1}{2}(q_{s+1}-r_{s+1})}\left(z^{-\frac{1}{2}(q_s-s)}(0)\right) + \sum_{\alpha=1}^{s} z^{-\frac{1}{2}(q_{s+1}-r_{s+1})}\operatorname{span}\left(e_{\frac{1}{2}(q_s-s)+\alpha}\right) \\
&= z^{-\frac{1}{2}(q_{s+1}-(s+1))}(0) + \sum_{\alpha=1}^{s}\operatorname{span}\left(e_{\frac{1}{2}(q_s-s)+\alpha+\frac{1}{2}(q_{s+1}-r_{s+1})}\right) \\
&= z^{-\frac{1}{2}(q_{s+1}-(s+1))}(0) + \sum_{\alpha=1}^{s}\operatorname{span}\left(e_{\frac{1}{2}(q_s-(s+1))+\alpha}\right),
\end{aligned}
$$

where we used the relation $r_{s+1} = q_s + 1$. The second summand simplifies to

$$
z^{-\frac{1}{2}(q_{s+1}-r_{s+1})}\operatorname{span}(e_{r_{s+1}+s+1}) = \operatorname{span}(e_{\frac{1}{2}(q_{s+1}+(s+1))})
$$

and hence we get

$$
F_{q_{s+1}} = z^{-\frac{1}{2}(q_{s+1}-(s+1))}(0) + \sum_{\alpha=1}^{s+1}\operatorname{span}\left(e_{\frac{1}{2}(q_s-(s+1))+\alpha}\right)
$$

which proves the claim.

Inserting $s = n - 2k$ into (11) yields

$$
\begin{aligned}
F_{i_m} = F_n &= z^{-k}(0) + \sum_{\alpha=1}^{s+1}\operatorname{span}(e_{k+\alpha}) \\
&= \operatorname{span}(e_1, ...e_k, f_1, ..., f_k) + \operatorname{span}(e_{k+1}, ..., e_{n-k}) \\
&= \operatorname{span}(e_1, ...e_{n-k}, f_1, ..., f_k),
\end{aligned}
$$

which finishes the proof of the lemma. □

For admissible $I = (i_1, ..., i_m)$ and $p \in \{1, ..., n\}$ such that $p-1, p+1 \in \{i_1, ..., i_m\}$, i.e. $p-1 = i_q$ and $p+1 = i_{q'}$ for some q and q', we can define a new admissible family denoted by $I - \{p, p+1\} = (i'_1, ..., i'_{m-(q'-q)})$, where

$$
i'_l := \begin{cases} i_l, & \text{if } l \leq q, \\ i_{l+(q'-q)} - 2 & \text{if } l > q. \end{cases}
$$

It is easy to verify that we have inequalities $0 < i'_l - i'_{l-1} \leq 2$ for all $l \in \{1, ..., m-(q'-q)\}$. Passing from I to $I - \{p, p+1\}$ can be understood combinatorially by deleting the symbols at positions p and $p+1$ in the dot-cross sequence corresponding to I.

Example 1.24. Consider $I = (1, 3, 4, 5, 6, 7)$ as in Example 1.4 with corresponding dot-cross sequence $\bullet \times \times \bullet \bullet \bullet \bullet$. If we choose $p = 2$, then $q - q' = 1$ and the family $I - \{2, 3\}$ is given by $(1, 2, 3, 4, 5)$. For $p = 4$ we have $q - q' = 2$ and the family $I - \{4, 5\}$ is given by $(1, 3, 4, 5)$. From the combinatorial point of view we have:

$$
I - \{2, 3\} = \bullet \bullet \bullet \bullet \bullet \qquad\qquad I - \{4, 5\} = \bullet \times \times \bullet \bullet.
$$

Since $p-1, p+1 \in \{i_1, ..., i_m\}$, it makes sense to define

$$
X_{I,p} := \{(F_{i_1}, ..., F_{i_m}) \in Y_I \mid F_{p+1} = z^{-1}F_{p-1}\}. \tag{12}
$$

Lemma 1.25. *The map* $\psi_{I,p} \colon X_{I,p} \to Y_{I-\{p,p+1\}}$ *given by the assignment*

$$(F_{i_1}, ..., F_{i_m}) \mapsto \left(F'_{i'_1}, ..., F'_{i'_{m-(q'-q)}} \right),$$

where

$$F'_{i'_l} := \begin{cases} F_{i_l}, & \text{if } l \leq q, \\ zF_{i_{l+(q'-q)}} & \text{if } l > q. \end{cases} \tag{13}$$

is a well-defined surjection.

Proof. Let $(F_{i_1}, ..., F_{i_m}) \in X_{I,p}$ be a flag and let

$$\psi_{I,p}(F_{i_1}, ..., F_{i_m}) = \left(F'_{i'_1}, ..., F'_{i'_{m-(q'-q)}} \right)$$

be its image. By Lemma 1.17 and Remark 1.18 we have

$$\dim\left(zF_{i_{l+(q'-q)}}\right) = \dim\left(F_{i_{l+(q'-q)}}\right) - 2 = i_{l+(q'-q)} - 2,$$

which implies $\dim(F'_{i'_l}) = i'_l$ for every $l \in \{1, ..., m-(q'-q)\}$ by the definition of $(i'_1, ..., i'_{m-(q'-q)})$ and (13). Thus the vector spaces of the flag $\psi_{I,p}(F_{i_1}, ..., F_{i_m})$ have the right dimensions.

Notice that we have inclusions

$$zF'_{i'_l} = z\left(zF_{i_{l+(q'-q)}}\right) \subset zF_{i_{l-1+(q'-q)}} = F'_{i'_{l-1}} \tag{14}$$

for every $l > q+1$ and

$$zF'_{i'_l} = zF_{i_l} \subset F_{i_{l-1}} = F'_{i'_{l-1}}, \tag{15}$$

if $l \leq q$. In particular, (14) and (15) together with

$$zF'_{i'_{q+1}} = zF_{i_{q'+1}} \subset zF_{i_{q'}} = F_{i_q} = F'_{i'_q}$$

show that there are inclusions $zF'_{i'_l} \subset F'_{i'_{l-1}}$ for every $l \in \{1, ..., m-(q'-q)\}$.

We omit the easy check that the flag $(F_{i_1}, ..., F_{i_m}) \in X_{I,p}$ given by

$$F_{i_l} = \begin{cases} F'_{i'_l}, & \text{if } l \leq q, \\ z^{-1}F'_{i'_{l-(q'-q)}} & \text{if } l > q, \end{cases}$$

defines a preimage of $(F'_{i'_1}, ..., F'_{i'_{m-(q'-q)}})$ under $\psi_{I,p}$. Hence $\psi_{I,p}$ is surjective. □

Remark 1.26. In order to tie in with the notation introduced in Remark 1.21, we will write $X_{n,p}$ and $\psi_{n,p}$ in the case of full flags.

The following lemma is a bit technical and not well-motivated at this point. In some sense it shows that $\psi_{I,p}$ respects the relations defining the irreducible components of $\mathcal{F}l_I^{n-k,k}$. Reading the proofs of Lemma 1.28 and Lemma 1.34, where everything is a bit more explicit, will probably be a great help for the reader in cherishing its generality and power.

Lemma 1.27. *Let* $(F_{i_1}, ..., F_{i_m}) \in X_{I,p}$ *be a flag and let* $i_{s_1}, i_{s_2} \in \{i_1, ..., i_m\} - \{p, p+1\}$ *such that* $s_1 < s_2$. *If we define*

$$r_j := \begin{cases} s_j & \text{if } s_j \leq q, \\ s_j - (q' - q) & \text{if } s_j > q, \end{cases}$$

for $j = 1, 2$, *then the following equivalence holds:*

$$F_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2} - i_{s_1} + 1)} F_{i_{s_1}} \quad \Leftrightarrow \quad F'_{i'_{r_2}} = z^{-\frac{1}{2}(i'_{r_2} - i'_{r_1} + 1)} F'_{i'_{r_1}}. \tag{16}$$

Moreover, if $i_s \in \{i_1, ..., i_m\} - \{p, p+1\}$ *and*

$$r := \begin{cases} s & \text{if } s \leq q, \\ s - (q' - q) & \text{if } s > q, \end{cases}$$

then there is an equivalence

$$F_{i_s} = F_{i_{s-1}} + \operatorname{span}\left(e_{\frac{1}{2}(i_s + \alpha)}\right) \quad \Leftrightarrow \quad F'_{i'_r} = F'_{i'_{r-1}} + \operatorname{span}\left(e_{\frac{1}{2}(i'_r + \alpha)}\right), \tag{17}$$

where α *is an integer such that* $i_s + \alpha$ *is even and* $\frac{1}{2}(i_s + \alpha) \leq N$.

Proof. The proof of equivalence (16) is divided into three cases:

- If $s_1 < s_2 \leq q$, then we have $r_j = s_j$ and

  $$F'_{i'_{r_j}} = F'_{i'_{s_j}} = F_{i_{s_j}}$$

  for $j = 1, 2$. In particular, both sides of (16) are exactly the same statements.

- If $s_1 \leq q < q' < s_2$, then

  $$F'_{i'_{r_1}} = F_{i_{s_1}} \quad \text{and} \quad F'_{i'_{r_2}} = F'_{i'_{s_2-(q'-q)}} \overset{(13)}{=} zF_{i_{s_2}}.$$

  Moreover, we have

  $$i'_{r_2} - i'_{r_1} = i'_{s_2-(q'-q)} - i'_{s_1} = i_{s_2} - i_{s_1} - 2.$$

  Thus we only have to show

  $$F_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2} - i_{s_1} + 1)} F_{i_{s_1}} \quad \Leftrightarrow \quad zF_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2} - i_{s_1} + 1) + 1} F_{i_{s_1}}$$

  to prove (16). But this follows by applying z (respectively z^{-1}) to the left (respectively right) side of the equivalence.

- If $q' < s_1 < s_2$, then we have

  $$F'_{i'_{r_j}} = F'_{i'_{s_j-(q'-q)}} = zF_{i_{s_j}}$$

  for $j = 1, 2$ and

  $$i'_{r_2} - i'_{r_1} = i'_{s_2-(q'-q)} - i'_{s_1-(q'-q)} = i_{s_2} - 2 - (i_{s_1} - 2) = i_{s_2} - i_{s_1}.$$

  Hence (16) is the same statement as

  $$F_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2} - i_{s_1} + 1)} F_{i_{s_1}} \quad \Leftrightarrow \quad zF_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2} - i_{s_1} + 1) + 1} F_{i_{s_1}},$$

  which we already treated in the case above.

In order to prove equivalence (17), two cases have to be considered.

- If $i_s \leq p-1$, i.e. $s \leq q$, there is nothing to show because both sides of the equivalence are the same statements similar to the first case in the proof of (16).
- If $i_s > p+1$, i.e. $s > q'$, we have to show the equivalence

$$F_{i_s} = F_{i_{s-1}} + \operatorname{span}\left(e_{\frac{1}{2}(i_s+\alpha)}\right) \Leftrightarrow F'_{i'_{s-(q'-q)}} = F'_{i'_{s-(q'-q)-1}} + \operatorname{span}\left(e_{\frac{1}{2}\left(i'_{s-(q'-q)}+\alpha\right)}\right), \tag{18}$$

because $r = s-(q'-q)$.

Now suppose the left hand side of (18) holds. Since $s-(q'-q) > q$ in the case under consideration, we obtain

$$\begin{aligned} F'_{i'_{s-(q'-q)}} &\overset{(13)}{=} zF_{i_s} = z\left(F_{i_{s-1}} + \operatorname{span}(e_{\frac{1}{2}(i_s+\alpha)})\right) \\ &\overset{(1.23)}{=} zF_{i_{s-1}} + z\left(\operatorname{span}(e_{\frac{1}{2}(i_s+\alpha)})\right). \end{aligned}$$

Since $i'_{s-(q'-q)} = i_s - 2$ we obtain

$$\begin{aligned} z\left(\operatorname{span}(e_{\frac{1}{2}(i_s+\alpha)})\right) &= \operatorname{span}(e_{\frac{1}{2}(i_s+\alpha)-1}) \\ &= \operatorname{span}(e_{\frac{1}{2}((i_s-2)+\alpha)}) \\ &= \operatorname{span}(e_{\frac{1}{2}(i'_{s-(q'-q)}+\alpha)}) \end{aligned}$$

and thus it remains to prove

$$zF_{i_{s-1}} = F'_{i'_{s-(q'-q)-1}},$$

in order to deduce the right hand side of equivalence (18). If $s > q'+1$ this is an immediate consequence of (13) and if $s = q'+1$ we calculate

$$F'_{i'_{s-(q'-q)-1}} = F'_{i'_q} = F_{i_q} = zF_{i_{q'}} = zF_{i_{s-1}},$$

where we used the fact that $(F_{i_1}, ..., F_{i_m}) \in X_{I,p}$ for the third equality. We leave it to the reader to check that the right hand side of (18) implies the left hand side (simply apply z^{-1} to the equation and argue similarly as above).

$\square$

1.3.2 Review of the Springer fiber case

In the following we only consider the special case of full flags. We prove that ϕ_n maps the irreducible component $K_a \subset \mathcal{F}l^{n-k,k}$ corresponding to the cup diagram $a \in B^{n-k,k}$ to the set $T_a \subset \left(\mathbb{P}^1\right)^n$ (cf. Proposition 1.32 below) defined by

$$T_a := \left\{(l_1, ..., l_n) \in \left(\mathbb{S}^2\right)^n \mid l_i^\perp = l_j \text{ if } (i,j) \in a \text{ and } l_i = \operatorname{span}(e) \text{ if } (i) \in a\right\}.$$

For a fixed cup diagram $a \in B^{n-k,k}$ with $k \geq 1$ there always exists $p \in \{1, ..., n-1\}$ such that the dots p and $p+1$ are connected by an arc in a. Removing this arc yields a new cup diagram in $B^{n-k-1,k-1}$ which we denote by $a - \{p, p+1\}$.

Corresponding to the choice of p consider the set

$$X_{n,p} := \{(F_1, ..., F_n) \in Y_n \mid F_{p+1} = z^{-1}F_{p-1}\}$$

and the map $\psi_{n,p}\colon X_{n,p} \to Y_{n-2}$ given by

$$(F_1, ..., F_n) \mapsto (F_1, ..., F_{p-1}, zF_{p+2}, ..., zF_n)$$

as introduced in (12) and Lemma 1.25. The following lemma generalizes [Weh09, Lemma 3.2] from the equal-row case to the general two-row case.

Lemma 1.28. *There is an equality of sets:*

$$K_a = \psi_{n,p}^{-1}(K_{a-\{p,p+1\}}).$$

Proof. By assumption there is an arc in a connecting the dots p and $p+1$. Hence the equality $F_{p+1} = z^{-1}F_{p-1}$ holds for every $(F_1, ..., F_n) \in K_a$ by Lemma 1.22 and K_a is contained in $X_{n,p}$.

Since $K_a = \psi_{n,p}^{-1}(K_{a-\{p,p+1\}})$ is equivalent to the statement

$$(F_1, ..., F_n) \in K_a \quad \Leftrightarrow \quad \psi_{n,p}(F_1, ..., F_n) \in K_{a-\{p,p+1\}},$$

we have to prove that a flag $(F_1, ..., F_n) \in X_{n,p}$ satisfies the conditions of Lemma 1.22 for the cup diagram a, if and only if the flag $(F'_1, ..., F'_{n-2})$ satisfies the conditions of the lemma for $a - \{p, p+1\}$. Being totally explicit, it remains to show the equivalence

$$F_{s_2} = z^{-\frac{1}{2}(s_2-s_1+1)}F_{s_1} \quad \Leftrightarrow \quad F'_{r_2} = z^{-\frac{1}{2}(r_2-r_1+1)}F'_{r_1},$$

where

$$r_j := \begin{cases} s_j & \text{if } s_j \leq p-1, \\ s_j - 2 & \text{if } s_j > p-1, \end{cases}$$

for $j = 1, 2$ and the equivalence

$$F_s = F_{s-1} + \operatorname{span}\left(e_{\frac{1}{2}(s+\rho_a(s))}\right) \quad \Leftrightarrow \quad F'_r = F'_{r-1} + \operatorname{span}\left(e_{\frac{1}{2}\left(r+\rho_{a-\{p,p+1\}}(r)\right)}\right).$$

where

$$r := \begin{cases} s & \text{if } s \leq p-1, \\ s-2 & \text{if } s > p-1. \end{cases}$$

The reader easily sees that this is just the statement of Lemma 1.27 (with $\alpha = \rho_a(s) = \rho_{a-\{p,p+1\}}(r)$) in the special case considered here. $\square$

In the following lemma we recall results contained in [CK08, Theorem 2.1], [Weh09, Lemma 2.4].

Lemma 1.29. *The homeomorphism ϕ_n maps $X_{n,p}$ to the set*

$$A_{n,p} := \{(l_1, ..., l_n) \in (\mathbb{P}^1)^n \mid l_{p+1} = l_p^\perp\}$$

and the following diagram commutes:

$$\begin{tikzcd} X_{n,p} \arrow[r, two heads, "\psi_{n,p}"] \arrow[d, "\phi_n|_{X_{n,p}}"', "\cong"] & Y_{n-2} \arrow[d, "\phi_{n-2}", "\cong"'] \\ A_{n,p} \arrow[r, two heads, "f_{n,p}|_{A_{n,p}}"'] & (\mathbb{P}^1)^{n-2} \end{tikzcd}$$

where $f_{n,p}\colon (\mathbb{P}^1)^n \twoheadrightarrow (\mathbb{P}^1)^{n-2}$ is the map which forgets the coordinates p and $p+1$.

Before we can prove the lemma we have to introduce some more linear algebra. The following result was first proved in [Weh09, Lemma 2.2].

Lemma 1.30. *Let $U \subset \mathbb{C}^{2N}$ be a z-stable subspace such that $ker(z) \subset U \subset im(z)$. Then z maps $z^{-1}U \cap U^\perp$ isomorphically to $U \cap (zU)^\perp$, and the following diagram commutes:*

$$\begin{tikzcd} z^{-1}U \cap U^\perp \arrow[rr, "z"] \arrow[dr, "C"'] & & U \cap (zU)^\perp \arrow[dl, "C"] \\ & \mathbb{C}^2 & \end{tikzcd}$$

Proof. In order to see that z actually maps elements in $z^{-1}U\cap U^\perp$ to elements of $U\cap(zU)^\perp$, notice that for $v \in z^{-1}U \cap U^\perp$ we clearly have $z(v) \in U$ (because in particular $v \in z^{-1}U$). For the proof that $z(v) \in (zU)^\perp$ pick any $u \in U$ and write

$$v = v_1 + ... + v_N, \qquad u = u_1 + ... + u_N$$

with $v_j, u_j \in \operatorname{span}(e_j, f_j)$. Since $\operatorname{span}(e_1, f_1) = \ker(z) \subset U$ by assumption and $v \in U^\perp$ we deduce that $v_1 = 0$. It is easy to see that $z : \operatorname{span}(e_j, f_j) \to \operatorname{span}(e_{j-1}, f_{j-1})$ is unitary for $j \geq 2$. Thus one gets

$$\langle zv, zu\rangle = \sum_{i=2}^{N}\langle zv_i, zu_i\rangle = \sum_{i=2}^{N}\langle v_i, u_i\rangle = \langle v, u\rangle = 0,$$

which proves $zv \in (zU)^\perp$.

To check the commutativity of the diagram let $v \in z^{-1}U \cap U^\perp$ and decompose $v = v_2 + ... + v_N$ with $v_j \in \operatorname{span}(e_j, f_j)$ as above. One easily checks that $C(v_j) = C(zv_j)$ for all $j \geq 2$ and hence obtains

$$C(zv) = \sum_{j=2}^{N} C(zv_j) = \sum_{j=2}^{N} C(v_j) = C(v).$$

Our assumptions together with Lemma 1.16 imply that $C\colon z^{-1}U \cap U^\perp \to \mathbb{C}^2$ is an isomorphism and so $z\colon z^{-1}U \cap U^\perp \to U \cap (zU)^\perp$ must be injective by the commutativity of the diagram. By Lemma 1.17 we have

$$\dim(U) = 2 + \dim(zU). \tag{19}$$

Basic linear algebra tells us that $U \cap (zU)^\perp \cong U/zU$ and hence $U \cap (zU)^\perp$ is two-dimensional by (19) and thus $z\colon z^{-1}U \cap U^\perp \to U \cap (zU)^\perp$ is an isomorphism. $\square$

Corollary 1.31. *Let $U \subset U' \subset \mathbb{C}^{2N}$ be two subspaces such that $\dim U' = \dim U + 1$ and $\ker(z) \subset U \subset \operatorname{im}(z)$. If U is z-stable and $zU' \subset U$, then*

$$C(U' \cap U^\perp) = C(zU' \cap (zU)^\perp).$$

Proof. By our assumptions we can apply Lemma 1.30 to U and obtain an isomorphism

$$z\colon z^{-1}U \cap U^\perp \xrightarrow{\cong} U \cap (zU)^\perp.$$

Notice that the inclusion $zU' \subset U$ together with the assumption on the dimension on U' imply that $U' \cap U^\perp \subset z^{-1}U \cap U^\perp$ is a one-dimensional subspace. The same is true for

$zU' \cap (zU)^\perp \subset U \cap (zU)^\perp$. Moreover, it is easy to see that for every $v \in U' \cap U^\perp$ we have $zv \in zU' \cap (zU)^\perp$ which implies $z(U' \cap U^\perp) = zU' \cap (zU)^\perp$ by the above dimensional considerations. Hence, one obtains the desired equality

$$C(U' \cap U^\perp) = C(z(U' \cap U^\perp)) = C(zU' \cap (zU)^\perp),$$

because $C(v) = C(z(v))$ for all $v \in z^{-1}U \cap U^\perp$ by Lemma 1.30. $\square$

Proof (Lemma 1.29). Let $(F_1, ..., F_n) \in X_{n,p}$ be a flag. Both vector spaces $F_{p+1} \cap F_p^\perp$ and $F_p \cap F_{p-1}^\perp$ are contained in $z^{-1}F_{p-1} \cap F_{p-1}^\perp$ and they are clearly orthogonal. By Lemma 1.16 the map

$$C\colon z^{-1}F_{p-1} \cap F_{p-1}^\perp \xrightarrow{\cong} \mathbb{C}^2$$

is a unitary isomorphism. Hence we deduce that the images $l_{p+1} = C(F_{p+1} \cap F_p^\perp)$ and $l_p = C(F_p \cap F_{p-1}^\perp)$ are orthogonal.

In order to check the commutativity of the square we calculate

$$\begin{aligned} f_{n,p}\left(\phi_n(F_1, ..., F_n)\right) &= f_{n,p}\Big(C(F_1 \cap F_0^\perp), C(F_2 \cap F_1^\perp), ..., C(F_n \cap F_{n-1}^\perp)\Big) \\ &= (l_1, ..., l_{n-2}) \in \left(\mathbb{P}^1\right)^{n-2}, \end{aligned}$$

where

$$l_j = \begin{cases} C(F_j \cap F_{j-1}^\perp) & \text{if } j < p, \\ C(F_{j+2} \cap F_{j+1}^\perp) & \text{if } j \geq p. \end{cases} \tag{20}$$

On the other hand one gets

$$\begin{aligned} \phi_{n-2}\left(\psi_{n,p}(F_1, ..., F_n)\right) &= \phi_{n-2}\left(F_1, ..., F_{p-1}, zF_{p+2}, ..., zF_n\right) \\ &= (l'_1, ..., l'_{n-2}) \in \left(\mathbb{P}^1\right)^{n-2}, \end{aligned}$$

where

$$l'_j = \begin{cases} C(F_j \cap F_{j-1}^\perp) & \text{if } j < p, \\ C(zF_{j+2} \cap (zF_{j+1})^\perp) & \text{if } j \geq p. \end{cases} \tag{21}$$

Notice that in the case $j = p$ we inserted the equality $F_{p+1} = z^{-1}F_{p-1}$ to obtain the above result.

Comparing (20) and (21), we see that it suffices to prove that

$$C(F_{j+2} \cap F_{j+1}^\perp) = C(zF_{j+2} \cap (zF_{j+1})^\perp)$$

for all $j \geq p$ to complete the argument. But this follows from Corollary 1.31 by setting $U := F_{j+1}$ and $U' := F_{j+2}$, because

$$\ker(z) = z^{-1}(0) \subset z^{-1}F_{p-1} = F_{p+1} \subset F_{j+1}$$

and $F_{j+1} \subset \mathrm{im}(z)$ by increasing N if necessary (all the other hypotheses of the lemma are obviously satisfied). $\square$

Proposition 1.32. *There is an equality of sets:*

$$\phi_n(K_a) = T_a.$$

Proof. Let $n \geq 1$ and consider the Springer fiber $\mathcal{F}l^{n,0}$. This variety has one irreducible component corresponding to the matching consisting of n rays and no arcs. The relations of Proposition 1.22 imply that this irreducible component consists of precisely one flag, namely

$$(\mathrm{span}(e_1), \mathrm{span}(e_1, e_2), ..., \mathrm{span}(e_1, ..., e_n)) .$$

Now it is easy to check that ϕ_n maps this flag to $(\mathrm{span}(e), ..., \mathrm{span}(e)) \in \left(\mathbb{P}^1\right)^n$.

Notice also that by Lemma 1.29 we obtain the claim for $\mathcal{F}l^{1,1}$.

