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# Foundations of Representation Theory
## —Exercise sheet 1—

Let $k$ be a field.

**Exercise 1.** Convince yourself that the set $A^\times = \{a \in A \mid \exists b \in A : \ ab = ba = 1\}$ of units of a $k$-algebra $A$ is a group under multiplication. A homomorphism $f : A \to B$ of $k$-algebras induces a homomorphism of groups $f^\times : A^\times \to B^\times$.

(i) Let $G$ be a group. The image of the map $G \to k[G],\ g \mapsto g$ is contained in the group of units of $k[G]$ and yields a homomorphism $\varphi : G \to k[G]^\times$ of groups. Show that for every $k$-algebra $A$ and every homomorphism $\psi : G \to A^\times$ of groups, there exists a unique homomorphism $f : k[G] \to A$ of $k$-algebras such that $f^\times \circ \varphi = \psi$.

(ii) Determine the group algebras $k[\mathbb{Z}^n]$ and $k[\mathbb{Z}/2\mathbb{Z}]$.

**Exercise 2.** Let $Q$ be the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-1}} n.$$

Show that the path algebra $kQ$ is isomorphic to the $k$-algebra $L_n$ of lower triangular $n \times n$-matrices.

**Exercise 3.** Let $A$ be a $k$-algebra.

(i) Let $M$ be a left $A$-module. Show that the map $\varphi_M : A \to \mathrm{End}_k(M)$ defined by $\varphi_M(a) : M \to M, x \mapsto ax$ is a homomorphism of $k$-algebras.

(ii) Let $V$ be a $k$-vector space and let $\varphi : A \to \mathrm{End}_k(V)$ be a homomorphism of $k$-algebras. Show that $V$ becomes a left $A$-module via $A \times V \to V,\ (a, x) \mapsto ax := (\varphi(a))(x)$. Denote this module by $V_\varphi$.

(iii) Let $V$ and $W$ be two $k$-vector spaces and let $\varphi : A \to \mathrm{End}_k(V)$ and $\psi : A \to \mathrm{End}_k(W)$ be homomorphisms of $k$-algebras. Let $f : V \to W$ be a $k$-linear map. Prove that $f$ is a homomorphism of $A$-modules if and only if $\psi(a) \circ f = f \circ \varphi(a)$ for every $a \in A$.

**Exercise 4.** Consider the algebra $A = M_{n\times n}(k)$ of $n \times n$-matrices over $k$.

(i) Show that $A$ has no non-zero proper two-sided ideals.

(ii) Let $M \neq 0$ be a left $A$-module. Prove that $\dim_k M \geq n$.

**Due on Friday, 19.10.2018, before the lecture.**

General information:

- The homepage of the lecture is www.math.uni-bonn.de/ag/stroppel/Franzen_GZD_1819.htmpl.
- The tutorials start in the second week.
- The exercise sheet will be published every Friday on the homepage and are due on the following Friday.
- You may submit your solutions in groups of two.