For the general case we proceed by induction, assuming that the claim of the proposition is true for the irreducible components of $\mathcal{F}l^{n-k-1,k-1}$. Let $a \in B^{n-k,k}$ be a cup diagram and fix an arc connecting dots p and $p+1$. By Lemma 1.29 there is an equality of maps

$$\psi_{n,p} \circ \phi_{n-2} = f_{n,p}|_{A_{n,p}} \circ \phi_n|_{X_{n,p}}.$$

In particular, we have an equality of sets

$$\phi_n|_{X_{n,p}} \left(\psi_{n,p}^{-1}(K_{a-\{p,p+1\}})\right) = f_{n,p}|_{A_{n,p}}^{-1} \left(\phi_{n-2}(K_{a-\{p,p+1\}})\right) .$$

Applying Lemma 1.28 and the induction hypothesis to this equality yields

$$\phi_n|_{X_{n,p}}(K_a) = f_{n,p}|_{A_{n,p}}^{-1}(T_{a-\{p,p+1\}}).$$

But we clearly have

$$f_{n,p}|_{A_{n,p}}^{-1}(T_{a-\{p,p+1\}}) = T_a,$$

which finishes the proof. $\square$

1.3.3 Extension to Spaltenstein varieties

Let $I = (i_1, ..., i_m)$ be admissible and let i_p be an integer in the sequence such that $i_p - i_{p-1} = 2$, i.e. there is a cross at i_p and $i_p - 1$ in the dot-cross sequence. By Lemma 1.25 we obtain a well-defined map $\psi_{I,i_p-1} \colon X_{I,i_p-1} \to Y_{I-\{i_p-1,i_p\}}$ given by

$$(F_{i_1}, ..., F_{i_m}) \mapsto \left(F_{i_1}, ..., F_{i_{p-1}}, zF_{i_{p+1}}, ..., zF_{i_m}\right) .$$

Lemma 1.33. *The following diagram commutes:*

$$\begin{tikzcd} X_{I,i_p-1} \arrow[rr, "\psi_{I,i_p-1}"] \arrow[dr, "\phi_I|_{X_{I,i_p-1}}"'] & & Y_{I-\{i_p-1,i_p\}} \arrow[dl, "\phi_{I-\{i_p-1,i_p\}}"] \\ & \left(\mathbb{P}^1\right)^{2m-n} & \end{tikzcd}$$

Proof. Let $(F_{i_1}, ..., F_{i_m}) \in X_{I,i_p-1}$ be a flag and let $(i_{j_1}, ..., i_{j_{2m-n}})$ be the subsequence of I consisting of all entries where $i_{j_l} - 1 = i_{j_l-1}$. Then we have

$$\begin{aligned} \phi_{I-\{i_p-1,i_p\}} \circ \psi_{I,i_p-1} (F_{i_1}, ..., F_{i_m}) &= \phi_{I-\{i_p-1,i_p\}} \left(F_{i_1}, ..., F_{i_{p-1}}, zF_{i_{p+1}}, ..., zF_{i_m}\right) \\ &= (l_1, ..., l_{2m-n}) \in \left(\mathbb{P}^1\right)^{2m-n}, \end{aligned}$$

where

$$l_s = \begin{cases} C(F_{i_{j_s}} \cap F_{i_{j_s-1}}^{\perp}) & \text{if } j_s \leq p-1, \\ C(zF_{i_{j_s}} \cap (zF_{i_{j_s-1}})^{\perp}) & \text{if } j_s \geq p+1. \end{cases} \tag{22}$$

The same argument as in the proof of Lemma 1.29 shows that

$$C(zF_{i_{j_s}} \cap (zF_{i_{j_s-1}})^{\perp}) = C(F_{i_{j_s}} \cap (F_{i_{j_s-1}})^{\perp}),$$

for $j_s \geq p+1$ and hence we have $\phi_I|_{X_{I,i_p-1}} = \phi_{I-\{i_p-1,i_p\}} \circ \psi_{I,i_p-1}$. $\square$

Lemma 1.34. *There is an equality of sets:*

$$K_a = \psi_{I,i_p-1}^{-1}\left(K_{a-\{i_p-1,i_p\}}\right).$$

Proof. Since there is an invisible arc in $\tilde{a}$ connecting the crosses at positions $i_p - 1$ and i_p in the dot cross sequence corresponding to I, we have $F_{i_p} = z^{-1}F_{i_{p-1}}$ for every flag $(F_{i_1}, ..., F_{i_m}) \in K_a$ and thus $K_a \subset X_{I,i_p-1}$.

Similar to the proof of Lemma 1.28 it remains to prove that a flag $(F_{i_1}, ..., F_{i_m}) \in X_{I,i_p-1}$ satisfies the conditions of Lemma 1.22 for the cup diagram $\tilde{a}$, if and only if the flag $(F'_{i'_1}, ..., F'_{i'_{m-1}})$ satisfies the conditions of the lemma for $\tilde{a} - \{i_p - 1, i_p\}$. More explicitly, we have to show the equivalence

$$F_{i_{s_2}} = z^{-\frac{1}{2}(i_{s_2}-i_{s_1}+1)}F_{i_{s_1}} \quad \Leftrightarrow \quad F'_{i'_{r_2}} = z^{-\frac{1}{2}(i'_{r_2}-i'_{r_1}+1)}F'_{i'_{r_1}},$$

where

$$r_j := \begin{cases} s_j & \text{if } s_j \leq p-1, \\ s_j - 1 & \text{if } s_j > p, \end{cases}$$

for $j = 1, 2$, and the equivalence

$$F_{i_s} = F_{i_{s-1}} + \operatorname{span}\left(e_{\frac{1}{2}(i_s+\rho_{\tilde{a}}(i_s))}\right) \quad \Leftrightarrow \quad F'_{i'_r} = F'_{i'_{r-1}} + \operatorname{span}\left(e_{\frac{1}{2}\left(i'_r+\rho_{\tilde{a}-\{i_p-1,i_p\}}(i'_r)\right)}\right).$$

where

$$r := \begin{cases} s & \text{if } s \leq p-1, \\ s - 1 & \text{if } s > p. \end{cases}$$

The reader easily sees that this is just the statement of Lemma 1.27 (with $\alpha = \rho_{\tilde{a}}(i_s) = \rho_{\tilde{a}-\{i_p-1,i_p\}}(i'_r)$) in the special case considered here. $\square$

Proposition 1.35. *There is an equality of sets:*

$$\phi_I(K_a) = T_{a_{red}}.$$

Proof. We induct on the number of indices in I satisfying $i_l - i_{l-1} = 2$. If $i_l - i_{l-1} = 1$ for all $l \in \{1, ..., m\}$ the claim was proved in Proposition 1.32.

Now let $a \in B_I^{n-k,k}$ be a cup diagram and let i_p be an index such that $i_p - i_{p-1} = 2$. Then we have

$$\phi_I(K_a) \overset{(1.34)}{=} \phi_I\left(\psi_{I,i_p-1}^{-1}\left(K_{a-\{i_p-1,i_p\}}\right)\right) \overset{(1.33)}{=} \phi_{I-\{i_p-1,i_p\}}\left(K_{a-\{i_p-1,i_p\}}\right),$$

and by the induction hypothesis we have

$$\phi_{I-\{i_p-1,i_p\}}\left(K_{a-\{i_p-1,i_p\}}\right) = T_{a_{\mathrm{red}}}.$$

$\square$

1.4 Gluing the irreducible components

In this section we finalize the proof of Theorem 1.15. The essential ingredient will be the construction of a homeomorphism $\gamma_{2m-n} \colon \left(\mathbb{P}^1\right)^{2m-n} \xrightarrow{\cong} \left(\mathbb{S}^2\right)^{2m-n}$ such that the composition

$$Y_I \xrightarrow{\phi_I} \left(\mathbb{P}^1\right)^{2m-n} \xrightarrow{\gamma_{2m-n}} \left(\mathbb{S}^2\right)^{2m-n}$$

maps the irreducible component K_a to the set S_a as defined at the end of section 1.1.

Consider the stereographic projection $\sigma\colon \mathbb{S}^2\backslash\{\mathbf{p}\} \to \mathbb{C}$ and its analog for projective space

$$\theta\colon \mathbb{P}^1\backslash\{\mathrm{span}(e)\} \to \mathbb{C}\,,\ \mathrm{span}(\alpha e+\beta f) \mapsto \frac{\alpha}{\beta}.$$

We can use σ and θ to define a homeomorphism $\Phi : \mathbb{P}^1 \to \mathbb{S}^2$ by

$$\mathrm{span}(\alpha e+\beta f) \mapsto \begin{cases} \sigma^{-1}\left(\theta\left(\mathrm{span}(\alpha e+\beta f)\right)\right) & \text{if } \mathrm{span}(\alpha e+\beta f) \neq \mathrm{span}(e), \\ \mathbf{p} = (0,0,1) & \text{if } \mathrm{span}(\alpha e+\beta f) = \mathrm{span}(e). \end{cases}$$

This induces a homeomorphism $\Phi_n\colon \left(\mathbb{P}^1\right)^n \to \left(\mathbb{S}^2\right)^n$ on the n-fold products by setting

$$\Phi_n(l_1,...,l_n) := \left(\Phi(l_1),...,\Phi(l_n)\right).$$

Moreover, define an involutive homeomorphism $I_n\colon \left(\mathbb{S}^2\right)^n \to \left(\mathbb{S}^2\right)^n$ by

$$(x_1,...,x_n) \mapsto (-x_1,x_2,-x_3,...,(-1)^n x_n)$$

and set $\gamma_n := I_n \circ \Phi_n$.

Proposition 1.36. *There is an equality of sets $\gamma_n(T_a) = S_a$ for every $a \in B^{n-k,k}$.*

Proof. From the definition of Φ_n it is easy to see that $\Phi_n(T_a)$ consists of precisely those elements which satisfy the conditions

- $x_i = -x_j$ if (i,j) is a pair in a
- $x_i = \mathbf{p}$ if (i) is a ray

Notice that if the dots i and j are connected by an arc, then either i is odd and j even or i is even and j is odd. Hence $I_n(\Phi_n(T_a))$ is the set of elements $(x_1,...,x_n) \in \left(\mathbb{S}^2\right)^n$ satisfying

- $x_i = x_j$ if (i,j) is a pair in a
- $x_i = (-1)^i\mathbf{p}$ if (i) is a ray

which shows $\gamma_n(T_a) = I_n(\Phi_n(K_a)) = S_a$. $\square$

Proof (Theorem 1.15). We can write the $(n-k,k)$-Spaltenstein variety of type I as the union of its irreducible components

$$\mathcal{F}l_I^{n-k,k} = \bigcup_{a\in B_I^{n-k,k}} K_a$$

and view it is a subvariety of Y_I. Pushing this union through the homeomorphism

$$Y_I \xrightarrow{\phi_I} \left(\mathbb{P}^1\right)^{2m-n} \xrightarrow{\gamma_{2m-n}} \left(\mathbb{S}^2\right)^{2m-n}$$

yields

$$\begin{aligned} \gamma_{2m-n}(\phi_I(\mathcal{F}l_I^{n-k,k})) &= \bigcup_{a\in B_I^{n-k,k}} \gamma_{2m-n}\left(\phi_I\left(K_a\right)\right) \overset{(1.35)}{=} \bigcup_{a\in B_I^{n-k,k}} \gamma_{2m-n}\left(T_{a_{\mathrm{red}}}\right) \\ &\overset{(1.36)}{=} \bigcup_{a\in B_I^{n-k,k}} S_{a_{\mathrm{red}}}, \end{aligned}$$

and hence we obtain the desired homeomorphism

$$\mathcal{F}l_I^{n-k,k} \cong \bigcup_{a \in B_I^{n-k,k}} S_{a_{\mathrm{red}}}.$$

$\square$

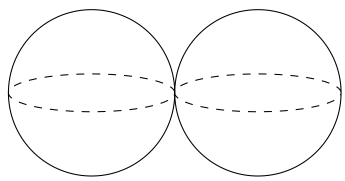
Example 1.37. Let $I = (1,3,4,6,7)$ and consider $\mathcal{F}l_I^x$ where x is of Jordan type $(4,3)$. Then the dot-cross sequence of I is given by $\bullet \times \times \bullet \times \times \bullet$ and the set $B_I^{4,3}$ consists of the following cup diagrams:

If we delete the crosses we obtain the following reduced diagrams:

Thus the Spaltenstein variety $\mathcal{F}l_I^x$ is homeomorphic the union of the following two sets:

$$S_{a_{\mathrm{red}}} = \{(x,x,-\mathbf{p}) \mid x \in \mathbb{S}^2\} \subset \left(\mathbb{S}^2\right)^3 \qquad S_{b_{\mathrm{red}}} = \{(-\mathbf{p},x,x) \mid x \in \mathbb{S}^2\} \subset \left(\mathbb{S}^2\right)^3.$$

Each of these submanifolds is homeomorphic to a two-sphere and their intersection is given by $S_{a_{\mathrm{red}}} \cap S_{b_{\mathrm{red}}} = \{(-\mathbf{p},-\mathbf{p},-\mathbf{p})\}$. Thus, topologically, the Spaltenstein variety $\mathcal{F}l_I^{4,3}$ is a wedge of two spheres:



2 A 2d TQFT via Spaltenstein varieties

The following material is at the heart of this work. After discussing the combinatorics of tangle and circle diagrams in the first subsection we use these diagrams to define subvarieties inside a finite product of Spaltenstein varieties in the second subsection. If two diagrams are related by a certain graphical operation called local surgery, then the corresponding varieties are related by an inclusion map, i.e. one of them sits inside the other one. Finally, in the last subsection, we compute the pullback and pushforward of these inclusions in cohomology and relate them to the TQFT associated with the ring of dual numbers equipped with a Frobenius algebra structure. The obtained results provide the basis for the geometric constructions and applications in the third part of this work.

2.1 Combinatorics of tangle and circle diagrams

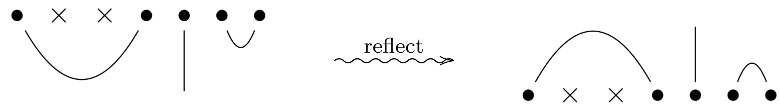
We begin by extending our combinatorial tool kit, introduce cap diagrams and explain how to combine them with cup diagrams in order to get tangle and circle diagrams. Similar combinatorial structures also occur in the work of Brundan and Stroppel [BS10, BS11].

Let us remark right from the beginning that whenever we draw cup, cap, tangle and circle diagrams we think of these as means of visualizing combinatorial structures. In particular, we do not distinguish between diagrams related by a planar isotopy leaving the dot-cross sequences in the diagrams fixed.

In the following $I = (i_1, \ldots, i_m)$ and $I' = (i'_1, \ldots, i'_{m'})$ are admissible sequences with highest integer n and n' respectively, i.e. $i_m = n$ and $i'_{m'} = n'$. We say that the sequences I and I' have length m and length m', respectively.

Definition 2.1. Let $a \in B_I^{n-k,k}$ be a cup diagram. Then we obtain the corresponding *cap diagram* $\overline{a}$ by reflecting the diagram a in the horizontal line containing the dots and crosses.

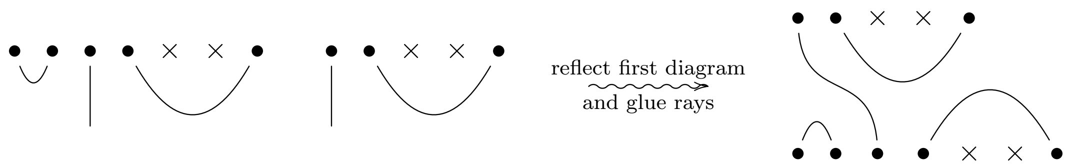
Example 2.2. The diagram on the right side is the cap diagram corresponding to the cup diagram on the left:



Notice that the order of the symbols in a dot-cross sequence induces an order on the rays of a cup diagram on this dot-cross sequence. In particular, if two cup diagrams $a \in B_I^{n-k,k}$, $b \in B_{I'}^{n'-k',k'}$ have the same number of rays, i.e. the equality $n - 2k = n' - 2k'$ holds, then there is a unique order-preserving bijection between the rays in the two diagrams.

Definition 2.3. Let $a \in B_I^{n-k,k}$, $b \in B_{I'}^{n'-k',k'}$ be two cup diagrams having the same number of rays. Then we define a new diagram $b\overline{a}$ by placing the cup diagram b on top of the cap diagram $\overline{a}$ (with both dot-cross sequences left-aligned) and connect the loose ends of the rays in b with the ones in $\overline{a}$ pairwise according to the order-preserving bijection. The resulting diagram $b\overline{a}$ can be drawn in the plane without crossings and it is called a *(combinatorial) tangle diagram of type* (I, I'). The glued pairs of rays are referred to as the *strands* of the tangle diagram.

Example 2.4. The following picture shows the process of building a combinatorial tangle diagram from two given cup diagrams as explained in Definition 2.3:

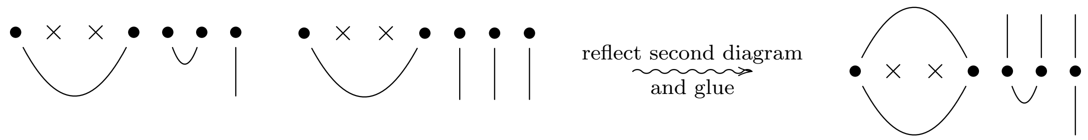


The rays in the two cup diagrams are glued at their endpoints to obtain the strand connecting the third dot on the bottom with the first dot on top.

Definition 2.5. Let $a \in B_I^{n-k,k}$, $b \in B_I^{n-k',k'}$ be cup diagrams. We define $\overline{b}a$ to be the diagram obtained by sticking the cap diagram $\overline{b}$ on top of the cup diagram a, i.e. we glue the two diagrams along their common dot-cross sequence (respecting the ordering of the sequence). This diagram is called a *circle diagram of type* I.

Remark 2.6. The term "circle diagram" might be a bit deceiving because in general circle diagrams consist of both circles and line segments (cf. the following example). A circle diagram does not contain any line segments if and only if the two cup diagrams involved in its construction do not contain any rays.

Example 2.7. Here is an example illustrating the gluing process of a cup and cap diagram in order to obtain a circle diagram:



The resulting diagram consists of one circle and two line segments.

Once we know how to stick cup diagrams on top of cap diagrams and vice versa we can easily construct more complicated diagrams by iterating the gluing process explained in Definition 2.3 and Definition 2.5.

For the remaining subsection we fix a collection $I_1, \dots, I_s$ of at least two admissible sequences. Let $m_1, m_2, \dots, m_s$ be the respective lengths of the admissible sequences and $n_1, \dots, n_s$ denote the respective heighest integers.

Definition 2.8. Let $b_i\overline{a_i}$ be a combinatorial tangle diagrams of type (I_i, I_{i+1}) for every $i \in \{1, \dots, s-1\}$. Using the gluing procedure from Definition 2.5 we can define a diagram

$$b_{s-1}\overline{a_{s-1}} \dots b_2\overline{a_2}b_1\overline{a_1}$$

called a *(combinatorial) tangle diagram of type* $(I_1, \dots, I_s)$. The set of all combinatorial tangle diagrams of type $(I_1, \dots, I_s)$ is denoted by $\mathcal{T}(I_1, ..., I_s)$.

Notice that for $s = 2$ the set $\mathcal{T}(I_1, I_2)$ consists of the diagrams from Definiton 2.3.

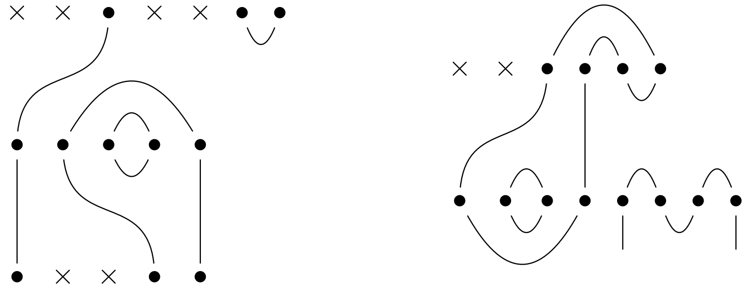
Definition 2.9. Let $b_{s-1}\overline{a_{s-1}} \dots b_2\overline{a_2}b_1\overline{a_1} \in \mathcal{T}(I_1, ..., I_s)$ be a combinatorial tangle diagram and let $a \in B_{I_1}^{n_1-k_1,k_1}, b \in B_{I_s}^{n_s-k_s,k_s}$ be cup diagrams. Then we can glue a to the bottom of t and $\overline{b}$ onto the top to obtain a diagram

$$\overline{b}ta = \overline{b}b_{s-1}\overline{a_{s-1}} \dots b_2\overline{a_2}b_1\overline{a_1}a$$

called a *circle diagram of type* $(I_1, \dots, I_s)$. The set of all circle diagrams of type $(I_1, \dots, I_s)$ is denoted by $\mathcal{C}(I_1, \dots, I_s)$. We write $\mathcal{C}(I)$ for the set of all circle diagrams of type I as in Definition 2.5.

Remark 2.10. As always, if I_i is a sequence consisting of all integers between 1 and n_i we will use the short notation $\mathcal{T}(\dots, n_i, \dots)$, respectively $\mathcal{C}(\dots, n_i, \dots)$, instead of writing down the whole admissible sequence.

Example 2.11. The diagram depicted on the left is a combinatorial tangle diagram in $\mathcal{T}((1,3,4,5), 5, (2,3,5,6,7))$ and on the right we have a circle diagram in $\mathcal{C}(8, (2,3,4,5,6))$:

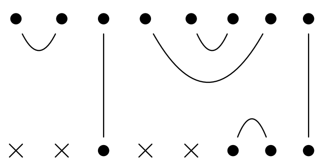


Remark 2.12. When we refer to the symbol (α,β) in a tangle or circle diagram, we mean the β-th symbol (counted from left to right) in the α-th dot cross sequence (counted from bottom to top). We think of the pairs of integers as being ordered lexicographically (the order in the first and second component comes from the canonical order of the natural numbers). By identifying the symbols with pairs of integers we obtain a total order on the set of all symbols in a tangle or circle diagram. This will be useful at the end of subsection 2.2.

The following subset of $\mathcal{T}(I_1,\dots,I_s)$ plays an important role in the next subsection (cf. Definition 2.21 and Remark 2.23):

Definition 2.13. Define $\mathbb{T}(I_1,...I_s)$ to be the set consisting of all tangle diagrams $t\in\mathcal{T}(I_1,\dots,I_s)$ which satisfy the following property: Whenever there is a strand in t connecting the dots (α,β) and $(\alpha+1,\beta')$, it follows that $\beta=\beta'$, i.e. every strand in t is vertical.

Example 2.14. Here is an example of a tangle diagram contained in $\mathbb{T}((2,3,5,6,7,8),8)$:



The tangle diagram from Example 2.11 provides a non-example because it is clearly not contained in the subset $\mathbb{T}((1,3,4,5),5,(2,3,5,6,7))$.

Notice that we can easily extend the reduction operation for cup diagrams as introduced in the first section to tangle and circle diagrams:

Definition 2.15. Given a tangle diagram $t=b_{s-1}\overline{a_{s-1}}\dots b_1\overline{a_1}\in\mathcal{T}(I_1,\dots,I_s)$ we define a reduced diagram t_{red} by setting

$$t_{\mathrm{red}}:=(b_{s-1})_{\mathrm{red}}\overline{(a_{s-1})_{\mathrm{red}}}\dots(b_1)_{\mathrm{red}}\overline{(a_1)_{\mathrm{red}}},$$

where, on the right side, the subscript "red" refers to the reduction of cup diagrams from Definition 1.13. Thus we obtain a reduction map

$$\mathrm{red}\colon\mathcal{T}(I_1,\dots,I_s)\to\mathcal{T}(2m_1-n_1,\dots,2m_s-n_s)$$

by sending a tangle diagram to its reduced diagram.

Similarly, given a circle diagram $\overline{b}ta\in\mathcal{C}(I_1,\dots,I_s)$, where $t=b_{s-1}\overline{a_{s-1}}\dots b_1\overline{a_1}\in\mathcal{T}(I_1,\dots,I_s)$, $a\in B_{I_1}^{n_1-k_1,k_1}$ and $b\in B_{I_s}^{n_s-k_s,k_s}$, we define

$$(\overline{b}ta)_{\mathrm{red}}:=\overline{b_{\mathrm{red}}}(b_{s-1})_{\mathrm{red}}\overline{(a_{s-1})_{\mathrm{red}}}\dots(b_1)_{\mathrm{red}}\overline{(a_1)_{\mathrm{red}}}a_{\mathrm{red}}$$

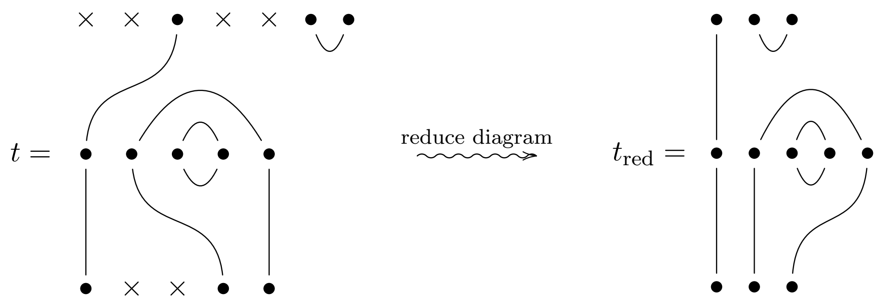
and thus we also have a map

$$\mathrm{red}\colon\mathcal{C}(I_1,\dots,I_s)\to\mathcal{C}(2m_1-n_1,\dots,2m_s-n_s).$$

For the simple circle diagrams $\overline{b}a\in\mathcal{C}(I)$ as in Definition 2.5, where I is an admissible sequence, we set $(\overline{b}a)_{\mathrm{red}}:=\overline{b_{\mathrm{red}}}a_{\mathrm{red}}$.

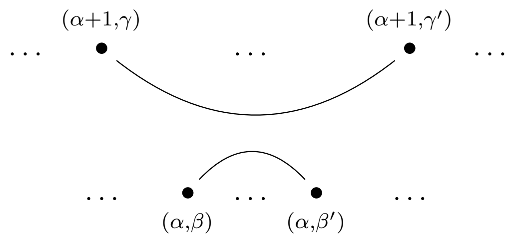
Remark 2.16. Let $t \in \mathcal{T}(I_1, \dots, I_s)$ be a tangle diagram. Suppose there is an arc in t connecting the dots (α, β) and (α, β'). Then there is a *corresponding arc* in t_{red} connecting the dots (α, γ) and (α, γ'), where the number γ (respectively γ') is the difference of β (respectively β') and the number of crosses which are left of β (respectively β') in the dot-cross sequence associated to I_α. This defines a canonical bijection between the arcs in t and the arcs in t_{red} (excluding of course all invisible arcs). Similarly, given a strand in t connecting the dots (α, β) and $(\alpha+1, \beta')$, there is a *corresponding strand* in t_{red} connecting the dots (α, γ) and $(\alpha+1, \gamma')$, where γ and γ' can be calculated as in the case of arcs. Again, this defines a canonical bijection between the strands in t and the strands in t_{red}. Obviously, all these notions still make sense if we replace the tangle diagram by a circle diagram. We will come back to this in the proof of Proposition 2.26.

Example 2.17. The following picture shows the reduction operation applied to the tangle diagram from Example 2.11:

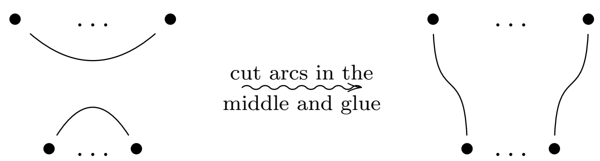


The resulting diagram is an element of $\mathcal{T}(3, 5, 3)$. The cup connecting the dots $(3, 6)$ and $(3, 6)$ in t corresponds (in the sense of Remark 2.16) to the cup connecting the dots $(3, 2)$ and $(3, 3)$ in t_{red} and the strand from $(1, 4)$ to $(2, 2)$ in t corresponds to the strand from $(1, 2)$ to $(2, 2)$ in t_{red}.

Definition 2.18. Let $t = b_{s-1}\overline{a_{s-1}} \dots b_1\overline{a_1} \in \mathcal{T}(I_1, ..., I_s)$ be a tangle diagram and fix $\alpha \in \{1, \dots, s-1\}$. Suppose there is an outermost arc in the cup diagram a_α connecting the dots at positions β and β' as well as an outermost arc in b_α connecting the dots γ and γ' such that $\rho_{a_\alpha}(\beta) = \rho_{b_\alpha}(\gamma)$, i.e. the number of rays left of β in a_α equals the number of rays left of γ in b_α (cf. Definition 1.9). So locally (using the indexing convention of Remark 2.12) the tangle diagram t looks as follows:



The assumption $\rho_{a_\alpha}(\beta) = \rho_{b_\alpha}(\gamma)$ guarantees that there is neither a strand connecting a dot left of (α, β) with a dot right of $(\alpha+1, \gamma')$ nor a strand connecting a dot left of $(\alpha+1, \gamma)$ with a dot right of (α, β'). In particular, we can perform the following local surgery operation



i.e. we cut the two arcs in the middle and reglue the resulting rays to obtain two strands. The rest of the diagram remains unchanged. In particular, by the assumption $\rho_{a_\alpha}(\beta) = \rho_{b_\alpha}(\gamma)$, the result of this surgery operation is again an element in $\mathcal{T}(I_1, ..., I_s)$.

Remark 2.19. Given a tangle diagram from the set $\mathbb{T}(I_1, \dots, I_s)$ a local surgery will often result in a diagram which is not contained in $\mathbb{T}(I_1, \dots, I_s)$ because a non-vertical strand is created. However, if we restrict to surgeries along cups which are opposite of each other, i.e. one cup connects the dots (α, β) and (α, β') while the other one connects the dots $(\alpha+1, \beta)$ and $(\alpha+1, \beta')$, then this problem does not occur.

Example 2.20. The following picture shows a concrete example of a local surgery. For the reader's convenience the arcs and strands involved in the surgery are doubled:

2.2 Varieties and manifolds associated with tangle diagrams

In the following we use the combinatorics of tangle and circle diagrams from the last subsection to define subvarieties inside a finite product of Spaltenstein varieties. We also provide a topological model which is related to these varieties by a homeomorphism (cf. Proposition 2.26). Moreover, the relationship between varieties associated to diagrams which differ by a local surgery is examined in Proposition 2.35.

Throughout this subsection we fix the following data: As in the previous subsection let $I_1, \dots, I_s$ be a family of admissible sequences. This time we additionally assume that all sequences have the same heighest integer n which is supposed to be even, i.e. we have $n = 2k$ for some positive integer k. Let $m_1, m_2, \dots, m_s$ denote the respective lengths of the admissible sequences $I_1, \dots, I_s$. Independent of these data we also consider a collection of even positive integers $n_1, \dots, n_s$, i.e. $n_i = 2k_i$ for every $i \in \{1, \dots, s\}$.

Let $N \geq \max(m_1, \dots, m_s)$ be a large integer. Then we can consider the smooth projective varieties $Y_{I_1}, \dots, Y_{I_s}$ of partial flags inside $\mathbb{C}^{2N}$ (we use the same N for every variety) fixed by the nilpotent operator $z \colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ as defined in subsection 1.2.2. At this point the reader is advised to also recall the definition of the map $C \colon \mathbb{C}^{2N} \to \mathbb{C}^2$ and the hermitian products on the source and target space (cf. the beginning of subsection 1.2). Notice that we obtain an embedding

$$\mathcal{F}l^{k,k}_{I_1} \times ... \times \mathcal{F}l^{k,k}_{I_s} \subset Y_{I_1} \times ... \times Y_{I_s}$$

by using the identification (4) from subsection 1.2.2 for every factor of the product. For the rest of this section we write $\mathcal{F}l^{k,k}_{I_1} \times ... \times \mathcal{F}l^{k,k}_{I_s}$ for this embedded product of Spaltenstein varieties. For an s-tuple of flags $F \in \mathcal{F}l^{k,k}_{I_1} \times ... \times \mathcal{F}l^{k,k}_{I_s}$ we use the notation $F_{\alpha,\beta}$ to denote the β-dimensional vector space in the α-th flag of F.

Since all the vector spaces $F_{\alpha,\beta}$ are subspaces of the same vector space $\mathbb{C}^{2N}$ it is possible to define relations between vector spaces in different flags of the s-tuple F, too. Relations which produce interesting subvarieties of $\mathcal{F}l^{k,k}_{I_1} \times ... \times \mathcal{F}l^{k,k}_{I_s}$ are encoded in the combinatorics of tangle and circle diagrams.

Definition 2.21. Let $t \in \mathbb{T}(I_1, \dots, I_s)$ be a tangle diagram. Given two cup diagrams $a \in B^{k,k}_{I_1}$, $b \in B^{k,k}_{I_s}$, we assign a subvariety

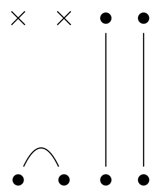
$$_bK(t)_a \subset \mathcal{F}l^{k,k}_{I_1} \times \cdots \times \mathcal{F}l^{k,k}_{I_s}$$

to these data as follows: By definition ${}_bK(t)_a$ consists of precisely those s-tuples of flags $F \in \mathcal{F}l^{k,k}_{I_1} \times ... \times \mathcal{F}l^{k,k}_{I_s}$ which satisfy the relations:

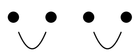
(R1) $F_{\alpha,\beta'} = z^{-\frac{1}{2}(\beta'-\beta+1)} F_{\alpha,\beta-1}$, if the symbols at position (α,β) and (α,β'), $\beta' > \beta$, are connected by a (possibly invisible) arc in the circle diagram $\overline{b}ta$,

(R2) $F_{\alpha,\beta} = F_{\alpha+1,\beta}$, if the dots at positions (α,β) and $(\alpha+1,\beta)$ are connected by a vertical strand in $\overline{b}ta$.

Example 2.22. Consider the tangle diagram

$t =$  $\in \mathbb{T}(4,(2,3,4))$

together with the following two cup diagrams:

$a =$  $\in B^{2,2}$ $\qquad$ $b =$  $\in B^{2,2}_{(2,3,4)}$

Then the subvariety ${}_bK(t)_a$ consists of all flags

$$((F_{1,1},F_{1,2},F_{1,3},F_{1,4}),(F_{2,2},F_{2,3},F_{2,4})) \in \mathcal{F}l^{2,2} \times \mathcal{F}l^{2,2}_{(2,3,4)}$$

satisfying the conditions

$$F_{1,2} = z^{-1}(0) \qquad F_{1,3} = F_{2,3} \qquad F_{1,4} = F_{2,4} \qquad F_{2,2} = z^{-1}(0)$$

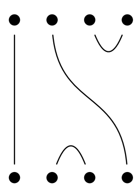
imposed by the diagram t as well as the additional relations

$$\left[F_{1,2} = z^{-1}(0)\right] \qquad F_{1,4} = z^{-1}(F_{1,2}) \qquad \left[F_{2,2} = z^{-1}(0)\right] \qquad F_{2,4} = z^{-1}(F_{2,2})$$

coming from the diagrams a and b (the relations in brackets are redundant). All in all, we have

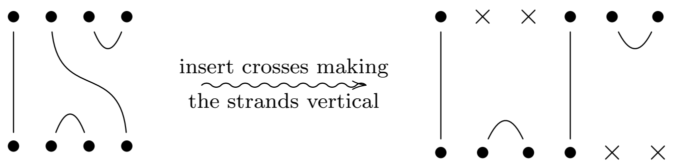
$${}_bK(t)_a = \left\{ \left((F_{1,1}, z^{-1}(0), F_{1,3}, z^{-2}(0)\right), \left(z^{-1}(0), F_{1,3}, z^{-2}(0)\right) \right\} \subset \mathcal{F}l^{2,2} \times \mathcal{F}l^{2,2}_{(2,3,4)}.$$

Remark 2.23. We would like to point out that Definition 2.21 does not extend to arbitrary diagrams in $\mathcal{T}(I_1,\ldots,I_s)$, i.e. the restriction to tangle diagrams from the set $\mathbb{T}(I_1,\ldots,I_s)$ is necessary. In order to illustrate this consider the following diagram:

$t =$  $\in \mathcal{T}(4,4)$

Since there is a strand connecting the fourth dot on the bottom with the second dot on the top the strand relation **(R2)** would tell us to make the identification $F_{1,4} = F_{2,2}$ in order to obtain the variety ${}_bK(t)_a$ (for some fixed cup diagrams $a \in B^{2,2}$, $b \in B^{2,2}$). Obviously, this is nonsense because $\dim F_{1,4} \neq \dim F_{2,2}$.

However, tangle diagrams with non-vertical strands will become extremely important in subsections 3.2 and 3.3. Hence, it is desirable to be able to assign varieties to these diagrams, too. There seem to be several possible solutions to this problem, e.g. one might try to alter relation **(R2)** and replace it with a more complicated one which makes sense for non-vertical strands, too. Our idea is to keep the simple relation and instead introduce crosses (playing the role of "placeholders") in the combinatorics which make the strands vertical, e.g. in the case of the above diagram we would make the replacement



and work with the right diagram instead of the left one. In the geometric world the introduction of crosses in the combinatorics corresponds to passing from Springer fibers to Spaltenstein varieties.

We can extend the topological model of Spaltenstein varieties from the first part of this work to an easy topological model of the varieties from Definition 2.21. One of the advantages of working in the topological setting is that the issues explained in Remark 2.23 above do not occur, i.e. it is possible to define manifolds inside a finite product of topological Springer fibers for diagrams with non-vertical strands by only making simple coordinate identifications.

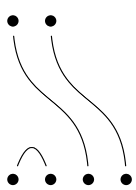
Definition 2.24. Given a tangle diagram $t \in \mathcal{T}(n_1,\ldots,n_s)$ and cup diagrams $a \in B^{k_1,k_1}$,$b \in B^{k_s,k_s}$, we define a manifold ${}_bS(t)_a \subset \mathcal{S}^{k_1,k_1} \times ... \times \mathcal{S}^{k_s,k_s}$ as the set of all elements

$$((x_{1,1},\ldots,x_{1,n_1}),\ldots,(x_{s,1},\ldots,x_{s,n_s})) \in \mathcal{S}^{k_1,k_1} \times ... \times \mathcal{S}^{k_s,k_s}$$

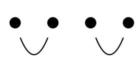
satisfying the coordinate equations

- **(R1')** $x_{\alpha,\beta} = x_{\alpha,\beta'}$, if the dots at position (α,β) and (α,β') are connected by an arc in $\overline{b}ta$,
- **(R2')** $x_{\alpha,\beta} = x_{\alpha+1,\beta'}$, if the dots at positions (α,β) and $(\alpha+1,\beta')$ are connected by a strand in $\overline{b}ta$.

Example 2.25. Consider the following combinatorial tangle diagram

$t =$  $\in \mathcal{T}(4,2)$

which is turned into a circle diagram by closing it up on top and bottom with the diagrams

$a =$  $\in B^{2,2}$ $\qquad b =$  $\in B^{1,1}$

respectively. Thus the manifold ${}_bS(t)_a$ consists of all elements

$$((x_{1,1},x_{1,2},x_{1,3},x_{1,4}),(x_{2,1},x_{2,2})) \in \mathcal{S}^{2,2} \times \mathcal{S}^{1,1}$$

satisfying the relations

$$x_{1,1} = x_{1,2} \qquad x_{1,3} = x_{2,1} \qquad x_{1,4} = x_{2,2}$$

(coming from the arcs and rays in t) and

$$[x_{1,1} = x_{1,2}] \qquad x_{1,3} = x_{1,4} \qquad x_{2,1} = x_{2,2}$$

(coming from the diagrams a and b). Thus we have

$$_bS(t)_a = ((x,x,y,y),(y,y)) \subset \mathcal{S}^{2,2} \times \mathcal{S}^{1,1}.$$

The next goal is to prove that the varieties from Definition 2.21 are related to the manifolds from Definition 2.24 via a homeomorphism (cf. Proposition 2.26 below), thereby justifying the term "topological model".

Notice that we obtain a homeomorphism

$$Y_{I_1} \times \ldots \times Y_{I_s} \xrightarrow{\phi_{I_1} \times \ldots \times \phi_{I_s}} \left(\mathbb{P}^1\right)^{2m_1-n} \times \ldots \times \left(\mathbb{P}^1\right)^{2m_s-n} \tag{23}$$

as the product of the homeomorphisms from Corollary 1.20 and a homeomorphism

$$\left(\mathbb{P}^1\right)^{2m_1-n} \times \ldots \times \left(\mathbb{P}^1\right)^{2m_s-n} \xrightarrow{\gamma_{2m_1-n} \times \ldots \times \gamma_{2m_s-n}} \left(\mathbb{S}^2\right)^{2m_1-n} \times \ldots \times \left(\mathbb{S}^2\right)^{2m_s-n} \tag{24}$$

as the product of maps defined in subsection 1.4. In the following, we will write $\phi_{I_1,\ldots,I_s}$ for the map (23) and $\gamma_{2m_1-n,\ldots,2m_s-n}$ for the map (24).

It follows immediately from Theorem 1.15 that the composition

$$\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s} = \left(\gamma_{2m_1-n} \circ \phi_{I_1}\right) \times \left(\gamma_{2m_2-n} \circ \phi_{I_2}\right) \times \ldots \times \left(\gamma_{2m_s-n} \circ \phi_{I_s}\right)$$

restricts to a homeomorphism

$$\mathcal{F}l^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_s} \xrightarrow{\cong} \mathcal{S}^{m_1-k,m_1-k} \times \ldots \times \mathcal{S}^{m_s-k,m_s-k}. \tag{25}$$

Our task is to identify the image of the varieties $_bK(t)_a \subset \mathcal{F}l^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_s}$ from Definition 2.21 under this homeomorphism.

Proposition 2.26. *Let $t = b_{s-1}\overline{a_{s-1}} \ldots b_1\overline{a_1} \in \mathbb{T}(I_1,\ldots,I_s)$ be a tangle diagram. Then the homeomorphism (25) restricts to a homeomorphism*

$$\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}\big|_{_bK(t)_a} : {_bK(t)_a} \xrightarrow{\cong} {_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}}$$

for every choice of cup diagrams $a \in B^{k,k}_{I_1}$, $b \in B^{k,k}_{I_s}$.

Proof. Let us fix a s-tuple of flags $F \in {_bK(t)_a} \subset \mathcal{F}l^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_s}$ satisfying the relations **(R1)** and **(R2)** with respect to the diagram $\overline{b}ta$ and let

$$\mathbf{x} = ((x_{1,1},\ldots,x_{1,2m_1-n}),\ldots,(x_{s,1},\ldots,x_{s,2m_s-n})) = \gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}(F)$$

denote its image. In order to prove $\mathbf{x} \in {_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}}$ we have to show that this element satisfies the relations **(R1')** and **(R2')** imposed by the diagram $(\overline{b}ta)_{\mathrm{red}}$.

- Suppose there is an arc in $(\overline{b}ta)_{\mathrm{red}}$ connecting the dots (α,γ) and (α,γ'). Then there is a corresponding arc in $\overline{b}ta$ connecting the dots (α,β) and (α,β') (cf. Remark 2.16). Since $\mathcal{F}l^{k,k}_{I_\alpha}$ is the union of its irreducible components there exists a cup diagram $c \in B^{k,k}_{I_\alpha}$ such that the α-th flag of our fixed s-tuple lies in the component K_c. The assumed relations imply that we can choose a c which has an arc connecting the dots β and β'. In particular, we have

$$F \in \mathcal{F}l^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_{i-1}} \times K_c \times \mathcal{F}l^{k,k}_{I_{i+1}} \times \ldots \times \mathcal{F}l^{k,k}_{I_s}$$

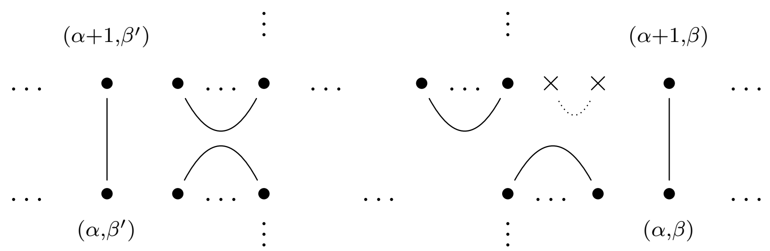
and by Theorem 1.15 it follows that

$$\mathbf{x} \in \mathcal{S}^{m_1-k,m_1-k} \times \ldots \times S_{c_{\mathrm{red}}} \times \ldots \times \mathcal{S}^{m_s-k,m_s-k}.$$

Since there is an arc in c_{red} connecting the dots γ and γ' the definition of $S_{c_{\mathrm{red}}}$ immediately implies the equality $x_{\alpha,\gamma} = x_{\alpha,\gamma'}$, i.e relation **(R1')** for the arc under consideration.

- Next, assume that there is a strand in $(\overline{b}ta)_{\mathrm{red}}$ connecting the dot (α,γ) with the dot $(\alpha+1,\gamma')$. Then there is a corresponding vertical strand in $\overline{b}ta$ connecting the dots (α,β) and $(\alpha+1,\beta)$, i.e. we have $F_{\alpha,\beta} = F_{\alpha+1,\beta}$ for the respective vector spaces in the s-tuple of flags F by **(R2)**.

  We claim that we also have an equality $F_{\alpha,\beta-1} = F_{\alpha+1,\beta-1}$. In order to see this consider an outermost sequence of (possibly invisible) arcs in the cup diagrams a_α and b_α between the ray starting at position β and the preceding ray, say at position β' (since $t = b_{s-1}\overline{a_{s-1}}\ldots b_1\overline{a_1} \in \mathbb{T}(I_1,\ldots,I_s)$ all the rays in a_α and b_α start at the same positions). Thus, locally the tangle diagram t looks as follows:



  By inductively inserting all the relations **(R1)** for the outermost arcs (similar as in the proof of Lemma 1.22) and by using the equality $F_{\alpha,\beta'} = F_{\alpha+1,\beta'}$ (coming from relation **(R2)**) it follows that

$$F_{\alpha,\beta-1} = z^{-\frac{1}{2}(\beta-\beta'-1)}F_{\alpha,\beta'} = z^{-\frac{1}{2}(\beta-\beta'-1)}F_{\alpha+1,\beta'} = F_{\alpha+1,\beta-1}$$

  which is what we claimed.

  Now the equations $F_{\alpha,\beta-1} = F_{\alpha+1,\beta-1}$ and $F_{\alpha,\beta} = F_{\alpha+1,\beta}$ together obviously imply

$$C(F_{\alpha,\beta} \cap F^{\perp}_{\alpha,\beta-1}) = C(F_{\alpha+1,\beta} \cap F^{\perp}_{\alpha+1,\beta-1}), \tag{26}$$

  where $C\colon \mathbb{C}^{2N} \to \mathbb{C}^2$ is the map introduced at the beginning of subsection 1.2. By the definition of the homeomorphism $\phi_{I_1,\ldots,I_s}$ this equality is equivalent to saying

$$\phi_{I_1,\ldots,I_s}(F)_{\alpha,\gamma} = \phi_{I_1,\ldots,I_s}(F)_{\alpha+1,\gamma'},$$

where the subscript α, γ picks out the respective component, i.e. the one corresponding to the γ-th $\mathbb{P}^1$ in the α-th factor of the product $(\mathbb{P}^1)^{2m_1-n} \times \ldots \times (\mathbb{P}^1)^{2m_s-n}$.

Finally, recall that the map $\gamma_{2m_1-n,\ldots,2m_s-n}$ first identifies $\phi_{I_1,\ldots,I_s}(F)_{\alpha,\gamma} \in \mathbb{P}^1$ with a point on the sphere $\mathbb{S}^2$ which is then sent to itself if γ is even and to its antipode if γ is odd. Hence, it follows that

$$\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}(F)_{\alpha,\gamma} = \gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}(F)_{\alpha+1,\gamma'}$$

because γ and γ' are easily seen to be either both even or both odd.

This proves $x_{\alpha,\gamma} = x_{\alpha+1,\gamma'}$, i.e. relation **(R2')** for the strand under consideration.

All in all, we thus have an inclusion

$$\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}({}_bK(t)_a) \subset {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}. \tag{27}$$

In order to complete the argument let $F \in \mathcal{F}l^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_s}$ be a s-tuple of flags (not necessarily contained in ${}_bK(t)_a$) whose image

$$\mathbf{x} = ((x_{1,1}, \ldots, x_{1,2m_1-n}), \ldots, (x_{s,1}, \ldots, x_{s,2m_s-n})) = \gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}(F)$$

is assumed to be contained in ${}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}$. We want to prove that this implies $F \in {}_bK(t)_a$, i.e. we have to show that F satisfies the relations **(R1)** and **(R2)** imposed by the diagram $\bar{b}ta$.

- Suppose there is an arc in $\bar{b}ta$ connecting the dots (α, β) and (α, β'). If this arc is invisible there is nothing to show because the relations for invisible arcs are automatically satisfied for every element $F \in Fl^{k,k}_{I_1} \times \ldots \times \mathcal{F}l^{k,k}_{I_s}$. Hence, assume that the arc is not invisible. Then there is a corresponding arc in $(\bar{b}ta)_{\mathrm{red}}$ connecting the dots (α, γ) and (α, γ'). Then we argue similarly as in the proof of "**(R1)** $\Rightarrow$ **(R1')**" above, i.e. there exists $c \in B^{k,k}_{I_\alpha}$ which has an arc connecting the dots β and β' such that

  $$\mathbf{x} \in \mathcal{S}^{m_1-k,m_1-k} \times \cdots \times S_{c_{\mathrm{red}}} \times \cdots \times \mathcal{S}^{m_s-k,m_s-k}$$

  and hence we have

  $$F \in \mathcal{F}l^{k,k}_{I_1} \times \cdots \times \mathcal{F}l^{k,k}_{I_{i-1}} \times K_c \times \mathcal{F}l^{k,k}_{I_{i+1}} \times \cdots \times \mathcal{F}l^{k,k}_{I_s}$$

  by Theorem 1.15 which proves $F_{\alpha,\beta'} = z^{-\frac{1}{2}(\beta'-\beta+1)} F_{\alpha,\beta-1}$, i.e **(R1)** for the arc under consideration.

- Assume there is a strand in $\bar{b}ta$ connecting the dots (α, β) and $(\alpha+1, \beta)$. Then there is a corresponding strand in $(\bar{b}ta)_{\mathrm{red}}$ connecting the dots (α, γ) and $(\alpha+1, \gamma')$. In particular, we have $x_{\alpha,\gamma} = x_{\alpha+1,\gamma'}$ by **(R2')** which implies equation (26), i.e.

  $$C(F_{\alpha,\beta} \cap F^{\perp}_{\alpha,\beta-1}) = C(F_{\alpha+1,\beta} \cap F^{\perp}_{\alpha+1,\beta-1}),$$

  by reversing the argumentation from the case "**(R2)** $\Rightarrow$ **(R2')**" above (check that every implication was in fact an equivalence).

  We have already seen that the s-tuple F satisfies the arc relations **(R1)** for the diagram $\bar{b}ta$. Moreover, by induction, we assume that the relation (**R2**) holds for all strands which are left of the one under consideration, i.e. we have equalities $F_{\alpha,\beta'} = F_{\alpha+1,\beta'}$ whenever there is a vertical strand in $\bar{b}ta$ connecting the dots (α, β')

and $(\alpha+1,\beta')$, where $\beta' < \beta$. Thus, we can use an outermost sequence argument (cf. the proof of the implication "**(R2)** $\Rightarrow$ **(R2')**") to deduce the equality $F_{\alpha,\beta-1} = F_{\alpha+1,\beta-1}$. If we combine this with equation (26) we get

$$C(F_{\alpha,\beta} \cap F_{\alpha,\beta-1}^{\perp}) = C(F_{\alpha+1,\beta} \cap F_{\alpha,\beta-1}^{\perp}). \tag{28}$$

By Lemma 1.16 C restricts to a unitary isomorphism on $z^{-1}F_{\alpha,\beta-1} \cap F_{\alpha,\beta-1}^{\perp}$. Since $F_{\alpha,\beta}$ and $F_{\alpha+1,\beta}$ are both contained in $z^{-1}F_{\alpha,\beta-1} \cap F_{\alpha,\beta-1}^{\perp}$ the equation (28) implies that $F_{\alpha,\beta} = F_{\alpha+1,\beta}$, thereby proving relation **(R2)** for the strand under consideration.

This shows that the inclusion (27) is in fact an equality which finishes the proof. □

Example 2.27. Notice that the diagrams in Example 2.25 are the reduced diagrams of the ones from Example 2.22. In particular, by Proposition 2.26, the complex variety from Example 2.22 is homeomorphic to the manifold from Example 2.25.

Finally, we ask for a relationship between the varieties (respectively manifolds) associated to two tangle diagrams which only differ by a single local surgery. The answer is provided in Proposition 2.35 below. As a preparation we introduce some more definitions first.

For $t \in \mathcal{T}(n_1,\dots,n_s)$ a tangle diagram and cup diagrams $a \in B^{k_1,k_1}, b \in B^{k_s,k_s}$ (notice there are only arcs in a and b and no rays) the circle diagram $\overline{b}ta$ consists of circles only and there are no loose endpoints in the diagram.

Definition 2.28. Let $t \in \mathcal{T}(n_1,\dots,n_s)$ be a tangle diagram and let $a \in B^{k_1,k_1}$ and $b \in B^{k_s,k_s}$ be cup diagrams. Then we write $c(\overline{b}ta)$ to denote the number of circles in the diagram $\overline{b}ta$.

Two dots in $\overline{b}ta$ are said to be *equivalent* if there is a sequence of arcs and strands in the diagram which connects the two dots. This clearly defines an equivalence relation on the set of dots and the equivalence classes correspond bijectively to the circles in the diagram. Henceforth, we will often be a bit sloppy and use the term "circle" when we actually mean an equivalence class of dots. However, this should not cause any confusion.

Recall that there is a total order on the set of all dots in a circle diagram by identifying the dots with pairs of positive integers (α,β) as explained in Remark 2.12.

Definition 2.29. A dot is called a *distinguished representative* of a circle in $\overline{b}ta$ if it is minimal (with respect to the order from Remark 2.12) among all the dots lying on the circle under consideration.

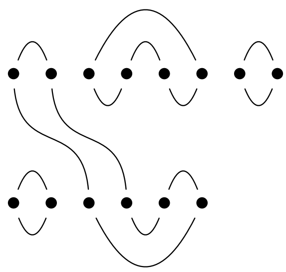
This notion of a distinguished representative can be used to define a total order on the set of all circles in $\overline{b}ta$.

Definition 2.30. Let C_1 and C_2 be two distinct circles in the diagram $\overline{b}ta$. We say that C_1 is smaller than C_2 if the distinguished representative of C_1 is smaller than the one of C_2.

From now on we stick to the following conventions: When we write $C_1,\dots,C_{c(\overline{b}ta)}$ for the collection of circles of a circle diagram $\overline{b}ta$ we assume that this collection is already ordered in the sense of Definition 2.30, i.e. we have $C_1 < C_2 < \dots < C_{c(\overline{b}ta)}$. Moreover, whenever we refer to the i-th circle in a circle diagram $\overline{b}ta$ we always mean the i-th circle with respect to this ordering.

Remark 2.31. Notice that the above notions also make sense for the simplest kind of circle diagram obtained by gluing a cap diagram on top of a cup diagram without a tangle diagram sitting between the two (cf. Definition 2.5). In particular, a circle C_1 in such a diagram is smaller than a circle C_2 if the leftmost dot on the circle C_1 is left of the leftmost dot on the circle C_2. Again, this yields a total order on the collection of all circles.

Example 2.32. Consider the circle diagram:



The set of dots is partitioned into the following four equivalence classes each of which corresponds to a circle in the diagram:

$$C_1 = \{(1,1),(1,2)\} \quad C_2 = \{(1,3),(1,4),(1,5),(1,6),(2,1),(2,2)\}$$

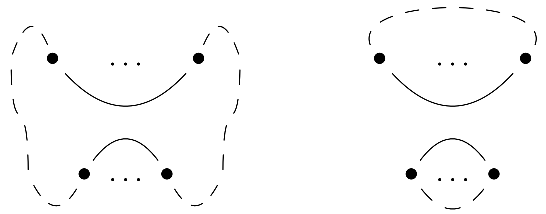
$$C_3 = \{(2,3),(2,4),(2,5),(2,6)\} \quad C_4 = \{(2,7),(2,8)\}.$$

The distinguished representatives as in Definition 2.29 are $(1,1)$, $(1,3)$, $(2,3)$ and $(2,7)$ respectively. In particular, the circles are already indexed correctly according to the total order from Definition 2.30, i.e. we have $C_1 < C_2 < C_3 < C_4$.

The following lemma describes to what extent a local surgery operation impacts the combinatorial structure of a tangle diagram.

Lemma 2.33. *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be tangle diagrams and suppose that t' is obtained from t by performing a single local surgery. Then, for fixed cup diagrams $a \in B^{k_1,k_1}$ and $b \in B^{k_s,k_s}$, there exists a pair of positive integers $i < j$ such that either the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}t'a$, due to the local surgery, or the i-th and j-th circle in $\overline{b}ta$ merge and become the i-th circle in $\overline{b}t'a$ while the rest of the circles remain unchanged.*

Proof. There are two cases to be considered: Either the two arcs involved in the surgery are parts of the same circle in $\overline{b}ta$ or they are not. The two cases can be visualized as follows:

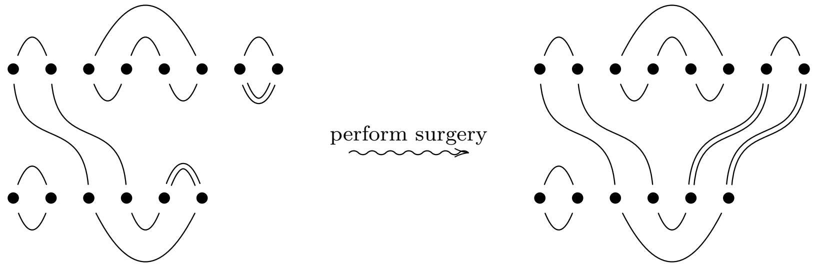


The dashed connection lines represent the remaining arcs and strands on the respective circles.

Let us treat the first case (left picture): Let us assume that the circle which contains the two arcs involved in the surgery is at the i-th position in the ordering of all circles in $\overline{b}ta$. Then the combinatorial structure of the diagram clearly forces the circle to split. Since the rest of the circles remain unchanged one of the two created circles is still at position i in the ordering of the circles in $\overline{b}t'a$ (namely the circle which contains the minimal representative of the old circle) while the other one is at position j for some $j > i$.

In the second case the two arcs involved in the local surgery lie on two different circles, say the i-th and j-th circle, where $i < j$. The surgery operation clearly forces the two circles to merge. In particular, the new circle is the easily seen to be the i-th circle in the diagram $\overline{b}ta$ because it contains the minimal representative of the i-th circle in $\overline{b}ta$ and the remaining circles are unchanged by the surgery. $\square$

Example 2.34. Here is an example of a local surgery which merges the second and fourth circle in the diagram from Example 2.32 into the second circle of the diagram resulting from the surgery, i.e. we have $i = 2$ and $j = 4$. As in Example 2.20 the arcs and strands involved in the surgery operation are doubled in the following picture:



Proposition 2.35. *Let $t, t' \in \mathbb{T}(I_1, \ldots, I_s)$ be two tangle diagrams, where t' is obtained from t by a single local surgery, and let $a \in B^{k,k}_{I_1}$, $b \in B^{k,k}_{I_s}$ be two fixed cup diagrams. Then we either have an inclusion*

$$ {}_bK(t)_a \subset {}_bK(t')_a \qquad \text{or} \qquad {}_bK(t')_a \subset {}_bK(t)_a $$

depending on whether a circle splits or two circles merge by passing from $(\overline{b}ta)_{\mathrm{red}}$ to $(\overline{b}t'a)_{\mathrm{red}}$ (a splitting circle corresponds to the first and two merging circles to the second inclusion).

Analogously, let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be two tangle diagrams, where t' is obtained from t by a single local surgery. Then for fixed cup diagrams $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$ we either have an inclusion

$$ {}_bS(t)_a \subset {}_bS(t')_a \qquad \text{or} \qquad {}_bS(t')_a \subset {}_bS(t)_a $$

depending on whether a circle splits or two circles merge by passing from $\overline{b}ta$ to $\overline{b}t'a$ (again, a splitting circle corresponds to the first and two merging circles to the second inclusion).

Proof. We start by proving the claim in the topological setting: Let $t, t' \in \mathbb{T}(n_1, \ldots, n_s)$ be tangle diagrams as in the proposition. By Lemma 2.33 there exist integers $i < j$ such that the local surgery either merges the i-th and j-th circle in $\overline{b}ta$ into the i-th circle in $\overline{b}t'a$ or the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}ta$. Without loss of generality assume that the first case is true (otherwise exchange t and t' in all that follows).

Let $C_1, \ldots, C_{c(\overline{b}ta)}$ be the classes of dots representing the respective circle in $\overline{b}ta$. Then the sets

$$ C_1, \ldots, C_i \cup C_j, \ldots, C_{j-1}, C_{j+1}, \ldots, C_{c(\overline{b}ta)} \tag{29} $$

represent the circles in $\overline{b}t'a$. It is a straightforward consequence of Definition 2.24 that ${}_bS(t)_a$ (respectively ${}_bS(t')_a$) consists of precisely those elements

$$ ((x_{1,1}, \ldots, x_{1,n_1}), \ldots, (x_{s,1}, \ldots, x_{s,n_s})) \in \left(\mathbb{S}^2\right)^{n_1} \times \ldots \times \left(\mathbb{S}^2\right)^{n_s} $$

which satisfy coordinate equations $x_{\alpha,\beta} = x_{\alpha',\beta'}$ whenever the dots corresponding to (α, β) and (α', β') lie on the same circle in $\overline{b}ta$ (respectively $\overline{b}t'a$). Thus the circles partition the set of coordinates into groups of coordinates which are necessarily equal. Since the partition corresponding to the circle diagram $\overline{b}ta$ is clearly finer than the one of $\overline{b}t'a$ (see (29) above) we deduce that ${}_bS(t')_a \subset {}_bS(t')_a$.

In the algebro-geometric setting we can deduce the claim from the topological case as follows: Let $t, t' \in \mathcal{T}(I_1, \ldots, I_s)$ be related by a local surgery as in the proposition. In particular, $t_{\mathrm{red}}, t'_{\mathrm{red}}$ are related by a local surgery and we have

$$_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}} \subset {}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \qquad \text{or} \qquad {}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \subset {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}$$

by the above argumentation. Without loss of generality assume that the first-mentioned inclusion holds (otherwise exchange t and t' in the notation). Then the injection obtained as the composition

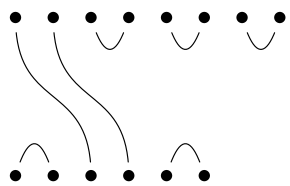
$$_bK(t)_a \xrightarrow{\cong} {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}} \hookrightarrow {}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \xrightarrow{\cong} {}_bK(t')_a,$$

where the outer maps are given by the homeomorphisms

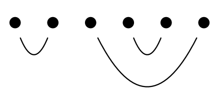
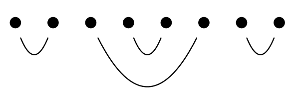
$$\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s} \qquad \text{and} \qquad \left(\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}\right)^{-1}$$

restricted to ${}_bK(t)_a$ and ${}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}}$ respectively, clearly sends every s-tuple of flags to itself. Hence, we deduce ${}_bK(t)_a \subset {}_bK(t')_a$. $\square$

Example 2.36. Consider the tangle diagram

$t =$  $\in \mathcal{T}(6,8)$

and the following two cup diagrams:

$a =$  $\in B^{3,3}$ $\qquad b =$  $\in B^{4,4}$

The circle diagram $\overline{b}ta$ associated with these choices is depicted in Example 2.34. The reader also finds a picture of the circle diagram $\overline{b}t'a$ in this example, where t' is the tangle diagram obtained by performing a local surgery on the two rightmost arcs of t.

Corresponding to these combinatorial data we have manifolds

$$_bS(t)_a = \left\{ ((x,x,y,y,y,y),(y,y,z,z,z,z,w,w)) \mid x,y,z,w \in \mathbb{S}^2 \right\} \tag{30}$$

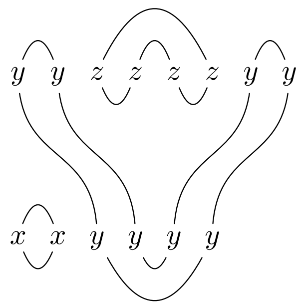
and

$$_bS(t')_a = \left\{ ((x,x,y,y,y,y),(y,y,z,z,z,z,y,y)) \mid x,y,z \in \mathbb{S}^2 \right\} \tag{31}$$

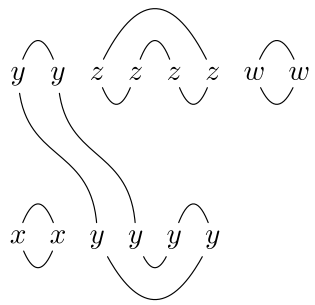
which are clearly related by an inclusion ${}_bS(t')_a \subset {}_bS(t)_a$. Notice that this inclusion agrees with the statement of Proposition 2.35 because the second and fourth circle in $\overline{b}ta$ merge into the second circle in $\overline{b}t'a$.

Since every dot in the circle diagram $\overline{b}ta$ (respectively $\overline{b}t'a$) corresponds to a sphere in the product $\left(\mathbb{S}^2\right)^6 \times \left(\mathbb{S}^2\right)^8$ we sometimes use the notation

$${}_bS(t')_a =$$



$$\hookrightarrow \quad {}_bS(t)_a =$$



instead of writing down the sets as in (30) and (31). Each one of the small letters x, y, z, w replacing the dots in the diagrams stands for the choice of an element in $\mathbb{S}^2$. This notation is very intuitive and emphasizes the combinatorial nature of the manifolds ${}_bS(t)_a$ and ${}_bS(t')_a$.

2.3 A 2d TQFT via pushforward and pullback maps in cohomology

The goal of this subsection is to provide an explicit description of the pullback and pushforward map in cohomology of the inclusions from Proposition 2.35. As a main result (cf. Theorem 2.50 and Theorem 2.51) we prove that these maps can be described via a $2d$ TQFT. This provides a first connection between the geometry and topology of Spaltenstein varieties and Khovanov's TQFT-based construction of tangle homology [Kho02].

We begin by establishing some notation and conventions which are used throughout this subsection: Given a positive integer N, let $\mathrm{Sym}(\{1, 2, \dots, N\})$ be the symmetric group of all permutations of the set $\{1, \dots, N\}$. Moreover, let X be a topological space and V a n-dimensional vector space with fixed basis $b_1, b_2, \dots, b_n$. We say that V together with this basis is a *based vector space.*

The group $\mathrm{Sym}(\{1, 2, \dots, N\})$ acts on the N-fold cartesian product of X with itself, i.e. a permutation σ induces a homeomorphism

$$\sigma\colon X^N \to X^N\ ,\ (x_1, x_2, \dots, x_N) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}) \tag{32}$$

by permuting the coordinates. We also have an induced linear automorphism of the N-fold tensor product of the based vector space V with itself given by

$$\sigma\colon V^{\otimes N} \to V^{\otimes N}\ ,\ b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_N} \mapsto b_{i_{\sigma(1)}} \otimes \cdots \otimes b_{i_{\sigma(N)}}, \tag{33}$$

where $(i_1, \dots, i_N)$ runs through all N-tuples of elements from the set $\{1, \dots, n\}$, thereby specifying the linear map σ completely. By abuse of notation we will always denote the induced maps with the same letter as the permutation. Notice that the isomorphism (33) is clearly grading-preserving if V is a graded vector space.

Let $\tau_i \in \mathrm{Sym}(\{1, 2, \dots, N\})$ be the transposition interchanging i and $i+1$. For a triple of positive integers $i < j \leq N$ we define a permutation $\sigma_{i,j} \in \mathrm{Sym}(\{1, 2, \dots, N\})$ as the following composition of transpositions

$$\sigma_{i,j} := \tau_{j-1} \circ \cdots \circ \tau_{i+2} \circ \tau_{i+1}. \tag{34}$$

If $j = i+1$ this is the empty composition, i.e. we set $\sigma_{i,i+1} = \mathrm{id}_{\{1,2,\dots,N\}}$ in this case. The permutation $\sigma_{i,j}$ will play a crucial in the following section.

Last but not least, we also fix a collection $n_1, \dots, n_s$ of even positive integers, i.e. $n_i = 2k_i$ for all $i \in \{1, \dots, s\}$.

2.3.1 The 2d TQFT associated with the ring of dual numbers

Let us consider the truncated polynomial ring $\mathbb{F}_2[X]/(X^2)$, i.e. the ring of dual numbers. As a $\mathbb{F}_2$-vector space this ring has a basis given by 1 and X. Throughout this work we will always view the ring of dual numbers as a graded $\mathbb{F}_2$-algebra with $\deg(X) = 2$ and $\deg(1) = 0$. In [Kho02], Khovanov works with a graded vector space

$$\mathcal{A} := \mathbb{F}_2[X]/(X^2)\{-1\},$$

obtained by shifting the grading down by 1, i.e. $\deg(X) = 1$ (this is not a graded algebra anymore). This grading is motivated by knot theory since the graded dimension of $\mathcal{A}$ is precisely the Jones polynomial (suitably normalized) of the unknot (cf. subsection 3.3 for more on this).

We can equip $\mathcal{A}$ with the structure of a commutative Frobenius algebra with trace form

$$\varepsilon\colon \mathcal{A} \to \mathbb{F}_2\,,\ \ 1 \mapsto 0\,,\ \ X \mapsto 1$$

and comultiplication

$$\delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\,,\ \ 1 \mapsto X \otimes 1 + 1 \otimes X\,,\ \ X \mapsto X \otimes X.$$

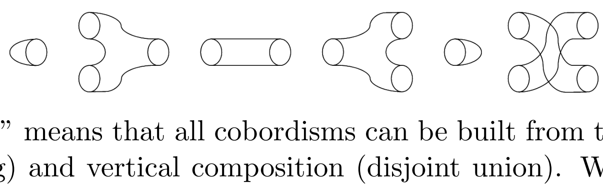
In particular, we obtain a $2d$ TQFT associated with this algebra, i.e. a symmetric monoidal functor $\mathcal{F}$ from the two-dimensional cobordism category **2Cob** to the category of finite-dimensional (graded) $\mathbb{F}_2$-vector spaces (cf. [Abr96,Koc03] for details on Frobenius algebras and their relation to TQFTs).

The objects of **2Cob** are finite, ordered disjoint unions of the smooth manifold $\mathbb{S}^1$ which is assumed to be equipped with a fixed orientation. We write $\mathbf{n}$ for the disjoint union consisting of n copies of $\mathbb{S}^1$ (the empty set $\mathbf{0}$ is also allowed). A morphism $\mathbf{n} \to \mathbf{m}$ is given by an equivalence class of (not necessarily connected) compact oriented surfaces (smooth 2-manifolds) which are equipped with an orientation-preserving diffeomorphism from their boundary to the disjoint union $\mathbf{n}^* \sqcup \mathbf{m}$ (the upper star denotes a reversal of orientation of all connected components). Two such surfaces Σ, Σ' are called *equivalent* if there exists an orientation-preserving diffeomorphism $\phi\colon \Sigma \to \Sigma'$ making the following diagram commute:

$$\begin{tikzcd} \partial\Sigma \arrow[rr, "\phi|_{\partial\Sigma}"] \arrow[dr, "\cong"'] & & \partial\Sigma' \arrow[dl, "\cong"] \\ & \mathbf{n}^* \sqcup \mathbf{m} & \end{tikzcd}$$

The reversal of orientation is used to distinguish between the in- and outboundary of the manifold (notice that in order to have a morphism we need a well-defined source and target). In the following we will always use the term "cobordism" to refer to a morphism in **2Cob**. The composition of morphisms is defined by gluing cobordisms (more precisely the representing surfaces) along their boundary and the monoidal structure is given by ordered disjoint union.

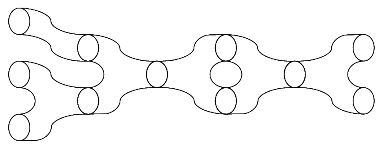
It can be shown (e.g. by using Morse theory) that the category **2Cob** is generated by the following *elementary cobordisms* (cf. e.g. [Abr96, Proposition 12] or [Koc03, §1.4.13]):



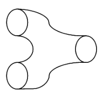
The word "generates" means that all cobordisms can be built from these elementary ones by horizontal (gluing) and vertical composition (disjoint union). We use the convention that cobordism pictures are supposed to be read from bottom to top and left to right. More precisely, this means that the circles on the left side of a picture belong to the inboundary and the circles on the right to the outboundary. Moreover, the i-th connected component of an object in **2Cob** is represented by the i-th circle counted from bottom to top. Notice that the category **2Cob** is combinatorial by nature. According to the classification of oriented surfaces with boundary (see e.g. [Hir76, Theorem 3.11]) a cobordism is completely determined once we know the number of in- and outboundary components and the genus of a representating surface. In particular, the pictures provide a precise graphical tool in the sense that they capture all the information from differential topology necessary in order to specify a cobordism uniquely.

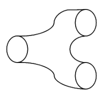
The picture of the twist cobordism which seems to penetrate itself in the middle is supposed to symbolize the fact that our surfaces are not embedded in an ambient space. Hence, the notion of “over” and “under” does not exist. In particular, the two connected components represented by the picture do not really intersect. Instead the drawing emphasizes the fact that we do not prefer one component (the category **2Cob** is symmetric and not just braided).

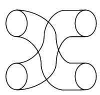
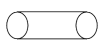
Example 2.37. Here is an example of a cobordism $\mathbf{3} \to \mathbf{2}$ of genus 1 decomposed into elementary cobordisms:



Let $\eta\colon \mathbb{F}_2 \to \mathcal{A}$ be the unit map sending 1 to 1 and let $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be the multiplication in $\mathcal{A}$. Moreover, $\tau\colon \mathcal{A} \otimes A \to \mathcal{A} \otimes \mathcal{A}$ denotes the twist map given by $\alpha \otimes \beta \mapsto \beta \otimes \alpha$. Then the TQFT $\mathcal{F}$ associated with the Frobenius algebra $\mathcal{A}$ sends a disjoint union of N circles (compact 1-manifolds) to the N-fold tensor product $\mathcal{A}^{\otimes N}$. On elementary cobordisms $\mathcal{F}$ is given by

 $\longmapsto m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$  $\longmapsto \eta\colon \mathbb{F}_2 \to \mathcal{A}$

 $\longmapsto \delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$  $\longmapsto \varepsilon\colon \mathcal{A} \to \mathbb{F}_2$

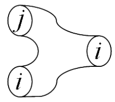
 $\longmapsto \tau\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$  $\longmapsto \mathrm{id}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$

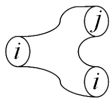
Notice that the monoidal functor $\mathcal{F}$ is completely determined by the above table. It requires some more work to see that it is also well-defined. However, this follows from the well-known folk theorem that in the case of a Frobenius algebra the relations between compositions and tensor products of the linear maps in the above table precisely correspond to the topological relations between cobordisms built from the elementary pieces by gluing and disjoint union (cf. [Abr96, Theorem 3] or [Koc03, §3.3]).

Example 2.38. Applying the TQFT functor $\mathcal{F}$ to the cobordism from Example 2.37 yields the linear map

$$\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{m \otimes \mathrm{id}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\delta} \mathcal{A} \otimes \mathcal{A}.$$

Definition 2.39. For a fixed triple $i < j \leq N$ of positive integers we define two cobordisms which we denote by

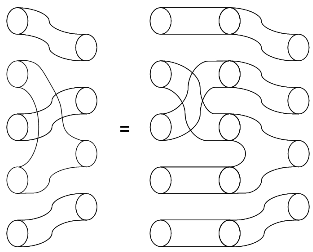


and


(35)

as follows: The cobordism on the right merges the i-th and j-th circle in a disjoint union of N circles into the the i-th circle in a disjoint union of $N-1$ circles via a pair of pants. The remaining circles are connected by identity cobordisms in an order-preserving manner. Similarly, the right cobordism splits the i-th circle in a disjoint union of $N-1$ circles into the i-th and j-th circle in a disjoint union of N circles. Again, the remaining circles are connected by identity cobordisms. We refer to these cobordisms as *surgery cobordisms* (the choice of terminology is justified below).

Example 2.40. The following picture shows the above cobordism (including a decomposition into elementary pieces) for the triple $2 < 4 \leq 5$:



Definition 2.41. Given a triple $i < j \leq N$ of positive integers we define linear maps $m_{i,j} \colon \mathcal{A}^{\otimes N} \to \mathcal{A}^{\otimes N-1}$ by

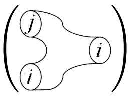
$$m_{i,j} = \left(\mathrm{id}^{\otimes i-1} \otimes m \otimes \mathrm{id}^{\otimes N-i-1}\right) \circ \sigma_{i,j}^{-1}$$

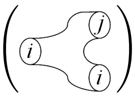
and $\delta_{i,j} \colon \mathcal{A}^{\otimes N-1} \to \mathcal{A}^{\otimes N}$ by

$$\delta_{i,j} := \sigma_{i,j} \circ \left(\mathrm{id}^{\otimes i-1} \otimes \delta \otimes \mathrm{id}^{\otimes N-i-1}\right),$$

where $\sigma_{i,j}$ is the permutation (34) introduced at the beginning of this section (it acts on $\mathcal{A}^{\otimes N}$ as in (33) with respect to the basis 1, X of $\mathcal{A}$).

Lemma 2.42. *Let $i < j \leq N$ be a triple of positive integers. Then the cobordisms from Definition 2.39 induce the linear maps $m_{i,j}$ and $\delta_{i,j}$ respectively, i.e. we have*

$\mathcal{F}\Bigg($ 

$\Bigg) = m_{i,j}$ *and* $\mathcal{F}\Bigg($ 

$\Bigg) = \delta_{i,j}.$

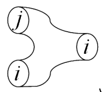
Proof. This follows immediately by decomposing the respective cobordisms into elementary pieces. □

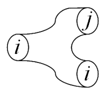
We finish our discussion of $2d$ TQFTs by relating the cobordisms from (35) to the combinatorics of tangle diagrams. Given a tangle diagram $t \in \mathcal{T}(n_1, \ldots, n_s)$ and cup diagrams $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$, the diagram $\overline{b}ta$ consists of an ordered collection of circles $C_1, \ldots, C_{c(\overline{b}ta)}$. Thus it makes sense to apply the functor $\mathcal{F}$ to the diagram $\overline{b}ta$ (after identifying $\overline{b}ta$ with the $c(\overline{b}ta)$-fold disjoint union of $\mathbb{S}^1$) to obtain $\mathcal{F}(\overline{b}ta) = \mathcal{A}^{\otimes c(\overline{b}ta)}$, where the i-th tensor factor of $\mathcal{A}^{\otimes N}$ corresponds to the circle C_i.

In particular, if $t' \in \mathcal{T}(n_1, \ldots, n_s)$ is obtained from t by a single local surgery, then there exist positive integers $i < j$ such that either the ith and j-th circle in $\overline{b}ta$ merge and become the i-th circle in $\overline{b}t'a$ or the i-th circle splits into the i-th and j-th circle in $\overline{b}t'a$ (cf. Lemma 2.33). So the circle diagrams $\overline{b}ta$ and $\overline{b}t'a$ are naturally connected via the surgery cobordisms from Definition 2.39. For future reference we summarize these ideas in the following lemma.

Lemma 2.43. *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be two tangle diagrams, where t' is obtained from t by a single local surgery, and let $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$ be cup diagrams.*

If the i-th and j-th circle in $\overline{b}ta$ merge and become the i-th circle in $\overline{b}t'a$ due to the local surgery, then the circle diagrams $\overline{b}ta$ and $\overline{b}t'a$ are naturally connected by the surgery cobordism





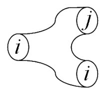
$$\overline{b}ta \longrightarrow \overline{b}t'a \qquad \text{or} \qquad \overline{b}t'a \longrightarrow \overline{b}ta,$$

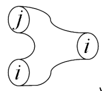
depending on whether we choose $\overline{b}ta$ or $\overline{b}t'a$ as domain. Applying the TQFT $\mathcal{F}$ to these cobordisms induces the maps

$$\mathcal{A}^{\otimes c(\overline{b}ta)} \xrightarrow{m_{i,j}} \mathcal{A}^{\otimes c(\overline{b}t'a)} \qquad \text{and} \qquad \mathcal{A}^{\otimes c(\overline{b}t'a)} \xrightarrow{\delta_{i,j}} \mathcal{A}^{\otimes c(\overline{b}ta)}$$

associated with the triple $i < j \le c(\overline{b}ta)$, respectively.

If the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}t'a$, then the circle diagrams $\overline{b}ta$ and $\overline{b}t'a$ are naturally connected by the surgery cobordism





$$\overline{b}ta \longrightarrow \overline{b}t'a \qquad \text{or} \qquad \overline{b}t'a \longrightarrow \overline{b}ta,$$

depending on whether we choose $\overline{b}ta$ or $\overline{b}t'a$ as domain. Applying the TQFT $\mathcal{F}$ to these cobordisms induces the maps

$$\mathcal{A}^{\otimes c(\overline{b}ta)} \xrightarrow{\delta_{i,j}} \mathcal{A}^{\otimes c(\overline{b}t'a)} \qquad \text{and} \qquad \mathcal{A}^{\otimes c(\overline{b}t'a)} \xrightarrow{m_{i,j}} \mathcal{A}^{\otimes c(\overline{b}ta)}$$

associated with the triple $i < j \le c(\overline{b}t'a)$, respectively.

Proof. This is just a reformulation of Lemma 2.42. □

2.3.2 Explicit calculation of pullback and pushforward maps

Finally, we want to compute the pullback and pushforward in cohomology of the inclusions from Proposition 2.35. In doing so, an important role is played by the diagonal embedding

$$\Delta\colon \mathbb{S}^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{S}^2\,,\ x \mapsto (x,x).$$

and the following generalization:

Definition 2.44. Given a triple $i < j \le N$ of positive integers, define

$$\Delta_{i,j}\colon (\mathbb{S}^2)^{N-1} \to (\mathbb{S}^2)^N$$

to be the map which embeds the i-th factor of $(\mathbb{S}^2)^{N-1}$ diagonally into the i-th and j-th factor of $(\mathbb{S}^2)^N$. More precisely, let $\sigma_{i,j} \in \mathrm{Sym}(\{1, \ldots, N\})$ be the permutation (34) introduced at the beginning of subsection 2.3. Then we define

$$\Delta_{i,j} := \sigma_{i,j} \circ \left(\mathrm{id}^{i-1} \times \Delta \times \mathrm{id}^{N-i-1}\right),$$

where $\sigma_{i,j}$ acts on $(\mathbb{S}^2)^N$ as in (32) by permuting coordinates.

Example 2.45. If one chooses $i = 2$, $j = 4$ and $N = 4$ then the map $\Delta_{2,4}\colon \left(\mathbb{S}^2\right)^3 \to \left(\mathbb{S}^2\right)^4$ is given by

$$(x, y, z) \mapsto (x, y, z, y).$$

Let $t \in \mathcal{T}(n_1, \ldots, n_s)$ be a tangle diagram and let $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$ be cup diagrams. Moreover, let $\nu_1, \ldots, \nu_{c(\overline{b}ta)}$ be the minimal representatives of the circles in the diagram $\overline{b}ta$ (cf. Definition 2.29). Thus it makes sense to define a homeomorphism $\xi_{\overline{b}ta}\colon {}_bS(t)_a \xrightarrow{\cong} \left(\mathbb{S}^2\right)^{c(\overline{b}ta)}$

$$((x_{1,1}, x_{1,2}, \ldots, x_{1,n_1}), \ldots, (x_{s,1}, \ldots, x_{s,n_s})) \mapsto (x_{\nu_1}, \ldots, x_{\nu_{c(\overline{b}ta)}})$$

by throwing away redundant coordinates.

The following lemma shows that the inclusion maps from Proposition 2.35 are related to the maps $\Delta_{i,j}$ via the homeomorphism $\xi_{\overline{b}ta}$.

Lemma 2.46. *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be two tangle diagrams, where t' is obtained from t by a single local surgery, and let $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$ be two cup diagrams.*

If the i-th and j-th circle in $\overline{b}ta$ merge and become the i-th circle in $\overline{b}t'a$ due to the local surgery, then we have a commutative diagram

$$\begin{tikzcd}
{}_bS(t')_a \arrow[r, hook] \arrow[d, "\xi_{\overline{b}t'a}"', "\cong"] & {}_bS(t)_a \arrow[d, "\cong"', "\xi_{\overline{b}ta}"] \\
\left(\mathbb{S}^2\right)^{c(\overline{b}t'a)} \arrow[r, "\Delta_{i,j}"] & \left(\mathbb{S}^2\right)^{c(\overline{b}ta)}
\end{tikzcd}$$

where $\Delta_{i,j}$ is defined with respect to the triple $i < j \leq c(\overline{b}ta)$.

If the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}t'a$, then the following diagram commutes

$$\begin{tikzcd}
{}_bS(t)_a \arrow[r, hook] \arrow[d, "\xi_{\overline{b}ta}"', "\cong"] & {}_bS(t')_a \arrow[d, "\cong"', "\xi_{\overline{b}t'a}"] \\
\left(\mathbb{S}^2\right)^{c(\overline{b}ta)} \arrow[r, "\Delta_{i,j}"] & \left(\mathbb{S}^2\right)^{c(\overline{b}t'a)}
\end{tikzcd}$$

where $\Delta_{i,j}$ is defined with respect to the triple $i < j \leq c(\overline{b}t'a)$.

Proof. Let us assume that the i-th and j-th circle in $\overline{b}ta$ merge into the i-th circle in $\overline{b}t'a$ and let $\nu_1, \ldots, \nu_{c(\overline{b}ta)}$ be the minimal representatives of the circles in $\overline{b}ta$. Then the minimal representatives of the circles in $\overline{b}t'a$ are given by $\nu_1, \ldots, \nu_{j-1}, \nu_{j+1}, \ldots, \nu_{c(\overline{b}ta)}$, i.e. ν_j is deleted.

Given an element $\mathbf{x} = ((x_{1,1}, \ldots, x_{1,n_1}), \ldots, (x_{s,1}, \ldots, x_{s,n_s})) \in {}_bS(t')_a$ we have

$$\xi_{\overline{b}ta}(\mathbf{x}) = \left(x_{\nu_1}, \ldots, x_{\nu_{c(\overline{b}ta)}}\right), \tag{36}$$

where $x_{v_i} = x_{v_j}$ (because the dots ν_i and ν_j lie on the same circle in $\overline{b}t'a$).

On the other hand we obtain

$$\begin{aligned}
\Delta_{i,j} \circ \xi_{\overline{b}t'a}(\mathbf{x}) &= \Delta_{i,j}\left(x_{\nu_1}, \ldots, x_{\nu_{j-1}}, x_{\nu_{j+1}}, \ldots, x_{\nu_{c(\overline{b}ta)}}\right) \\
&= \left(x_{\nu_1}, \ldots, x_{\nu_{j-1}}, x_{\nu_i}, x_{\nu_{j+1}}, \ldots, x_{\nu_{c(\overline{b}ta)}}\right)
\end{aligned}$$

which is the same as the $c(\overline{b}ta)$-tuple in (36). Hence, the first diagram in the lemma commutes.

In case that the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}t'a$ one can simply copy the above proof after replacing t with t' and thus obtain the commutativity of the second diagram. $\square$

The above lemma suggests to compute the pullback and pushforward of the maps $\Delta_{i,j}$ (this is done in Proposition 2.47 and Proposition 2.48 below) and then relate the result to the pushforward and pullback of the inclusions via the induced isomorphisms

$$H^*\left((\mathbb{S}^2)^{c(\overline{b}ta)}\right) \xrightarrow{\xi^*_{\overline{b}ta}} H^*\left({}_bS(t)_a\right) \qquad \text{and} \qquad H^*\left((\mathbb{S}^2)^{c(\overline{b}t'a)}\right) \xrightarrow{\xi^*_{\overline{b}t'a}} H^*\left({}_bS(t')_a\right).$$

Let N be a positive integer. By standard algebraic topology we have a natural isomorphism of graded $\mathbb{F}_2$-algebras

$$\varphi_N\colon H^*\left((\mathbb{S}^2)^N\right) \cong H^*\left(\mathbb{S}^2\right)^{\otimes N} \cong \mathbb{F}_2[X]/(X^2)^{\otimes N}, \tag{37}$$

where the first isomorphism is the inverse of the cross-product map K from the Künneth theorem and the second one is the N-fold tensor product of the unique isomorphism $H^*(\mathbb{S}^2) \cong \mathbb{F}_2[X]/(X^2)$ of graded $\mathbb{F}_2$-algebras sending the non-vanishing top cohomology class to X.

In the following we fix the $\mathbb{F}_2$-basis 1, X of the algebra $\mathbb{F}_2[X]/(X^2)$. We also equip $H^*(\mathbb{S}^2)$ with the structure of a based vector space by taking the preimages of 1 and X under the isomorphism $H^*(\mathbb{S}^2) \cong \mathbb{F}_2[X]/(X^2)$. In particular, we obtain induced maps

$$\sigma_{i,j}\colon \mathbb{F}_2[X]/(X^2)^N \to \mathbb{F}_2[X]/(X^2)^N \qquad \text{and} \qquad \sigma_{i,j}\colon H^*(\mathbb{S}^2)^N \to H^*(\mathbb{S}^2)^N$$

as defined in (33) for every triple $i < j \leq N$.

We are ready to observe that that the pullback $\Delta^*_{i,j}$ and the map $m_{i,j}$ from Definition 2.41 are compatible via the isomorphism (37) for any fixed triple $i < j \leq N$.

Proposition 2.47. *Given a triple of positive integers $i < j \leq N$, the following diagram commutes:*

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^N) \arrow[r, "\Delta^*_{i,j}"] \arrow[d, "\varphi_N"', "\cong"] & H^*((\mathbb{S}^2)^{N-1}) \arrow[d, "\cong"', "\varphi_{N-1}"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "m_{i,j}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N-1}
\end{tikzcd}$$

Proof. We prove the claim by showing the commutativity of the following diagram:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^N) \arrow[r, "\sigma^*_{i,j}"] \arrow[d, "K^{-1}"', "\cong"] & H^*((\mathbb{S}^2)^N) \arrow[r, "(\mathrm{id}^{i-1}\times\Delta\times\mathrm{id}^{N-i-1})^*"] \arrow[d, "K^{-1}"', "\cong"] & H^*((\mathbb{S}^2)^{N-1}) \arrow[d, "\cong"', "K^{-1}"] \\
H^*(\mathbb{S}^2)^{\otimes N} \arrow[r, "\sigma^{-1}_{i,j}"] \arrow[d, "\cong"'] & H^*(\mathbb{S}^2)^{\otimes N} \arrow[r, "\mathrm{id}^{\otimes i-1}\otimes\cup\otimes\mathrm{id}^{\otimes N-i-1}"] \arrow[d, "\cong"] & H^*((\mathbb{S}^2)^{N-1}) \arrow[d, "\cong"'] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "\sigma^{-1}_{i,j}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "\mathrm{id}^{\otimes i-1}\otimes m\otimes\mathrm{id}^{\otimes N-i-1}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N-1}
\end{tikzcd}$$

Notice that the outer square is just the diagram from the proposition broken up into four pieces.

It is an easy calculation to prove that the cross-product isomorphism respects twists, i.e. we have $K \circ \tau_l \circ K^{-1} = \tau_l$, where τ_l is a simple transposition, $1 \leq l < N$ (notice that the transposition on the right acts as in (32) and the one one the left as in (33)). By contravariance of the cohomology functor we thus obtain

$$\sigma_{i,j}^* = \tau_{i+1}^* \circ ... \circ \tau_{j-1}^* = K \circ \tau_{i+1} \circ ... \circ \tau_{j-1} \circ K^{-1} = K \circ \sigma_{i,j}^{-1} \circ K^{-1}$$

and hence the commutativity of the left square in the top row.

The commutativity of the top square on the right is an immediate consequence of combining the following two facts: Firstly, by standard algebraic topology, we have a commutative diagram

$$\begin{tikzcd} H^*(\mathbb{S}^2 \times \mathbb{S}^2) \arrow[rr, "\Delta^*"] \arrow[dr, "\cong", "K^{-1}"'] & & H^*(\mathbb{S}^2) \\ & H^*(\mathbb{S}^2) \otimes H^*(\mathbb{S}^2) \arrow[ur, "\cup"'] & \end{tikzcd}$$

Secondly, the cross product isomorphism is natural in the sense that the pullback of a cartesian product of maps corresponds to the tensor product of the pullbacks of the factors via K.

The commutativity of the two squares on the bottom of the diagram is clear (respectively a straightforward calculation). □

For a triple $i < j \leq N$ of positive integers we can also ask for the pushforward

$$(\Delta_{i,j})_! \colon H^*((\mathbb{S}^2)^{N-1}) \to H^*((\mathbb{S}^2)^N)$$

in cohomology.

For compact manifolds X, Y and a continuous map $f \colon X \to Y$ the pushforward $f_!$ is defined as the composition

$$H^*(X) \xrightarrow{P_X} H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{P_Y^{-1}} H^*(Y),$$

where f_* is the usual pushforward in homology and

$$P_X \colon H^*(X) \xrightarrow{\cong} H_*(X)$$

is the Poincaré isomorphism given by cap-product with the unique (since we work over $\mathbb{F}_2$) non-vanishing top homology class. Notice that pushforward in cohomology is functorial, i.e. for continuous maps $f \colon X \to Y$, $g \colon Y \to Z$ between compact topological manifolds we clearly have

$$(g \circ f)_! = g_! \circ f_!. \tag{38}$$

Proposition 2.48. *Given a triple of positive integers $i < j \leq N$, the following diagram commutes:*

$$\begin{tikzcd} H^*((\mathbb{S}^2)^{N-1}) \arrow[r, "(\Delta_{i,j})_!"] \arrow[d, "\varphi_{N-1}"', "\cong"] & H^*((\mathbb{S}^2)^N) \arrow[d, "\cong"', "\varphi_N"] \\ \mathbb{F}_2[X]/(X^2)^{\otimes N-1} \arrow[r, "\delta_{i,j}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N} \end{tikzcd}$$

Before we can prove this proposition we introduce some more isomorphisms and closely examine the Poincaré isomorphism in the case of a finite product of two-dimensional spheres (cf. Lemma 2.49 below).

Notice that the choice of basis 1, X for $\mathbb{F}_2[X]/(X^2)$ yields a distinguished basis of the $2k$-th graded component of the tensor product $\mathbb{F}_2[X]/(X^2)^{\otimes N}$ which we denote by A_{2k}. This basis is given by the following set of elementary tensors

$$\big\{\alpha_1 \otimes ... \otimes \alpha_N \mid \alpha_l \in \{1, X\} \text{ and } \#\{\alpha_l \mid \alpha_l = X, 1 \leq l \leq N\} = k\big\}. \tag{39}$$

The operator $\#$ in (39) returns the cardinality of a set. Corresponding to this basis we obtain a linear isomorphism

$$(\ )^\vee \colon A_{2k} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}_2}(A_{2k}, \mathbb{F}_2)\ ,\ \ \alpha_1 \otimes ... \otimes \alpha_N \mapsto (\alpha_1 \otimes ... \otimes \alpha_N)^\vee\ ,$$

sending a basis vector to its dual, i.e. we have

$$(\alpha_1 \otimes ... \otimes \alpha_N)^\vee (\beta_1 \otimes ... \otimes \beta_N) := \begin{cases} 1 & \text{if } \alpha_l = \beta_l\ \ \forall 1 \leq l \leq N, \\ 0 & \text{else.} \end{cases}$$

Next, recall the Kronecker pairing

$$\langle -, - \rangle \colon H^{2k}((\mathbb{S}^2)^N) \otimes H_{2k}((\mathbb{S}^2)^N) \to \mathbb{F}_2\ ,\ \ [f] \otimes [x] \mapsto \langle [f], [x] \rangle := f(x) \tag{40}$$

satisfying the crucial property

$$\langle [f], [g] \cap [x] \rangle = \langle [f] \cup [g], [x] \rangle \tag{41}$$

for all $[f], [g] \in H^{2k}((\mathbb{S}^2)^N)$ and $[x] \in H_{2k}((\mathbb{S}^2)^N)$. One can check that in our case this pairing is non-degenerate. In particular, the map

$$\phi \colon H_{2k}((\mathbb{S}^2)^N) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}_2}\Big(H^{2k}((\mathbb{S}^2)^N), \mathbb{F}_2\Big)\ ,\ \ [x] \mapsto ([f] \mapsto f(x)) \tag{42}$$

is an isomorphism of vector spaces for all $k \in \{0, ..., N\}$.

In order to state the next lemma, we introduce a linear involution

$$\mathrm{inv} \colon \mathbb{F}_2[X]/(X^2) \to \mathbb{F}_2[X]/(X^2)$$

given by $1 \mapsto X$ and $X \mapsto 1$. This yields an involution

$$\mathrm{inv}^{\otimes N} \colon \mathbb{F}_2[X]/(X^2)^{\otimes N} \to \mathbb{F}_2[X]/(X^2)^{\otimes N}.$$

This map sends elements of degree $2k$ to elements of degree $2(N-k)$.

Lemma 2.49. *For fixed $k \in \{0, ..., N\}$, the following diagram commutes:*

$$\begin{tikzcd}
H^{2k}((\mathbb{S}^2)^N) \arrow[r, "P"] & H_{2(N-k)}((\mathbb{S}^2)^N) \arrow[r, "\phi"] & \mathrm{Hom}_{\mathbb{F}_2}(H^{2(N-k)}((\mathbb{S}^2)^N), \mathbb{F}_2) \\
A_{2k} \arrow[u, "\varphi_N^{-1}"] \arrow[r, "\mathrm{inv}^{\otimes N}"] & A_{2(N-k)} \arrow[r, "(\ )^\vee"] & \mathrm{Hom}_{\mathbb{F}_2}(A_{2(N-k)}, \mathbb{F}_2) \arrow[u, "\mathrm{Hom}_{\mathbb{F}_2}(\varphi_N, \mathbb{F}_2)"']
\end{tikzcd}$$

Proof. By linearity we only need to check the commutativity of the diagram on the basis (39) of A_{2k}. So let $\alpha_1 \otimes ... \otimes \alpha_N \in A_{2k}$ be a basis element and let $[(\mathbb{S}^2)^N] \in H_{2N}((\mathbb{S}^2)^N)$ denote the generator of the top homology. In order to compute the linear functional

$$\phi \circ P \circ \varphi_N^{-1}(\alpha_1 \otimes ... \otimes \alpha_N) \in \mathrm{Hom}_{\mathbb{F}_2}(H^{2(N-k)}((\mathbb{S}^2)^N), \mathbb{F})$$

it suffices to see what it does on a basis of $H^{2(N-k)}((\mathbb{S}^2)^N)$. Let us fix the basis consisting of the images of the basis vectors of $A_{2(N-k)}$ as defined in (39) under the isomorphism φ_N^{-1}. Hence a typical basis element has the form $\varphi_N^{-1}(\beta_1\otimes ...\otimes\beta_N)$, where $\beta_1\otimes ...\otimes\beta_N \in A_{2(N-k)}$ is a tensor such that

$$\#\{\beta_l \mid \beta_l = X \, , \, 1 \leq l \leq N\} = N - k.$$

We can thus calculate

$$\begin{aligned}
&\phi \circ P \circ \varphi_N^{-1}(\alpha_1 \otimes ... \otimes \alpha_N) \left(\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N)\right) \\
=&\langle\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N), \varphi_N^{-1}(\alpha_1 \otimes ... \otimes \alpha_N) \cap [(\mathbb{S}^2)^N]\rangle \\
\overset{(41)}{=}&\langle\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N) \cup \varphi_N^{-1}(\alpha_1 \otimes ... \otimes \alpha_N), [(\mathbb{S}^2)^N]\rangle \\
=&\langle\varphi_N^{-1}(\alpha_1\beta_1 \otimes ... \otimes \alpha_N\beta_N), [(\mathbb{S}^2)^N]\rangle,
\end{aligned}$$

where the last equation follows because φ_N is an isomorphism of algebras. Notice that

$$\alpha_1\beta_1 \otimes ... \otimes \alpha_N\beta_N = \begin{cases} X^{\otimes N} & \text{if } \operatorname{inv}(\alpha_l) = \beta_l \;\; \forall 1 \leq l \leq N, \\ 0 & \text{else.} \end{cases}$$

Hence we obtain

$$\begin{aligned}
\phi \circ P \circ \varphi_N^{-1}(\alpha_1 \otimes ... \otimes \alpha_N) \left(\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N)\right) &= \langle\varphi_N^{-1}(\alpha_1\beta_1 \otimes ... \otimes \alpha_N\beta_N), [(\mathbb{S}^2)^N]\rangle \\
&= \begin{cases} 1 & \text{if } \operatorname{inv}(\alpha_l) = \beta_l \;\; \forall 1 \leq l \leq N, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

On the other hand one gets

$$\begin{aligned}
&\left(\operatorname{Hom}_{\mathbb{F}_2}(\varphi_N, \mathbb{F}_2) \circ (\;)^\vee \circ \operatorname{inv}^{\otimes N}(\alpha_1 \otimes ... \otimes \alpha_N)\right) \left(\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N)\right) \\
=& \left((\operatorname{inv}(\alpha_1) \otimes ... \otimes \operatorname{inv}(\alpha_N))^\vee \circ \varphi_N\right) \left(\varphi_N^{-1}(\beta_1 \otimes ... \otimes \beta_N)\right) \\
=&(\operatorname{inv}(\alpha_1) \otimes ... \otimes \operatorname{inv}(\alpha_N))^\vee(\beta_1 \otimes ... \otimes \beta_N) \\
=& \begin{cases} 1 & \text{if } \operatorname{inv}(\alpha_l) = \beta_l \;\; \forall 1 \leq l \leq N, \\ 0 & \text{else,} \end{cases}
\end{aligned}$$

which proves our claim. □

Proof (Proposition 2.48). The proof is subdivided into two parts.

Claim 1: The following diagram commutes:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^{N-1}) \arrow[r, "(\mathrm{id}^{i-1}\times\Delta\times\mathrm{id}^{N-i-1})_!"] \arrow[d, "\varphi_{N-1}"', "\cong"] & H^*((\mathbb{S}^2)^N) \arrow[d, "\cong"', "\varphi_N"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N-1} \arrow[r, "\mathrm{id}^{\otimes i-1}\otimes\delta\otimes\mathrm{id}^{\otimes N-i-1}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N}
\end{tikzcd}$$

Proof of Claim 1. By applying the Hom-functor to the diagram from Proposition 2.47 (in the case $j = i + 1$) we obtain a commutative square:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbb{F}_2}(H^*((\mathbb{S}^2)^{N-1}), \mathbb{F}_2) \arrow[r, "\operatorname{Hom}_{\mathbb{F}_2}(\Delta^*_{i,i+1}{,}\mathbb{F}_2)"] & \operatorname{Hom}_{\mathbb{F}_2}(H^*((\mathbb{S}^2)^N), \mathbb{F}_2) \\
\operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[X]/(X^2)^{\otimes N-1}, \mathbb{F}_2) \arrow[u, "\operatorname{Hom}_{\mathbb{F}_2}(\varphi_{N-1}{,}\mathbb{F}_2)"] \arrow[r, "\operatorname{Hom}_{\mathbb{F}_2}(m_{i,i+1}{,}\mathbb{F}_2)"] & \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[X]/(X^2)^{\otimes N}, \mathbb{F}_2) \arrow[u, "\operatorname{Hom}_{\mathbb{F}_2}(\varphi_N{,}\mathbb{F}_2)"']
\end{tikzcd}$$

If we combine this diagram with Lemma 2.49 we get a commutative diagram:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^{N-1}) \arrow[rr, "P^{-1}\circ\phi^{-1}\circ\mathrm{Hom}_{\mathbb{F}_2}(\Delta^*_{i,i+1},\mathbb{F}_2)\circ\phi\circ P"] \arrow[d, "\varphi_{N-1}"', "\cong"] & & H^*((\mathbb{S}^2)^{N}) \arrow[d, "\cong"', "\varphi_N"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N-1} \arrow[rr, "\mathrm{inv}^{\otimes N}\circ(()^\vee)^{-1}\circ\mathrm{Hom}_{\mathbb{F}_2}(m_{i,i+1},\mathbb{F}_2)\circ()^\vee\circ\mathrm{inv}^{\otimes N-1}"] & & \mathbb{F}_2[X]/(X^2)^{\otimes N-1}
\end{tikzcd}$$

By standard algebraic topology (homology and cohomology are dual) the map

$$H_*\left((\mathbb{S}^2)^{N-1}\right) \xrightarrow{\phi^{-1}\circ\mathrm{Hom}_{\mathbb{F}_2}(\Delta^*_{i,i+1},\mathbb{F}_2)\circ\phi} H_*\left((\mathbb{S}^2)^{N}\right)$$

is precisely the map $(\Delta_{i,i+1})_*$. In particular, the composition in the upper row is the pushforward $(\Delta_{i,i+1})_!$ in cohomology.

Thus, it remains to calculate the composition in the lower row. An easy dualization exercise shows that the map

$$\mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2[X]/(X^2)^{\otimes N-1},\mathbb{F}_2\right) \xrightarrow{\mathrm{Hom}_{\mathbb{F}_2}(m_{i,i+1},\mathbb{F}_2)} \mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2[X]/(X^2)^{\otimes N},\mathbb{F}_2\right)$$

is explicitly given by

$$(\alpha_1\otimes\ldots\otimes\alpha_{N-1})^\vee \mapsto \begin{cases} (\alpha_1\otimes\ldots\otimes\alpha_{i-1}\otimes 1\otimes 1\otimes\alpha_{i+1}\otimes\ldots\otimes\alpha_{N-1})^\vee & \text{if } \alpha_i=1, \\ (\alpha_1\otimes\ldots\otimes\alpha_{i-1}\otimes X\otimes 1\otimes\alpha_{i+1}\otimes\ldots\otimes\alpha_{N-1})^\vee & \\ +(\alpha_1\otimes\ldots\otimes\alpha_{i-1}\otimes 1\otimes X\otimes\alpha_{i+1}\otimes\ldots\otimes\alpha_{N-1})^\vee & \text{if } \alpha_i=X. \end{cases}$$

Another simple calculation using this result shows that the composition

$$((\ )^\vee\circ\mathrm{inv}^{\otimes N})^{-1}\circ\mathrm{Hom}_{\mathbb{F}_2}(m_{i,i+1},\mathbb{F}_2)\circ(\ )^\vee\circ\mathrm{inv}^{\otimes N-1}$$

is precisely the map $\delta_{i,i+1}=\mathrm{id}^{i-1}\otimes\delta\otimes\mathrm{id}^{N-i-1}$ which proves the above claim.

Claim 2: We have a commutative diagram:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^{N}) \arrow[r, "(\sigma^{-1}_{i,j})_!"] \arrow[d, "\varphi_N"', "\cong"] & H^*((\mathbb{S}^2)^{N}) \arrow[d, "\cong"', "\varphi_N"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "\sigma^{-1}_{i,j}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N}
\end{tikzcd}$$

Proof of Claim 2. In order to see this notice that the map

$$\mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2[X]/(X^2)^{\otimes N},\mathbb{F}_2\right) \xrightarrow{\mathrm{Hom}_{\mathbb{F}_2}(\tau_l,\mathbb{F}_2)} \mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2[X]/(X^2)^{\otimes N},\mathbb{F}_2\right),$$

where τ_l is a simple transposition interchanging l and $l+1$, is given by

$$(\alpha_1\otimes\ldots\otimes\alpha_N)^\vee \mapsto (\alpha_1\otimes\cdots\otimes\alpha_{l+1}\otimes\alpha_l\otimes\cdots\otimes\alpha_N)^\vee,$$

i.e. the vectors α_l and α_{l+1} change places. This easily implies that the map

$$((\ )^\vee\circ\mathrm{inv}^{\otimes N})^{-1}\circ\mathrm{Hom}_{\mathbb{F}_2}(\tau_l,\mathbb{F}_2)\circ(\ )^\vee\circ\mathrm{inv}^{\otimes N}$$

is given by τ_l and by Lemma 2.49 we have a commutative diagram:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^N) \arrow[r, "(\tau_l)_!"] \arrow[d, "\varphi_N"', "\cong"] & H^*((\mathbb{S}^2)^N) \arrow[d, "\cong"', "\varphi_N"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "\tau_l"] & \mathbb{F}_2[X]/(X^2)^{\otimes N}
\end{tikzcd}$$

Using the functoriality (38) of pushforward we obtain the commutativity of the diagram from the second claim.

All in all, putting together the first and second claim, the following diagram commutes:

$$\begin{tikzcd}
H^*((\mathbb{S}^2)^{N-1}) \arrow[r, "(\mathrm{id}^{i-1}\times\Delta\times\mathrm{id}^{N-i-1})_!"] \arrow[d, "\varphi_{N-1}"', "\cong"] & H^*((\mathbb{S}^2)^N) \arrow[r, "(\sigma_{i,j})_!"] \arrow[d, "\cong"', "\varphi_N"] & H^*((\mathbb{S}^2)^N) \arrow[d, "\cong"', "\varphi_N"] \\
\mathbb{F}_2[X]/(X^2)^{\otimes N-1} \arrow[r, "\mathrm{id}^{\otimes i-1}\otimes\delta\otimes\mathrm{id}^{\otimes N-i-1}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N} \arrow[r, "\sigma_{i,j}"] & \mathbb{F}_2[X]/(X^2)^{\otimes N}
\end{tikzcd}$$

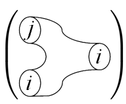
This finishes the proof of Proposition 2.48. $\square$

The observations of this section are summarized in the following main result:

Theorem 2.50 (Topological version). *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be two tangle diagrams where t' is obtained from t by a single local surgery and let $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$ be cup diagrams. We distinguish between two cases:*

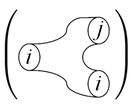
- *Suppose that the i-th and j-th circle in $\overline{b}ta$ merge and become the i-th circle in $\overline{b}t'a$ due to the local surgery. Then we have a commutative diagram*

$$\begin{tikzcd}
H^*({}_bS(t)_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)^*"] \arrow[d, "\varphi_{c(\overline{b}ta)}\circ\left(\xi^{-1}_{\overline{b}ta}\right)^*"', "\cong"] & H^*({}_bS(t')_a) \arrow[d, "\cong"', "\varphi_{c(\overline{b}t'a)}\circ\left(\xi^{-1}_{\overline{b}t'a}\right)^*"] \\
\mathcal{F}\left(\overline{b}ta\right)\{c(\overline{b}ta)\} \arrow[r, "\mathcal{F}"] & \mathcal{F}\left(\overline{b}t'a\right)\{c(\overline{b}t'a)\}
\end{tikzcd}$$



of vector spaces. If we exchange the domain and codomain we obtain a commutative diagram

$$\begin{tikzcd}
H^*({}_bS(t')_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)_!"] \arrow[d, "\varphi_{c(\overline{b}t'a)}\circ\left(\xi^{-1}_{\overline{b}t'a}\right)^*"', "\cong"] & H^*({}_bS(t)_a) \arrow[d, "\cong"', "\varphi_{c(\overline{b}ta)}\circ\left(\xi^{-1}_{\overline{b}ta}\right)^*"] \\
\mathcal{F}\left(\overline{b}t'a\right)\{c(\overline{b}t'a)\} \arrow[r, "\mathcal{F}"] & \mathcal{F}\left(\overline{b}ta\right)\{c(\overline{b}ta)\}
\end{tikzcd}$$



of vector spaces.

- *Suppose that the i-th circle in $\overline{b}ta$ splits into the i-th and j-th circle in $\overline{b}t'a$ due to the surgery. Then we obtain the same commutative diagrams as in the case above with the only difference that we have to exchange t and t' in both diagrams.*

Proof. Without loss of generality we assume that the i-th and j-th circle in the diagram $\bar{b}ta$ merge and become the i-th circle in $\bar{b}t'a$ due to the local surgery. For the case of a splitting circle one can simply copy the proof after exchanging t and t' in the notation.

If we apply the cohomology functor to the first commutative diagram in Lemma 2.46 we obtain the following commutative diagram:

$$\begin{tikzcd} H^*({}_bS(t)_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)^*"] \arrow[d, "\left(\xi_{\bar{b}ta}^{-1}\right)^*"', "\cong"] & H^*({}_bS(t')_a) \arrow[d, "\cong"', "(\xi_{\bar{b}t'a}^{-1})^*"] \\ H^*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}ta)}\right) \arrow[r, "\Delta_{i,j}^*"] & H^*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}t'a)}\right) \end{tikzcd}$$

Sticking this diagram on top of the one from Proposition 2.47 proves the commutativity of the diagram

$$\begin{tikzcd} H^*({}_bS(t)_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)^*"] \arrow[d, "\varphi_{c(\bar{b}ta)}\circ\left(\xi_{\bar{b}ta}^{-1}\right)^*"', "\cong"] & H^*({}_bS(t')_a) \arrow[d, "\cong"', "\varphi_{c(\bar{b}t'a)}\circ(\xi_{\bar{b}t'a}^{-1})^*"] \\ \mathcal{A}^{\otimes c(\bar{b}ta)}\{c(\bar{b}ta)\} \arrow[r, "m_{i,j}"] & \mathcal{A}^{\otimes c(\bar{b}t'a)}\{c(\bar{b}t'a)\} \end{tikzcd}$$

where we used the obvious fact that $\mathbb{F}_2[X]/(X^2)^{\otimes c(\bar{b}ta)} = \mathcal{A}^{\otimes c(\bar{b}ta)}\{c(\bar{b}ta)\}$ as graded vector spaces. In particular, by Lemma 2.43, the lower map is precisely the map induced by the cobordism as claimed in the theorem.

On the other hand, we also have a commutative diagram:

$$\begin{tikzcd} H^*({}_bS(t')_a) \arrow[r, "P"] & H_*({}_bS(t')_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)_*"] \arrow[d, "(\xi_{\bar{b}t'a})_*"', "\cong"] & H_*({}_bS(t)_a) \arrow[r, "P^{-1}"] \arrow[d, "\cong"', "(\xi_{\bar{b}ta})_*"] & H^*({}_bS(t)_a) \\ H^*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}t'a)}\right) \arrow[r, "P"] \arrow[u, "(\xi_{\bar{b}t'a})^*", "\cong"'] & H_*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}t'a)}\right) \arrow[r, "(\Delta_{i,j})_*"] & H_*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}ta)}\right) \arrow[r, "P^{-1}"] & H^*\left(\left(\mathbb{S}^2\right)^{c(\bar{b}ta)}\right) \arrow[u, "\cong", "(\xi_{\bar{b}ta})^*"'] \end{tikzcd}$$

The middle square is obtained by applying the homology functor to the first diagram from Lemma 2.46 and the two outer squares commute by a simple calculation (use the simple fact that $(\xi_{\bar{b}t'a})_*$ and $(\xi_{\bar{b}ta})_*$ respect the top homology classes).

Notice that the composition in the upper row is the pushforward of the inclusion ${}_bS(t')_a \hookrightarrow {}_bS(t)_a$ and the composition in the lower row is the pushforward of $\Delta_{i,j}$. Thus it follows from Proposition 2.48 that we have a commutative diagram:

$$\begin{tikzcd} H^*({}_bS(t')_a) \arrow[r, "({}_bS(t')_a \hookrightarrow {}_bS(t)_a)_!"] \arrow[d, "\varphi_{c(\bar{b}t'a)}\circ\left(\xi_{\bar{b}t'a}^{-1}\right)^*"', "\cong"] & H^*({}_bS(t)_a) \arrow[d, "\cong"', "\varphi_{c(\bar{b}ta)}\circ(\xi_{\bar{b}ta}^{-1})^*"] \\ \mathcal{A}^{\otimes c(\bar{b}t'a)}\{c(\bar{b}t'a)\} \arrow[r, "\delta_{i,j}"] & \mathcal{A}^{\otimes c(\bar{b}ta)}\{c(\bar{b}ta)\} \end{tikzcd}$$

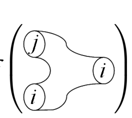
Again, Lemma 2.43 shows that the lower map in this diagram is precisely the map induced by the cobordism as stated in the theorem. This finishes the argument. □

Using the homeomorphism from Proposition 2.26 we can easily translate the above theorem into the algebro-geometric world of flag varieties without much further work. Let $I_1, \dots, I_s$ be a collection of admissible sequences each of which has highest integer $n = 2k$ and let $m_1, \dots, m_s$ be the respective lengths.

Theorem 2.51 (Algebro-geometric version). *Let $t, t' \in \mathbb{T}(I_1, \ldots, I_s)$ be two tangle diagrams where t' is obtained from t by a single local surgery and let $a \in B^{k,k}_{I_1}$, $b \in B^{k,k}_{I_s}$ be cup diagrams. We distinguish between two cases:*

- *Suppose that the i-th and j-th circle in $(\overline{b}ta)_{\mathrm{red}}$ merge and become the i-th circle in $(\overline{b}t'a)_{\mathrm{red}}$ due to the local surgery. Then we have a commutative diagram*

$$\begin{tikzcd} H^*({}_bK(t)_a) \arrow[r, "({}_bK(t')_a \hookrightarrow {}_bK(t)_a)^*"] \arrow[d, "\cong"] & H^*({}_bK(t')_a) \arrow[d, "\cong"] \\ \mathcal{F}\left((\overline{b}ta)_{\mathrm{red}}\right)\{c\left((\overline{b}ta)_{\mathrm{red}}\right)\} \arrow[r, "\mathcal{F}(\ldots)"] & \mathcal{F}\left((\overline{b}t'a)_{\mathrm{red}}\right)\{c\left((\overline{b}t'a)_{\mathrm{red}}\right)\} \end{tikzcd}$$



of vector spaces, where the vertical isomorphism are given by the maps

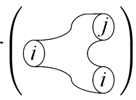
$$\varphi_{c\left((\overline{b}ta)_{\mathrm{red}}\right)} \circ \left(\xi^{-1}_{(\overline{b}ta)_{\mathrm{red}}}\right)^* \circ \left(\gamma^{-1}_{2m_1-n,\ldots,2m_s-n}\right)^* \circ \left(\phi^{-1}_{I_1,\ldots,I_s}\right)^* \tag{43}$$

and

$$\varphi_{c\left((\overline{b}t'a)_{\mathrm{red}}\right)} \circ \left(\xi^{-1}_{(\overline{b}t'a)_{\mathrm{red}}}\right)^* \circ \left(\gamma^{-1}_{2m_1-n,\ldots,2m_s-n}\right)^* \circ \left(\phi^{-1}_{I_1,\ldots,I_s}\right)^* \tag{44}$$

respectively. Moreover, we also have a commutative diagram

$$\begin{tikzcd} H^*({}_bK(t')_a) \arrow[r, "({}_bK(t')_a \hookrightarrow {}_bK(t)_a)_!"] \arrow[d, "\cong"] & H^*({}_bK(t)_a) \arrow[d, "\cong"] \\ \mathcal{F}\left((\overline{b}t'a)_{\mathrm{red}}\right)\{c\left((\overline{b}t'a)_{\mathrm{red}}\right)\} \arrow[r, "\mathcal{F}(\ldots)"] & \mathcal{F}\left((\overline{b}ta)_{\mathrm{red}}\right)\{c\left((\overline{b}ta)_{\mathrm{red}}\right)\} \end{tikzcd}$$



of vector spaces, where the right vertical isomorphism is given by (43) and the left one by (44).

- *Suppose that the i-th circle in $(\overline{b}ta)_{\mathrm{red}}$ splits into the i-th and j-th circle in $(\overline{b}t'a)_{\mathrm{red}}$ due to the surgery. Then we obtain the same commutative diagrams as in the case above with the only difference that we have to exchange t and t' in both diagrams.*

Proof. Let us assume that the i-th and j-th circle in $(\overline{b}ta)_{\mathrm{red}}$ merge and become the i-th circle in $(\overline{b}t'a)_{\mathrm{red}}$ (as in the proof of Theorem 2.50 the case of a splitting circle follows by exchanging t and t'). By Proposition 2.35 we have inclusions

$${}_bK(t')_a \subset {}_bK(t)_a \quad \text{and} \quad {}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \subset {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}$$

which are related by the homeomorphism from Proposition 2.26, i.e. the diagram on the left of the following picture, where the two vertical homeomorphisms are given by the respective restrictions of the map $\gamma_{2m_1-n,\ldots,2m_s-n} \circ \phi_{I_1,\ldots,I_s}$, clearly commutes and we obtain the commutative diagram on the right by applying the cohomology functor:

$$\begin{tikzcd} {}_bK(t)_a \arrow[d, "\cong"] & {}_bK(t')_a \arrow[l, hook', "\supset"'] \arrow[d, "\cong"] \\ {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}} & {}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \arrow[l, hook', "\supset"'] \end{tikzcd} \quad \overset{H^*}{\rightsquigarrow} \quad \begin{tikzcd} H^*({}_bK(t)_a) \arrow[r, "\subset^*"] & H^*({}_bK(t')_a) \\ H^*\left({}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[u, "\cong"] \arrow[r, "\subset^*"] & H^*\left({}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[u, "\cong"] \end{tikzcd}$$

Now the claim follows by sticking the diagram on the right on top of the diagram obtained by applying Theorem 2.50 to the tangle and cup diagrams t_{red}, t'_{red} and a_{red}, b_{red}.

Alternatively, we can consider the commutative diagram

$$\begin{tikzcd}
H^*({}_bK(t')_a) \arrow[r, "P"] & H_*({}_bK(t')_a) \arrow[r, "\subset_*"] \arrow[d, "\cong"] & H_*({}_bK(t)_a) \arrow[r, "P^{-1}"] \arrow[d, "\cong"'] & H^*({}_bS(t)_a) \\
H^*\left({}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[r, "P"] \arrow[u, "\cong"] & H_*\left({}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[r, "\subset_*"] & H_*\left({}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[r, "P^{-1}"] & H^*\left({}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}\right) \arrow[u, "\cong"']
\end{tikzcd}$$

where the vertical isomorphisms are the usual pushforward and pullback of the homeomorphism $\gamma_{2m_1-n,\dots,2m_s-n} \circ \phi_{I_1,\dots,I_s}$ in homology and cohomology. The commutativity of the middle square is clear and as in the proof of Theorem 2.50 the outer squares are easily seen to be commutative, too.

The map in the upper row is by definition the pushforward of the inclusion ${}_bK(t')_a \subset {}_bK(t)_a$ in cohomology. Similarly, the map in the lower row is the pushforward of the inclusion ${}_{b_{\mathrm{red}}}S(t'_{\mathrm{red}})_{a_{\mathrm{red}}} \subset {}_{b_{\mathrm{red}}}S(t_{\mathrm{red}})_{a_{\mathrm{red}}}$. Again, we can apply Theorem 2.50 to the tangle and cup diagrams t'_{red}, t_{red} and a_{red}, b_{red} and combine the resulting diagram with the one above in order to deduce the claim. $\square$

Remark 2.52. For completeness, we remark that all the results in this subsection are still true if we replace $\mathbb{F}_2$ by the more natural coefficient ring $\mathbb{Z}$ or $\mathbb{C}$. However, this requires a careful choice of orientations of all the involved manifolds in order to get the correct signs. This is rather technical and therefore avoided in this work.

3 A geometric construction of Khovanov homology

3.1 Khovanov's arc algebra as a convolution algebra

We begin by recalling the definition of the basic arc algebra as introduced in [Kho02, §2.4]. Interesting generalizations motivated by Lie theory can be found in [BS11] and [Str09]. We also review a result by Stroppel and Webster [SW12] who reconstructed these algebras geometrically by realizing them as convolution algebras using Springer fibers.

We set $\mathcal{H}^0 = \mathbb{F}_2$ (viewed as an algebra over itself). Given a positive integer k, it makes sense to define a graded vector space

$$\mathcal{H}^k := \bigoplus_{(a,b)\in\left(B^{k,k}\right)^2} {}_b(\mathcal{H}^k)_a,$$

where ${}_b(\mathcal{H}^k)_a := \mathcal{F}(\overline{b}a)\{k\}$, because $\overline{b}a$ is an ordered collection of circles (cf. Remark 2.31) to which we can apply the TQFT functor $\mathcal{F}$ as explained in subsection 2.3.1.

Let $a, b, c, d \in B^{k,k}$ be cup diagrams. In order to define a collection of multiplication maps

$$m_{d,c,b,a} \colon {}_d(\mathcal{H}^k)_c \otimes {}_b(\mathcal{H}^k)_a \to {}_d(\mathcal{H}^k)_a \tag{45}$$

turning $\mathcal{H}^k$ into an associative graded algebra with unit, set (45) to be the zero map unless $b = c$. In the latter case consider the tangle diagram $t_0 := b\overline{b}$ and choose an order on the cups of b compatible with the nesting. Then we inductively obtain a sequence of tangle diagrams $t_0, t_1, \dots, t_k$ as follows: The diagram t_i is obtained from the diagram t_{i-1} by performing a local surgery on the i-th cup of b. In particular, we obtain a sequence of circle diagrams $\overline{d}t_0a, \dots, \overline{d}t_ka$ which are connected by natural surgery cobordisms (cf.

Lemma 2.43). By applying the TQFT functor $\mathcal{F}$ to the resulting chain of cobordisms we obtain a map

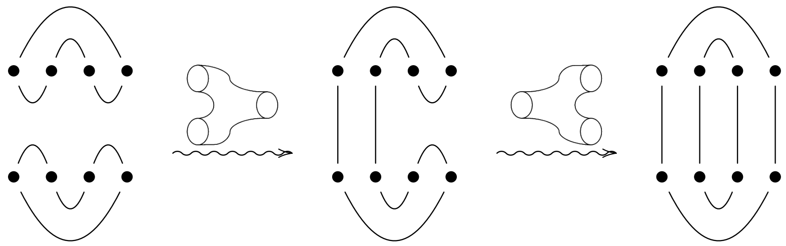
$$m_{d,b,b,a}\colon \mathcal{F}(\overline{d}b) \otimes \mathcal{F}(\overline{b}a) \cong \mathcal{F}(\overline{d}t_0 a) \to \mathcal{F}(\overline{d}t_k a) \cong \mathcal{F}(\overline{d}a).$$

The collection of maps $m_{d,c,b,a}$ defines the multiplication of Khovanov's algebra.

Example 3.1. If we choose

$a = d =$  $\qquad b = c =$ 

then the multiplication map $\mathcal{F}(\overline{a}b) \otimes \mathcal{F}(\overline{b}a) \to \mathcal{F}(\overline{a}a)$ for $\mathcal{H}^2$ is obtained by applying $\mathcal{F}$ to the following cobordism:



The induced linear map is given by

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\delta} \mathcal{A} \otimes \mathcal{A}.$$

In the work of Stroppel and Webster [SW12] the algebra $\mathcal{H}^k$ was redefined as a convolution algebra using the irreducible components of the (k,k)-Springer fibers. More precisely, they define a graded vector space by setting

$$\mathcal{H}^k_{\mathrm{Geo}} := \bigoplus_{(a,b)\in(B^{k,k})^2} {}_b(\mathcal{H}^k_{\mathrm{Geo}})_a,$$

where

$${}_b(\mathcal{H}^k_{\mathrm{Geo}})_a := H^*(S_a \cap S_b)\{k - \frac{1}{2}\dim(S_a \cap S_b)\}$$

and $\dim(S_a \cap S_b)$ denotes the dimension of $S_a \cap S_b$ viewed as a real manifold. The graded vector space $\mathcal{H}^k_{\mathrm{Geo}}$ can be equipped with a convolution product. In order to explain this consider the diagram

$$\begin{tikzcd}
H^*(S_a \cap S_b) \arrow[dr, "\iota^*_{ab}"] & & \\
& H^*(S_a \cap S_b \cap S_c) \arrow[r, "(\iota_{ac})_!"] & H^*(S_a \cap S_c) \\
H^*(S_b \cap S_c) \arrow[ur, "\iota^*_{bc}"'] & &
\end{tikzcd}$$

where the maps ι_{ab}, ι_{bc} and ι_{ac} are the respective inclusions and fix a cohomology class $f \in H^*(S_a \cap S_b \cap S_c)$. We define a linear map

$$H^*(S_a \cap S_b) \otimes H^*(S_b \cap S_c) \to H^*(S_a \cap S_c)$$

by the formula

$$\alpha *_f \beta := (\iota_{ac})_*(f \cup \iota_{ab}^*(\alpha) \cup \iota_{bc}^*(\beta)) \tag{46}$$

where $\alpha \in H^*(S_a \cap S_b)$ and $\beta \in H^*(S_b \cap S_c)$.

For general f this algebra behaves very badly, i.e. it might neither be associative nor graded (cf. [SW12, Section 4] for an interesting non-associative algebra corresponding to a "nested TQFT"). The following theorem states the existence of a particular nice choice of f. It is the topological equivalent to the algebro-geometric version [SW12, Theorem 35].

Theorem 3.2. *There exists a cohomology class $f \in H^*(S_a \cap S_b \cap S_c)$ such that the algebra $\mathcal{H}^k_{\mathrm{Geo}}$ with the convolution product (46) defined above is isomorphic to Khovanov's arc algebra $\mathcal{H}^k$ as a graded algebra.*

Remark 3.3. The class f is constructed inductively in the proof. Its precise geometric meaning remains mysterious to the author. It would be nice to have an explicit description in terms of characteristic classes of a certain vector bundle. It can be shown that the degree of f corresponds to the number of handles in Khovanov's cobordism. However, we refrain from making this precise.

Proof (Sketch). The general idea of the proof is to use Theorem 2.50 and interpret the chain of cobordism as a chain of pullbacks and pushforward maps in cohomology. Once this chain is constructed one can use the well-known clean intersection formula (cf. e.g. [Qui71] or [Ron80]) to inductively replace pullbacks with pushforwards and vice versa (this is where the cohomology class f comes from) until the composition can be described as a single pullback to the intersection followed by a pushforward. $\square$

3.2 Geometric bimodules via Spaltenstein varieties

Our goal is to use the general machinery developed in the second part of this work to extend the results from [SW12] by providing a geometric construction of what Khovanov calls "geometric bimodules" in [Kho02]. These geometric bimodules are certain combinatorially-defined vector spaces equipped with an action of a suitable arc algebra from the left and right, respectively. Despite their name, there is no geometry involved in the original construction of these bimodules. In Proposition 3.7 and Proposition 3.10 we realize these bimodules by defining a left and right action of an arc algebra on a sum of cohomology rings. Moreover, we construct some important bimodule maps using pushforward and pullback in cohomology (cf. Proposition 3.8 and Proposition 3.11).

Throughout this subsection we fix a collection $n_1 = 2k_1, \ldots, n_s = 2k_s$ of positive integers.

3.2.1 Khovanov's bimodules via a 2d TQFT

We begin by reviewing some of the basic definitions and results on geometric bimodules contained in [Kho02]. More general bimodules associated with the generalized arc algebras from [BS11] can be found in [BS10, §3].

Let $t \in \mathcal{T}(n_1, \ldots, n_s)$ be a tangle diagram. Then we define a graded vector space

$$\mathcal{F}(t) := \bigoplus_{a \in B^{k_1,k_1},\, b \in B^{k_s,k_s}} {}_b\mathcal{F}(t)_a,$$

where ${}_b\mathcal{F}(t)_a := \mathcal{F}(\bar{b}ta)\{k_1\}$, i.e. the summands of $\mathcal{F}(t)$ are obtained by applying the TQFT functor $\mathcal{F}$ to the circle diagram $\bar{b}ta$ as explained at the end of subsection 2.3.1.

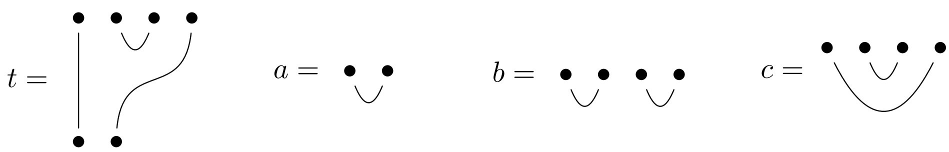
In [Kho02, §2.7], Khovanov defines a $(\mathcal{H}^{k_s}, \mathcal{H}^{k_1})$-bimodule structure on $\mathcal{F}(t)$. In order to explain the left $\mathcal{H}^{k_s}$-action we fix cup diagrams $a \in B^{k_1,k_1}$ and $b, c \in B^{k_s,k_s}$. Consider the tangle diagram $t_0 := b\overline{b}t$. By choosing an order on the cups of b compatible with the nesting (as in the definition of the multiplication of the arc algebra) we inductively obtain a sequence of tangle diagrams $t_0, \dots, t_{k_s}$, where the diagram t_i is obtained from the diagram t_{i-1} by performing a local surgery on the i-th cup of b. In particular, we obtain a sequence of circle diagrams $\overline{c}t_0 a, \dots, \overline{c}t_{k_s}a$. By Lemma 2.43 there is a natural surgery cobordism from the diagram $\overline{c}t_{i-1}a$ to the diagram $\overline{c}t_i a$. If we apply the TQFT $\mathcal{F}$ to this chain of cobordisms we obtain a linear map

$$l_{c,b,a} \colon {}_c(\mathcal{H}^{k_s})_b \otimes {}_b\mathcal{F}(t)_a \cong \mathcal{F}(\overline{c}t_0 a) \longrightarrow \mathcal{F}(\overline{c}t_{k_s}a) \cong {}_c\mathcal{F}(t)_a,$$

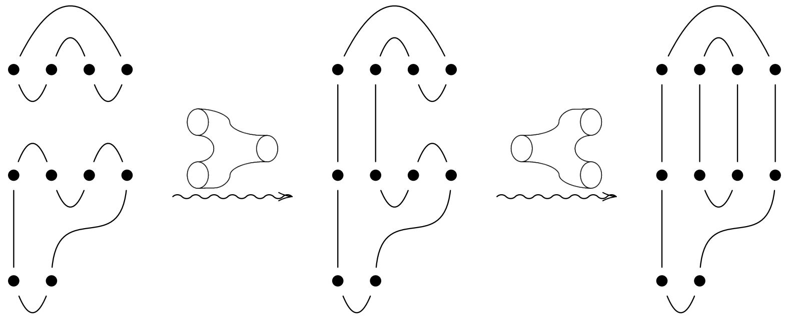
The collection of all such maps (varying over all choices of cup diagrams a, b, c) defines the left $\mathcal{H}^{k_s}$-action $l \colon \mathcal{H}^{k_s} \otimes \mathcal{F}(t) \to \mathcal{F}(t)$.

Similarly, the right $\mathcal{H}^{k_1}$-action on $\mathcal{F}(t)$ is defined. The resulting bimodule $\mathcal{F}(t)$ is finitely-generated and projective as a left and as a right module (cf. [Kho02, Prop.3]).

Example 3.4. Consider the following list of diagrams:



Then the map $l_{c,b,a} \colon {}_c(\mathcal{H}^2)_b \otimes {}_d\mathcal{F}(t)_c \to {}_d\mathcal{F}(t)_b$ associated with these data is obtained by applying the TQFT functor $\mathcal{F}$ to the following sequence of circle diagrams and surgery cobordisms:



Thus, being totally explicit, the map $l_{c,b,a}$ is precisely the composition $\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\delta} \mathcal{A} \otimes \mathcal{A}$.

Following Khovanov [Kho02, §2.8], we call a graded $(\mathcal{H}^k, \mathcal{H}^{k'})$-bimodule a *geometric bimodule* if there exists a grading-preserving isomorphism to a finite direct sum of bimodules $\mathcal{F}(t)$ (possibly with an extra grading shift), where t can be any tangle diagram whose bottom dot-cross sequence consists of $2k'$ dots and whose top dot-cross sequence consists of $2k$ dots. In particular, we have an additive category $\mathcal{GB}(k, k')$ whose objects are the geometric $(\mathcal{H}^k, \mathcal{H}^{k'})$-bimodules and whose morphisms are grading-preserving bimodule homomorphisms.

In subsection 3.3 we will be interested in bimodule homomorphisms induced by surgery cobordisms.

Proposition 3.5. *Let $t, t' \in \mathcal{T}(n_1, \dots, n_s)$ be tangle diagrams, where t' is obtained from t by a single local surgery. Then the direct sum of the linear maps $\mathcal{F}(\overline{b}ta)\{k_1\} \to \mathcal{F}(\overline{b}t'a)\{k_1\}$*

(one summand for every pair of cup diagrams $a \in B^{k_1,k_1}$, $b \in B^{k_s,k_s}$) induced by the natural surgery cobordism from the diagram $\bar{b}ta$ to the diagram $\bar{b}t'a$ (cf. Lemma 2.43) defines a grading-preserving homomorphism

$$\mathcal{F}(t) \to \mathcal{F}(t')\{-1\} \tag{47}$$

of $(\mathcal{H}^{k_s}, \mathcal{H}^{k_1})$-bimodules.

Similarly, we also obtain a grading-preserving bimodule homomorphism

$$\mathcal{F}(t') \to \mathcal{F}(t)\{-1\}. \tag{48}$$

in the other direction. In particular, the maps (47) and (48) are morphisms in the category $\mathcal{GB}(k_s, k_1)$.

Proof. It is clear that the linear maps (47) and (48) are indeed bimodule homomorphisms (see also the more general statement [Kho02, Proposition 5] in Khovanov's paper). Moreover, notice that the maps $\mathcal{F}(\bar{b}ta)\{k_1\} \to \mathcal{F}(\bar{b}t'a)\{k_1\}$ have degree 1 for every choice of cup diagrams $a \in B^{k_1,k_1}$ and $b \in B^{k_s,k_s}$ (the multiplication and comultiplication in $\mathcal{A}$ have degree 1). Hence, the direct sum of the maps is grading-preserving after shifting down the grading of the bimodule in the codomain by 1. □

3.2.2 Geometric and topological construction of bimodules

For the rest of this subsection we fix a tangle diagram $t \in \mathcal{T}(n_1, \ldots, n_s)$. The next goal is to reconstruct the associated bimodule $\mathcal{F}(t)$ topologically. In order to do that we define a graded vector space

$$\mathcal{G}(t) := \bigoplus_{a \in B^{k_1,k_1}, b \in B^{k_s,k_s}} {}_b\mathcal{G}(t)_a,$$

where the ${}_b\mathcal{G}(t)_a$ are the shifted cohomology rings

$${}_b\mathcal{G}(t)_a := H^*\left({}_bS(t)_a\right)\{k_1 - \frac{1}{2}\dim({}_bS(t)_a)\}$$

of the manifolds from Definition 2.24. In this context the word "dim" denotes the real dimension of the manifold enclosed in parentheses.

Lemma 3.6. *There is an isomorphism of graded vector spaces $\Phi_t\colon \mathcal{G}(t) \cong \mathcal{F}(t)$ which is explicitly given as the sum of the isomorphisms*

$$\varphi_{\dim({}_bS(t)_a)} \circ \left(\xi_{\bar{b}ta}^{-1}\right)^* : H^*\left({}_bS(t)_a\right)\{k_1 - \frac{1}{2}\dim({}_bS(t)_a)\} \xrightarrow{\cong} \mathcal{F}(\bar{b}ta)\{k_1\}. \tag{49}$$

Proof. This is part of Theorem 2.50. Since $\dim({}_bS(t)_a) = 2 \cdot c(\bar{b}ta)$ the linear isomorphisms (49) are grading-preserving for every choice of cup diagrams $a \in B^{k_1,k_1}$ and $b \in B^{k_s,k_s}$. □

We can use the isomorphism from Lemma 3.6 to define a left $\mathcal{H}^{k_s}$-action on $\mathcal{G}(t)$ as follows:

$$\mathcal{H}^{k_s} \otimes \mathcal{G}(t) \xrightarrow{\mathrm{id}\otimes\Phi_t} \mathcal{H}^{k_s} \otimes \mathcal{F}(t) \xrightarrow{l} \mathcal{F}(t) \xrightarrow{\Phi_t^{-1}} \mathcal{G}(t). \tag{50}$$

Similarly, we can also define a right $\mathcal{H}^{k_1}$ action, thereby turning the graded vector space $\mathcal{G}(t)$ into a $(\mathcal{H}^{k_s}, \mathcal{H}^{k_1})$-bimodule.

Proposition 3.7. *The isomorphism of graded vector spaces $\Phi_t\colon \mathcal{G}(t) \xrightarrow{\cong} \mathcal{F}(t)$ from Lemma 3.6 becomes an isomorphism of $(\mathcal{H}^{k_s}, \mathcal{H}^{k_1})$-bimodules if we equip $\mathcal{G}(t)$ with the bimodule structure defined above. In particular, Φ_t is an isomorphism in the category $\mathcal{GB}(k_s, k_1)$.*

Proof. Notice that we obviously have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}^{k_s} \otimes \mathcal{G}(t) \arrow[r, "\mathrm{id}\otimes\Phi_t"] \arrow[d, "\mathrm{id}\otimes\Phi_t"'] & \mathcal{H}^{k_s} \otimes \mathcal{F}(t) \arrow[r, "l"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}"'] & \mathcal{F}(t) \arrow[r, "\Phi_t^{-1}"] \arrow[d, "\mathrm{id}"] & \mathcal{G}(t) \arrow[d, "\Phi_t"] \\
\mathcal{H}^{k_s} \otimes \mathcal{F}(t) \arrow[r, "\mathrm{id}\otimes\mathrm{id}"] & \mathcal{H}^{k_s} \otimes \mathcal{F}(t) \arrow[r, "l"] & \mathcal{F}(t) \arrow[r, "\mathrm{id}"] & \mathcal{F}(t)
\end{tikzcd}$$

By the definition of the left $\mathcal{H}^{k_s}$-action (50) on $\mathcal{G}(t)$ the commutativity of this diagram is equivalent to saying that the map Φ_t is an isomorphism of left $\mathcal{H}^{k_s}$-modules. Proving that Φ_t is also an isomorphism of right H^{k_1}-modules follows completely analogously. $\square$

Proposition 3.8. *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be tangle diagrams, where t' is obtained from t by a single local surgery. Then we obtain a grading-preserving homomorphism of bimodules*

$$\mathcal{G}(t) \to \mathcal{G}(t')\{-1\} \tag{51}$$

as the direct sum of the linear maps

$$H^*\left({}_bS(t)_a\right)\{k_1 - \frac{1}{2}\dim({}_bS(t)_a)\} \to H^*\left({}_bS(t')_a\right)\{k_1 - \frac{1}{2}\dim({}_bS(t')_a)\}$$

obtained by pullback or pushforward in cohomology depending on whether ${}_bS(t')_a \subset {}_bS(t)_a$ or ${}_bS(t)_a \subset {}_bS(t')_a$.

Similarly, we also obtain a grading-preserving homomorphism of bimodules

$$\mathcal{G}(t') \to \mathcal{G}(t)\{-1\} \tag{52}$$

in the other direction. The maps (51) and (52) are morphisms in the category $\mathcal{GB}(k_s, k_1)$.

Proof. The following diagram is commutative:

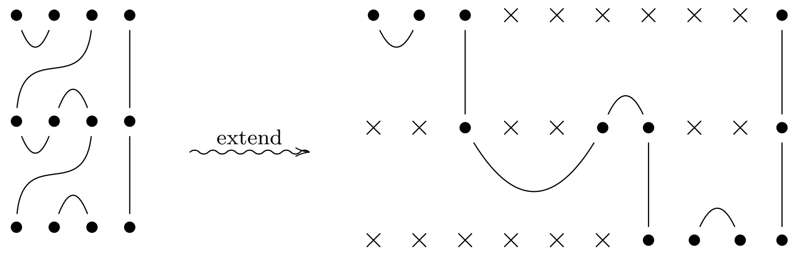
$$\begin{tikzcd}
\mathcal{G}(t) \arrow[r, "(51)"] \arrow[d, "\Phi_t"', "\cong"] & \mathcal{G}(t')\{-1\} \arrow[d, "\Phi_{t'}", "\cong"'] \\
\mathcal{F}(t) \arrow[r, "(47)"] & \mathcal{F}(t')\{-1\}
\end{tikzcd}$$

By linearity it suffices to check this on a summand of $\mathcal{G}(t)$. But then the claimed commutativity follows directly from Theorem 2.50. By Proposition 3.5 and Proposition 3.7 the map (47) and the isomorphism Φ_t are morphisms in the category of geometric bimodules. Hence, by the commutativity of the above diagram, so is the upper map $\mathcal{G}(t) \to \mathcal{G}(t')\{-1\}$. The same argument proves that the map (52) is a bimodule homomorphism, too. $\square$

For the reader who prefers to work with flag varieties we sketch how these bimodules can also be constructed in the algebro-geometric picture. Since all the statements can be deduced from the corresponding statements in the topological world by precomposing with the homeomorphism from Proposition 2.26 we omit the proofs (cf. the proof of Theorem 2.51 for a detailed example of such a reduction argument).

Fix a tangle diagram $t \in \mathcal{T}(n_1, \ldots, n_s)$. Then there exist an even positive integer $n = 2k$ and admissible sequences $I_1, \ldots, I_s$, each of which has heighest integer n, and a tangle diagram $t_{\text{ext}} \in \mathbb{T}(I_1, \ldots, I_s)$ with the property that $(t_{\text{ext}})_{\text{red}} = t$, i.e. the tangle diagram t_{ext} is obtained from t by adding pairs of crosses making the strands in the diagram vertical. We call t_{ext} an *extension of t* (this explains the subscript). It is clear that extensions always exist but they are obviously not unique, e.g. we can always add crosses at the end of the dot-cross sequences and still end up with an extension of the diagram we started with.

Example 3.9. On the left we have an example of a tangle diagram in $\mathcal{T}(4,4,4)$ and on the right we have an extension:



The extension is not minimal, e.g. it is possible to delete the fourth and fifth cross in all of the three dot-cross sequences and we still have a well-defined extension of the tangle diagram we started with. For reasons of efficiency it is preferable to work with minimal extensions. However, this is not necessary.

For the remaining subsection we fix an extension $t_{\mathrm{ext}} \in \mathbb{T}(I_1, \ldots, I_s)$ of the tangle diagram $t \in \mathcal{T}(n_1, \ldots, n_s)$. Define a graded vector space

$$\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}}) := \bigoplus_{a \in B_{I_1}^{\frac{n}{2},\frac{n}{2}},\, b \in B_{I_s}^{\frac{n}{2},\frac{n}{2}}} {}_b\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}})_a,$$

where the ${}_b\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}})_a$ are the shifted cohomology rings

$${}_b\mathcal{G}_{\mathrm{Geo}}(t)_a := H^*\left({}_bK(t_{\mathrm{ext}})_a\right)\{k_1 - \dim({}_bK(t_{\mathrm{ext}})_a)\}$$

of the varieties from Definition 2.21 (here, the word "dim" stands for the complex dimension of the respective variety).

As in the topological case we obtain a left and right action of an arc algebra on the vector space $\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}})$ via the isomorphism of graded vector spaces $\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}}) \cong \mathcal{F}(t)$ obtained as the sum of the grading-preserving linear isomorphisms

$$H^*\left({}_bK(t_{\mathrm{ext}})_a\right)\{k_1 - \dim({}_bK(t_{\mathrm{ext}})_a)\} \xrightarrow{\cong} \mathcal{F}(b_{\mathrm{red}} t a_{\mathrm{red}})\{k_1\} \tag{53}$$

as in Theorem 2.51, where $a \in B_{I_1}^{\frac{n}{2},\frac{n}{2}}$ and $b \in B_{I_s}^{\frac{n}{2},\frac{n}{2}}$. Thus, it makes sense to formulate the following algebro-geometric analog of Proposition 3.7:

Proposition 3.10. *The sum of the isomorpisms 53 of graded vector spaces defines an isomorphism of geometric bimodules* $\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}}) \cong \mathcal{F}(t)$.

We also have an analog of Proposition 3.5 and Proposition 3.8.

Proposition 3.11. *Let $t, t' \in \mathcal{T}(n_1, \ldots, n_s)$ be tangle diagrams, where t' is obtained from t by a single local surgery, and let $t_{\mathrm{ext}}, t'_{\mathrm{ext}} \in \mathbb{T}(I_1, \ldots, I_s)$ be extensions. Then we obtain a grading-preserving homomorphism of bimodules*

$$\mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}}) \to \mathcal{G}_{\mathrm{Geo}}(t'_{\mathrm{ext}})\{-1\} \tag{54}$$

as the direct sum of the linear maps

$$H^*\left({}_bK(t_{\mathrm{ext}})_a\right)\{k_1 - \dim({}_bK(t_{\mathrm{ext}})_a)\} \to H^*\left({}_bK(t'_{\mathrm{ext}})_a\right)\{k_1 - \dim({}_bK(t'_{\mathrm{ext}})_a)\}$$

(one summand for every pair of cup diagrams $a \in B_{I_1}^{\frac{n}{2},\frac{n}{2}}$, $b \in B_{I_s}^{\frac{n}{2},\frac{n}{2}}$) obtained by pullback or pushforward in cohomology depending on whether we have ${}_bK(t'_{\mathrm{ext}})_a \subset {}_bK(t_{\mathrm{ext}})_a$ or ${}_bK(t_{\mathrm{ext}})_a \subset {}_bK(t'_{\mathrm{ext}})_a$.

Similarly, we also obtain a grading-preserving homomorphism of bimodules

$$\mathcal{G}_{\mathrm{Geo}}(t'_{\mathrm{ext}}) \to \mathcal{G}_{\mathrm{Geo}}(t_{\mathrm{ext}})\{-1\} \tag{55}$$

in the other direction. The maps (54) and (55) are morphisms in the category $\mathcal{GB}(k_s, k_1)$.

Thus we have constructed a representative of each isomorphism class of objects in the category of geometric bimodules purely in terms of Spaltenstein varieties. The distinguished bimodule homomorphisms from Proposition 3.8 (respectively Proposition 3.11) play a crucial role in the following subsection.

3.3 A geometric construction of Khovanov homology

So far we have only dealt with planar tangle diagrams which can be embedded in $\mathbb{R}^2$. In this last section we add one dimension and study tangles in $\mathbb{R}^3$. Whereas the algebraic data associated with the two-dimensional world lives inside an additive category (the category of geometric bimodules) we have to pass to a triangulated category (the homotopy category of complexes of geometric bimodules) in order to describe the three-dimensional objects algebraically.

Due to the work of Khovanov [Kho02] we know how to assign a complex of geometric bimodules to a tangle in $\mathbb{R}^3$ in such a way that isotopy of tangles corresponds to isomorphism in the homotopy category of complexes. Thus, we have a tangle invariant (cf. also Theorem 3.30 below) which is known to categorify the Reshetikhin-Turaev invariant [RT90] associated with the quantum group $U_q(\mathfrak{sl}_2)$. In Theorem 3.34, the main result of this thesis, we prove that Khovanov's homological invariant can be reconstructed without a $2d$ TQFT only using cohomological methods and the topology of Spaltenstein varieties.

3.3.1 Tangles, planar projections and resolutions

We begin by recalling the notion of a tangle and the notion of a planar projection. Then we explain how to assign a collection of combinatorial tangle diagrams to a given planar projection. This connects the combinatorics of tangles in a three-dimensional space to the familiar planar combinatorics from subsection 2.1.

Definition 3.12. A *geometric tangle* is the image a proper, smooth embedding of a compact, smooth 1-manifold (this is just a finite disjoint union of circles $\mathbb{S}^1$ and intervals $[0,1]$) in $\mathbb{R}^2 \times [0,1]$. An *oriented geometric tangle* is a geometric tangle together with an orientation of each connected component.

By definition, a proper embedding maps the boundary to the boundary and the interior to the interior. In particular, the intersection of the image of the embedding with the set $\partial(\mathbb{R}^2 \times [0,1])$ consists of a finite number of points in $\mathbb{R}^2 \times \{0\}$, called the *bottom endpoints* of the geometric tangle, and a finite number of points in $\mathbb{R}^2 \times \{1\}$, called the *top endpoints*. Throughout this work we only consider geometric tangles with an even number of top and bottom endpoints.

Definition 3.13. A geometric tangle is called a *geometric (k,k')-tangle* if it has $2k'$ bottom and $2k$ top endpoints.

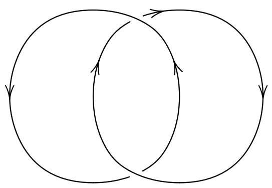
Remark 3.14. A geometric tangle without any endpoints is just a *knot* (if the domain of the embedding consists of a single $\mathbb{S}^1$) or a *link*.

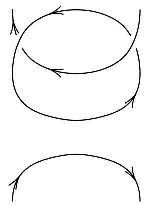
It is natural to identify geometric tangles which are related by a smooth isotopy.

Definition 3.15. A *tangle* as a smooth isotopy class of geometric tangles. An *oriented tangle* is a smooth isotopy class of oriented geometric tangles, where we additionally assume that the isotopies are orientation-preserving.

Let D be a generic plane projection of T. This is just a planar drawing of the tangle without "singularities" such as triple intersections, tangencies and cusps. If we are dealing with oriented tangles we will often decorate the pictures with orientation arrows.

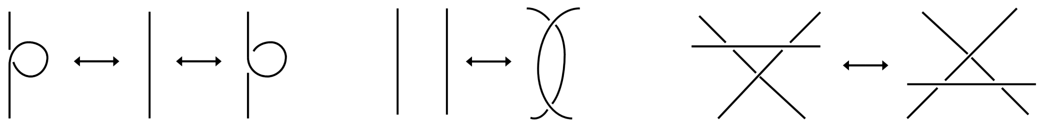
Example 3.16. The picture on the left is a plane projection of an oriented Hopf link. The one on the right represents a $(1, 1)$-tangle.





Obviously there are many different possible ways of representing a tangle by a planar projection. The answer to the question about a precise relationship between a tangle and its planar projections goes back to the early work of Reidemeister [Rei26].

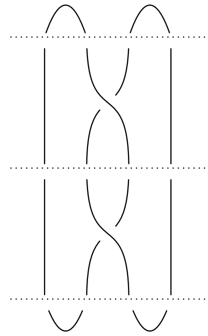
Proposition 3.17. *Two planar projections represent isotopic geometric tangles if and only if they are related by a planar isotopy and/or a finite sequence of Reidemeister moves:*



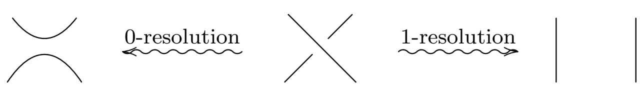
Proof. This is a well-known result from elementary knot theory. Detailed proofs can be found in [KT08, Kas95] or many other standard textbooks on the subject. □

A given planar projection D can be sliced up and written (after performing an appropriate planar isotopy if necessary) as a vertical composition of elementary projections, i.e. a plane projection of a tangle which does not contain any circles and has at most one crossing.

Example 3.18. Here is an example of a Hopf link sliced up into elementary projections:

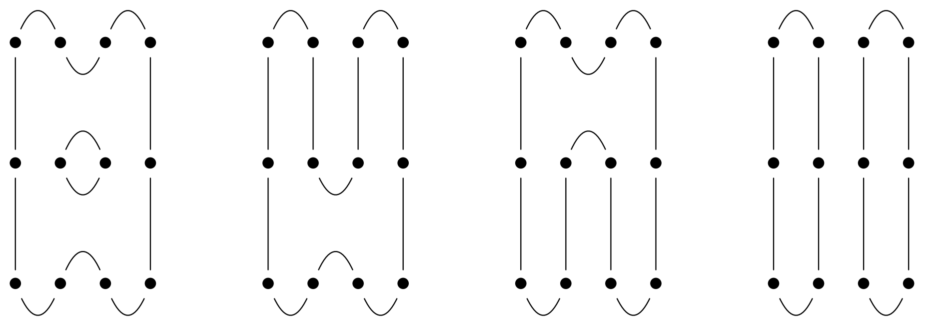


We would like to associate a collection of tangle diagrams (as in subsection 2.1) to a plane projection D with a fixed decomposition. Motivated by elementary skein-theory we can resolve the crossings of D in two possible ways:



A *resolution of* D is a resolution of each crossing of D. This yields a combinatorial tangle diagram (or a circle diagram if D represents a knot or link) if we mark the gluing points of the elementary projections in the decomposition with a dot.

Example 3.19. Here is a complete list of resolutions for a plane projection of a Hopf link decomposed as in Example 3.18:



3.3.2 The category of commutative cubes and a functor to chain complexes

We proceed by introducing a category of commutative cubes and explain how to assign a chain complex to an object of this category in a functorial way. Most of the material presented in this subsection is standard (see e.g. [Kho00, Kho02, BN02]), except maybe for the extensive use of categorical language (inspired by the work of Everitt and Turner [ET09] on generalized Khovanov cube constructions). The categorical viewpoint seems to be appropriate for our purposes because our focus of interest lies on the morphisms between commutative cubes and the induced morphisms between the chain complexes associated with these cubes rather than the cubes and complexes themselves.

Given a positive integer N, we can consider the set $\{1, 2, \ldots, N\}$ and its power set $\mathcal{P}(\{1, 2, \ldots, N\})$, i.e. the set of all subsets of $\{1, 2, \ldots, N\}$. We will always view the power set as a partially ordered set (poset), where the order relation is given by inclusion. For two sets $X, Y \in \mathcal{P}(\{1, \ldots, N\})$ we write $X \prec Y$ if Y covers X, i.e. Y can be obtained from X by adding a single element.

A useful tool for visualizing the structure of the poset $\mathcal{P}(\{1, \ldots, N\})$ is its Hasse diagram. This is the oriented graph which has a vertex for every subset of $\{1, \ldots, N\}$ and an edge between the vertices corresponding to $X, Y \in \mathcal{P}(\{1, \ldots, N\})$ whenever $X \prec Y$. The edges are oriented towards the subset with larger cardinality.

Example 3.20. Here is the Hasse diagram of the poset $\mathcal{P}(\{1, 2, 3\})$:

$$\begin{tikzcd}
& \{1\} \arrow[r] \arrow[dr] & \{1,2\} \arrow[dr] & \\
\emptyset \arrow[ur] \arrow[r] \arrow[dr] & \{2\} \arrow[ur] \arrow[dr] & \{1,3\} \arrow[r] & \{1,2,3\} \\
& \{3\} \arrow[r] \arrow[ur] & \{2,3\} \arrow[ur] &
\end{tikzcd}$$

Notice that we use the convention to arrange subsets of the same cardinality in columns when depicting a Hasse diagram. This is motivated by Bar-Natan's paper on Khovanov homology [BN02] (cf. also Example 3.27).

In the following, we will view the poset of subsets of $\{1, \dots, N\}$ as a category denoted by $\mathcal{P}os_N$. More precisely, the objects of $\mathcal{P}os_N$ are the elements of the power set $\mathcal{P}(\{1, \dots, N\})$ and the morphism space $\mathrm{Hom}_{\mathcal{P}os_N}(X, Y)$ between two objects $X, Y \in \mathcal{P}(\{1, \dots, N\})$ either consists of a single element, if X is a subset of Y, or is the empty set in all other cases. In terms of the Hasse diagram we can say that $\mathrm{Hom}_{\mathcal{P}os_N}(X, Y) \neq \emptyset$ if and only if there is an oriented path from X to Y.

The next definition can be found in [Kho00, Definition 1] or [Kho02, Definition 2] in a slightly disguised form.

Definition 3.21. Let $\mathcal{C}$ be a category. A *commutative N-cube in $\mathcal{C}$* is given by a functor $F\colon \mathcal{P}os_N \to \mathcal{C}$. This is equivalent to specifying an object $F(X) \in \mathcal{C}$ for every set $X \in \mathcal{P}(\{1, \dots, N\})$ and a morphism $F(X) \to F(Y)$ for every pair of sets $X \prec Y$. By functoriality these data must be subject to the following condition: The diagram

$$\begin{tikzcd}
 & F(X \cup \{x\}) \arrow[dr] & \\
F(X) \arrow[ur] \arrow[dr] & & F(X \cup \{x, y\}) \\
 & F(X \cup \{y\}) \arrow[ur] &
\end{tikzcd}$$

commutes for every triple (X, x, y), where $X \in \mathcal{P}(\{1, \dots, N\})$ and $x, y \in \{1, \dots, N\} \backslash X$, $x \neq y$.

Remark 3.22. The notion of a commutative N-cube is not as abstract as it sounds. Essentially, this is just a commutative diagram in the category $\mathcal{C}$ whose shape is given by the Hasse diagram of the poset $\mathcal{P}(\{1, \dots, N\})$.

We would like to go one step further than what is usually presented in the standard literature and define a category whose objects are the commutative N-cubes as defined above. A natural choice for such a category is the functor category $\mathbf{Fun}(\mathcal{P}os_N, \mathcal{C})$. Written out explicitly, we thus have the following morphisms in $\mathbf{Fun}(\mathcal{P}os_N, \mathcal{C})$:

Definition 3.23. A *morphism* between two commutative N-cubes F, G in $\mathcal{C}$ is a natural transformation of functors $\eta\colon F \to G$, i.e. we have a morphism $\eta_X\colon F(X) \to G(Y)$ in $\mathcal{C}$ for every set $X \in \mathcal{P}(\{1, \dots, N\})$ such that the diagram

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r] & G(Y)
\end{tikzcd}$$

is commutative for every pair $X \prec Y$.

In the following we set $\mathcal{C} = \mathcal{GB}(k, k')$, the category of geometric $(\mathcal{H}^k, \mathcal{H}^{k'})$-bimodules. Let $\mathbf{Ch}_{\mathcal{GB}(k,k')}$ denote the category of chain complexes of geometric bimodules.

Definition 3.24. We define a functor $\mathbf{Fun}(\mathcal{P}os_N, \mathcal{GB}(k, k')) \to \mathbf{Ch}_{\mathcal{GB}(k,k')}$ as follows:

- An object $F \in \mathbf{Fun}(\mathcal{P}os_N, \mathcal{GB}(k, k'))$, i.e. a commutative N-cube $F\colon \mathcal{P}os_N \to \mathcal{GB}(k, k')$, is sent to the complex

$$\cdots \to 0 \to F(\emptyset) \xrightarrow{\partial} \dots \xrightarrow{\partial} \underbrace{\bigoplus_{\substack{X \in \mathcal{P}(\{1,\dots,N\}),\\ \#(X) = i}} F(X)}_{\text{homological degree } i} \xrightarrow{\partial} \dots \xrightarrow{\partial} F(\{1, \dots, N\}) \to 0 \to \dots,$$

where the differential ∂^i in homological degree i is the sum of all maps in the cube from a bimodule associated with a set of cardinality i to a bimodule associated with a set of cardinality $i+1$.

- A morphism of cubes $\eta\colon F \to G$ is sent to the chain map whose i-th component is the direct sum of the natural transformation maps $\eta_X\colon F(X) \to G(X)$, where $\#(X) = i$:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i}} F(X) \arrow[r, "\partial"] \arrow[d, "\bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i}} \eta_X"'] & \bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i+1}} F(X) \arrow[r] \arrow[d, "\bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i+1}} \eta_X"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i}} G(X) \arrow[r, "\partial"] & \bigoplus_{\substack{X\in\mathcal{P}(\{1,\dots,N\}),\\ \#(X)=i+1}} G(X) \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 3.25. *The functor* $\mathbf{Fun}(\mathcal{P}os_N, \mathcal{GB}(k,k')) \to \mathbf{Ch}_{\mathcal{GB}(k,k')}$ *from Definition 3.24 is well-defined.*

Proof. In order to prove that the alleged chain complex associated to a commutative N-cube F is indeed an object of $\mathbf{Ch}_{\mathcal{GB}(k,k')}$ we have to show that $\partial \circ \partial = 0$. By linearity it suffices to check this on a summand of the i-th chain group. Let $\alpha \in F(X)$ be an element, where $X \in \mathcal{P}(\{1,\dots,N\}$ is a set with cardinality i. Then it follows from the commutativity relation in Definition 3.21 that

$$\partial \circ \partial(\alpha) \in \bigoplus_{Y\in\mathcal{P}(\{1,\dots,N\}),\, \#(Y)=i+2} F(Y)$$

is a sum of elements in which each summand occurs exactly twice. Since we work over $\mathbb{F}_2$ it follows that $\partial \circ \partial(\alpha) = 0$.

One easily sees that the map induced by a morphism of N-cubes is indeed a chain map. This follows immediately from the commutative square in the Definition of a morphism of N-cubes. Thus the functor from Definition 3.24 is well-defined on objects and morphisms.

We omit the easy verification that the functor respects compositions and identities. □

Remark 3.26. If we work over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ (or more generally a ring in which $1+1 \neq 0$) the above constructions still make sense after replacing commutative cubes with *skew-commutative cubes* as in Khovanov's original construction (cf. e.g. [Kho00, §3.3]). More precisely, we would have to add signs to some of the maps in the cube in a consistent way such that the summands cancel.

Example 3.27. As already mentioned in Remark 3.22 it is useful to think of a N-cube as a commutative diagram. If we follow the convention of Example 3.20 and draw the diagram in such a way that objects associated with sets of the same cardinality occur in columns, then the chain complex in Definition 3.24 can be obtained by simply summing the objects and morphisms along the columns. As an example consider a functor $F\colon \mathcal{P}os_3 \to \mathcal{GB}(k,k')$, i.e. a commutative diagram

$$\begin{tikzcd}
 & F(\{1\}) \arrow[r] \arrow[dr] & F(\{1,2\}) \arrow[dr] & \\
F(\emptyset) \arrow[ur] \arrow[r] \arrow[dr] & F(\{2\}) \arrow[ur] \arrow[dr] & F(\{1,3\}) \arrow[r] & F(\{1,2,3\}) \\
 & F(\{3\}) \arrow[r] \arrow[ur] & F(\{2,3\}) \arrow[ur] &
\end{tikzcd}$$

of objects and morphisms in $\mathcal{GB}(k,k')$. Applying the functor to the category of chain complexes yields

$$F(\emptyset) \to F(\{1\}) \oplus F(\{2\}) \oplus F(\{3\}) \to F(\{1,2\}) \oplus F(\{1,3\}) \oplus F(\{2,3\}) \to F(\{1,2,3\}).$$

More concrete examples and applications of these abstract notions are provided in the following subsection.

3.3.3 The complexes $\mathcal{CKh}$ and $\mathcal{CKh}_{\mathrm{Geo}}$ and an explicit isomorphism

In the following we explain two interesting ways of assigning a commutative cube to a plane projection of a tangle. The first one is a review of Khovanov's original construction [Kho02, §3.4] and the second one uses the cohomology of the manifolds introduced in Definition 2.24. The resulting cubes are compared and proven to be isomorphic in Proposition 3.33. This yields a new construction of Khovanov's chain complex (cf. Theorem 3.34; Theorem 3.37 is the algebro-geometric version).

For the remaining subsection we fix a generic plane projection D of a (k,k')-tangle together with a decomposition into elementary pieces. Let us assume that the projection D has N crossings. By labelling each crossing with a different element from the set $\{1,\dots,N\}$, i.e. we order the crossings of D, we obtain a bijection

$$\mathcal{P}(\{1,\dots,N\}) \xrightarrow{1:1} \{\text{resolutions of } D\} \tag{56}$$

by sending a set $X \in \mathcal{P}(\{1,\dots,N\})$ to the tangle diagram obtained by 1-resolving all the crossings of D whose label is contained in X and 0-resolving all the others. We write $t(X)$ for the tangle diagram associated with the set X via this bijection.

Definition 3.28. Define a commutative N-cube $\mathcal{K}h$-$\mathcal{C}$ube: $\mathcal{P}os_N \to \mathcal{GB}(k,k')$ in the category of geometric bimodules as follows:

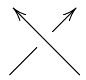
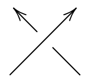
- The functor $\mathcal{K}h$-$\mathcal{C}$ube sends an object $X \in \mathcal{P}os_N$, i.e. a set $X \in \mathcal{P}(\{1,\dots,N\})$, to the geometric bimodule
$$\mathcal{K}h\text{-}\mathcal{C}\mathrm{ube}(X) := \mathcal{F}\left(t(X)\right)\{\#(X)\} \tag{57}$$
associated with the tangle diagram $t(X)$ (cf. subsection 3.2.1), where the grading is shifted according to the cardinality of X.

- Given a pair of objects $X, Y \in \mathcal{P}(\{1,\dots,N\})$ such that $X \prec Y$, then either the tangle diagrams $t(X)$ is obtained from $t(Y)$ by a single local surgery or vice versa (a 0-resolution in $t(X)$ is replaced by a 1-resolution in $t(Y)$). In any case, we obtain a homomorphism of bimodules
$$\mathcal{F}\left(t(X)\right)\{\#(X)\} \to \mathcal{F}\left(t(Y)\right)\{\#(Y)\} \tag{58}$$
as the sum of the linear maps induced by natural surgery cobordisms as in Proposition 3.5.

Lemma 3.29. *The functor $\mathcal{K}h$-$\mathcal{C}$ube: $\mathcal{P}os_N \to \mathcal{GB}(k,k')$ from Definition 3.28 is a well-defined commutative N-cube in the category of geometric bimodules.*

Proof. We already know that (57) is indeed a geometric bimodule and that (58) is a homomorphism of bimodules (cf. Proposition 3.5). Since $\#(Y) = \#(X)+1$ this homomorphism is grading-preserving and thus a morphism in $\mathcal{GB}(k,k')$. The commutativity condition in the definition of a cube follows immediately from the functoriality of the TQFT $\mathcal{F}$ (cf. [Lee05, §2.1.4] for a detailed case-by-case analysis using cobordism pictures). □

Hence, we can apply the functor from Definition 3.24 to the N-cube from Definition 3.28 and obtain a chain complex denoted by $\mathcal{CKh}(D)$. This is Khovanov's chain complex associated with the plane projection D as constructed in [Kho02, §3.4]. It becomes an invariant of oriented tangles after shifting the bimodule and homological grading according to the number of positive crossings $x(D)$ and the number of negative crossings $y(D)$ in D:

$x(D) = \#\Big($  $\Big)$ $\qquad$ $y(D) = \#\Big($  $\Big)$

More precisely, we have the following famous result due to Khovanov [Kho02, Theorem 2]:

Theorem 3.30. *Let D and D' be two plane projections of the same oriented tangle T, i.e. D and D' are related by a finite sequence of Reidemeister moves. Then there is a chain homotopy equivalence*

$$\mathcal{CKh}(D)[x(D)]\{2x(D) - y(D)\} \xrightarrow{\simeq} \mathcal{CKh}(D')[x(D')]\{2x(D') - y(D')\}$$

of complexes of $(\mathcal{H}^k, \mathcal{H}^{k'})$-bimodules. The number in squared brackets shifts the homological grading to the left. In particular, the isomorphism class in the homotopy category of complexes of geometric bimodules is independent of the chosen planar projection of an oriented tangle and therefore an invariant.

Proof. The proof consists of constructing explicit homotopy equivalences between the complexes associated with diagrams related by a Reidemeister move (cf. [Kho02, §4]). □

We want to see that this homological tangle invariant can be interpreted geometrically in terms of Spaltenstein varieties.

Definition 3.31. Define a commutative N-cube $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube: $\mathcal{P}os_N \to \mathcal{GB}(k,k')$ in the category of geometric bimodules as follows:

- The functor $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube sends an object $X \in \mathcal{P}os_N$, i.e. a set $X \in \mathcal{P}(\{1,\dots,N\})$, to the geometric bimodule
$$\mathcal{K}h_{\mathrm{Geo}}\text{-}\mathcal{C}\mathrm{ube}(X) := \mathcal{G}\left(t(X)\right)\{\#(X)\} \tag{59}$$
associated with the tangle diagram $t(X)$ (cf. subsection 3.2.2), where the grading is shifted according to the cardinality of X.

- Given a pair of objects $X, Y \in \mathcal{P}(\{1,\dots,N\})$ such that $X \prec Y$, then either the tangle diagrams $t(X)$ is obtained from $t(Y)$ by a single local surgery or vice versa. In any case, we obtain a homomorphism of bimodules
$$\mathcal{G}\left(t(X)\right)\{\#(X)\} \to \mathcal{G}\left(t(Y)\right)\{\#(Y)\} \tag{60}$$
as the sum of the linear maps induced by pullback or pushforward as in Proposition 3.8.

Lemma 3.32. *The functor $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube: $\mathcal{P}os_N \to \mathcal{GB}(k,k')$ from Definition 3.31 is a well-defined commutative N-cube in the category of geometric bimodules.*

Proof. By Proposition 3.7 the vector spaces (59) are objects in $\mathcal{GB}(k,k')$ and by Proposition 3.8 the maps (60) are morphisms in $\mathcal{GB}(k,k')$ (because $\#(Y) = \#(X)+1$). Hence, in order to prove the lemma, it remains to show the commutativity condition for every fixed triple (X,x,y) as in Definition 3.21.

Notice that this is equivalent to proving the commutativity of the following diagram because the outer square is precisely the

$$\begin{tikzcd}
\mathcal{G}\left(t(X)\right) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{G}\left(t(X)\right) \arrow[r, "(60)"] \arrow[d, "\Phi_{t(X)}"'] & \mathcal{G}\left(t(X\cup\{x\})\right) \arrow[r, "\mathrm{id}"] \arrow[d, "\Phi_{t(X\cup\{x\})}"] & \mathcal{G}\left(t(X\cup\{x\})\right) \arrow[d, "\mathrm{id}"] \\
\mathcal{G}\left(t(X)\right) \arrow[r, "\Phi_{t(X)}"] \arrow[d, "(60)"'] & \mathcal{F}\left(t(X)\right) \arrow[r, "(58)"] \arrow[d, "(58)"'] & \mathcal{F}\left(t(X\cup\{x\})\right) \arrow[d, "(58)"] & \mathcal{G}\left(t(X\cup\{x\})\right) \arrow[l, "\Phi_{t(X\cup\{x\})}"'] \arrow[d, "(60)"] \\
\mathcal{G}\left(t(X\cup\{y\})\right) \arrow[r, "\Phi_{t(X\cup\{y\})}"] \arrow[d, "\mathrm{id}"'] & \mathcal{F}\left(t(X\cup\{y\})\right) \arrow[r, "(58)"] & \mathcal{F}\left(t(X\cup\{x,y\})\right) & \mathcal{G}\left(t(X\cup\{x,y\})\right) \arrow[l, "\Phi_{t(X\cup\{x,y\})}"'] \arrow[d, "\mathrm{id}"] \\
\mathcal{G}\left(t(X\cup\{y\})\right) \arrow[r, "\mathrm{id}"] & \mathcal{G}\left(t(X\cup\{y\})\right) \arrow[r, "(60)"] \arrow[u, "\Phi_{t(X\cup\{y\})}"] & \mathcal{G}\left(t(X\cup\{x,y\})\right) \arrow[r, "\mathrm{id}"] \arrow[u, "\Phi_{t(X\cup\{x,y\})}"'] & \mathcal{G}\left(t(X\cup\{x,y\})\right)
\end{tikzcd}$$

The middle square commutes by Lemma 3.29 and the commutativity of the squares in the corners of the diagram is trivial. In order to check the commutativity of the remaining squares it suffices (by linearity) to check the commutativity summandwise. But by the definition of the involved maps this reduces the argument to the statement of Theorem 2.50 which finishes the proof. □

Proposition 3.33. *Let η_X denote the bimodule isomorphism associated with the tangle diagram $t(X)$ from Proposition 3.7, i.e. $\eta_X = \Phi_{t(X)}$. Then the collection of maps $\eta = (\eta_X)_{X\in\mathcal{P}(\{1,\dots,N\})}$ defines a natural isomorphism $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube $\to$ $\mathcal{K}h$-$\mathcal{C}$ube of functors, i.e. an isomorphism in the category $\mathbf{Fun}\,(\mathcal{P}os_N, \mathcal{GB}(k,k'))$ of commutative N-cubes in $\mathcal{GB}(k,k')$.*

Proof. Let $X, Y \in \mathcal{P}(\{1,\dots,N\})$ such that $X \prec Y$. We have already seen several times that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}(t(X))\{\#(X)\} \arrow[r, "(60)"] \arrow[d, "\eta_X=\Phi_{t(X)}"', "\cong"] & \mathcal{G}(t(Y))\{\#(Y)\} \arrow[d, "\cong"', "\eta_Y=\Phi_{t(Y)}"] \\
\mathcal{F}(t(X))\{\#(X)\} \arrow[r, "(58)"] & \mathcal{F}(t(Y))\{\#(Y)\}
\end{tikzcd}$$

Thus, the commutativity condition in the definition of a morphism of N-cubes is satisfied.

Since the maps η_X are isomorphisms in the category of geometric bimodules for every $X \in \mathcal{P}(\{1,\dots,N)$, it follows that $\eta\colon \mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube $\xrightarrow{\cong}$ $\mathcal{K}h$-$\mathcal{C}$ube is an isomorphism in $\mathbf{Fun}\,(\mathcal{P}os_N, \mathcal{GB}(k,k'))$. □

Theorem 3.34. *The chain complexes $\mathcal{CK}h(D)$ and $\mathcal{CK}h_{\mathrm{Geo}}(D)$ induced by applying the functor from Definition 3.24 to the N-cubes $\mathcal{K}h$-$\mathcal{C}$ube and $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube, respectively, are isomorphic in the category of chain complexes $\mathbf{Ch}_{\mathcal{GB}(k,k')}$ of geometric bimodules. In particular, the complexes $\mathcal{CK}h(D)$ and $\mathcal{CK}h_{\mathrm{Geo}}(D)$ are isomorphic in the homotopy category of chain complexes of geometric bimodules.*

Proof. By Proposition 3.33 we have an isomorphism of cubes $\mathcal{K}h_{\mathrm{Geo}}$-$\mathcal{C}$ube $\cong$ $\mathcal{K}h$-$\mathcal{C}$ube. Hence, applying the functor from Definition 3.24 induces an isomorphism of chain complexes $\mathcal{CK}h_{\mathrm{Geo}}(D) \cong \mathcal{CK}h(D)$. $\square$

Corollary 3.35. *The chain complex $\mathcal{CK}h_{\mathrm{Geo}}(D)$ is an invariant of oriented tangles in the homotopy category of complexes of geometric bimodules.*

Proof. This follows immediately from combining Theorem 3.30 with Theorem 3.34. $\square$

Thus, we have a description of Khovanov homology for tangles purely in terms of cohomology, pushforward and pullback maps. The $2d$ TQFT used in the original construction does not occur (at least not explicitly) in this picture anymore. In particular, Theorem 3.34 is our answer to the question from the introduction.

For completeness we also provide (without a proof) the analogous version of Theorem 3.34 using flag varieties. Consider the collection of tangle diagrams $t(X)$ obtained by resolving the crossings of D in all possible ways. Take the diagram $t(X)$ with the maximal number of strands and construct an extension $t_{\mathrm{ext}}(X) \in \mathbb{T}(I_1, \ldots, I_s)$ (as explained at the end of subsection 3.2.2), where $I_1, \ldots, I_s$ are appropriate admissible sequences with the same heighest integer $n = 2k$. Then it is easy to see that we also have extensions of the other diagrams in $\mathbb{T}(I_1, \ldots, I_s)$.

Now we can define a commutative N-cube as in the topological setting and prove that it is well-defined:

Definition 3.36. Define a commutative N-cube $\mathcal{K}h_{\mathrm{AlgGeo}}$-$\mathcal{C}$ube: $\mathcal{P}os_N \to \mathcal{GB}(k, k')$ in the category of geometric bimodules as follows:

- The functor $\mathcal{K}h_{\mathrm{AlgGeo}}$-$\mathcal{C}$ube sends an object $X \in \mathcal{P}os_N$, i.e. a set $X \in \mathcal{P}(\{1, \ldots, N\})$, to the geometric bimodule

$$\mathcal{K}h_{\mathrm{AlgGeo}}\text{-}\mathcal{C}\mathrm{ube}(X) := \mathcal{G}_{\mathrm{Geo}}\left(t_{\mathrm{ext}}(X)\right)\{\#(X)\} \tag{61}$$

  associated with the tangle diagram $t_{\mathrm{ext}}(X)$ (cf. subsection 3.2.2), where the grading is shifted according to the cardinality of X.

- Given a pair of objects $X, Y \in \mathcal{P}(\{1, \ldots, N\})$ such that $X \prec Y$, then either the tangle diagrams $t_{\mathrm{ext}}(X)$ is obtained from $t_{\mathrm{ext}}(Y)$ by a single local surgery or vice versa. In any case, we obtain a homomorphism of bimodules

$$\mathcal{G}_{\mathrm{Geo}}\left(t_{\mathrm{ext}}(X)\right)\{\#(X)\} \to \mathcal{G}_{\mathrm{Geo}}\left(t_{\mathrm{ext}}(Y)\right)\{\#(Y)\} \tag{62}$$

  as the sum of the linear maps induced by pullback or pushforward as in Proposition 3.11.

Theorem 3.37. *The chain complex $\mathcal{CK}h_{\mathrm{AlgGeo}}(D)$ obtained by applying the functor from Definition 3.24 to the commutative N-cube in Definition 3.36 is isomorphic to the chain complexes $\mathcal{CK}h_{\mathrm{Geo}}(D)$ and $\mathcal{CK}h(D)$ in the category of chain complexes of geometric bimodules.*

Proof. Essentially, this follows from the homeomorphism in Proposition 2.26. $\square$

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