Diplomarbeit

Drei Quanten-$\mathfrak{sl}_2$-Verallgemeinerungen des gefärbten Jonespolynoms in zwei Parametern

(*Three two-parameter quantum $\mathfrak{sl}_2$ generalisations of the coloured Jones polynomial*)

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Introduction

Deutsche Einleitung

Das Studium von Knoten beschäftigt sowohl Mathematiker als auch Physiker unterschiedlicher Fachrichtungen schon seit langer Zeit. Motiviert durch die verschiedensten Disziplinen wurden einige Methoden aufgestellt, um Knoten zu klassifizieren, unter anderem die Methode der Zuordnung eines Laurentpolynoms in einer oder mehreren Veränderlichen. Die Idee dahinter ist, einen beliebigen Knoten geschickt auf die Ebene zu projizieren und von unten nach oben zu lesen. Jeder Strang bekommt hierbei einen Vektorraum zugeordnet und jede Kreuzung und jede Schlaufe, die zwei Stränge von unten oder oben verbindet, steht für einen Morphismus zwischen Tensorprodukten dieser Räume. Die Verknüpfung aller Morphismen führt zu einer Abbildung vom Grundkörper der Laurentpolynome in sich selbst, die dem gelesenen Knoten auf eindeutige Weise ein Polynom zuordnet. Dieses Polynom muss gewisse Eigenschaften erfüllen, und zwar sollen Deformationen des Knotens durch Schlaufenbildung oder beliebiges „Ziehen“ an Strängen auf das Polynom keinen Einfluss haben. Das erste Polynom dieser Art entwickelte der amerikanische Mathematiker James W. Alexander im Jahre 1928 (siehe [Ale28]), später folgten beispielsweise das von Vaughan Jones entdeckte Jonespolynom (siehe [Jon85]) und das HOMFLY-Polynom[1].

Leider ist die Zuordnung von Polynomen zu Knoten nur in eine Richtung eindeutig, und zwar bekommen zwei (bis auf zulässige Deformation) gleiche Knoten auch das gleiche Polynom zugeordnet, die Umkehrung gilt jedoch keineswegs. Beispielsweise kann das Alexanderpolynom den rechtshändigen Trefoil-Knoten  und dessen Spiegelbild  nicht unterscheiden, allerdings gelingt dies unter Benutzung des Jonespolynoms (siehe [Jon85], S.105).

Die Motivation der aktuellen Forschung und auch dieser Arbeit ist es, ein neues Polynom zu finden, welches unter Umständen mächtiger ist als die bisher bekannten. Wir verfolgen hierbei den Ansatz, das schon existierende gefärbte Jonespolynom mit einem zweiten Parameter zu versehen und zu analysieren, welche Auswirkungen dies auf gewisse Standardbeispiele unter den Knoten hat. Die Färbung eines Stranges wird durch die Zuordnung einer natürlichen Zahl n realisiert, wobei diese in der Dimension des dem jeweiligen Strang zugeordneten Vektorraums codiert wird. Wir werden hierauf noch genauer eingehen. Motiviert wird die Einführung dieses zweiten Parameters durch die algebraische Kategorifizierung des Jonespolynoms (siehe beispielsweise [Kho00]). Auf kategorieller Ebene funktioniert der ursprüngliche Parameter q als in-

[1] Das HOMFLY-, HOMFLY-PT- oder auch verallgemeinerte Jonespolynom ist nach seinen Entdeckern Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, William B. R. Lickorish und David N. Yetter benannt. Der Zusatz „PT“ ehrt darauf aufbauende Erkenntnisse von Józef H. Przytycki und Paweł Traczyk. Siehe auch [FYH+85] und [PT85].

nere Gradverschiebung, wobei der neu einzuführende Parameter t eine Verschiebung im homologischen Grad beschreiben soll.

Wir führen im Folgenden drei verschiedene neue Polynome p_1, p_2 und p_3 in zwei Parametern ein. Die erste Version p_1 entsteht aus dem ursprünglichen Jonespolynom durch simple Skalierung mit gewissen Potenzen des Parameters t. Hierbei arbeitet man wie auch beim Einparameter-Jonespolynom mit Darstellungen der Quantengruppe $U_q(\mathfrak{sl}_2)$, jedoch über dem Grundkörper $\mathbb{C}(q,t)$. In der zweiten Variante p_2 haben wir den Parameter t subtiler eingebaut, sodass schon für einfache Knoten der neue Parameter nicht einfach wegfällt, wie es oft bei der ersten Variante der Fall ist. Hier ist es vonnöten, auf Darstellungen einer Zweiparameterversion der Quantengruppe zu arbeiten, die wir im Folgenden $U_{q,t}(\mathfrak{sl}_2)$ nennen. Diese geht ursprünglich auf Takeuchi [Tak90] zurück, ihre Darstellungstheorie wurde im Detail von Benkart und Witherspoon studiert (siehe [BW04a] und [BW04b]). Die dritte Variante p_3 schließlich geht von der Arbeit [Kho00] aus, wobei wir – anders als bei den anderen beiden Versionen – das Polynom von der Skeinrelation ausgehend abändern. Unsere Hauptresultate können in folgendem Theorem zusammengefasst werden:

Theorem. *Sei K ein beliebiger Knoten.*

(i) Die Zuordnungen $K \to p_i(K)$, $i = 1, 2, 3$, definieren Invarianten von orientierten Knoten (siehe 3.1.9, 4.2.7 und 5.0.5).

(ii) Die Polynome p_1 und p_2 lassen sich ferner durch Umskalierung als Tangleinvarianten schreiben (siehe 3.1.5 und 4.2.6).

(iii) Die Polynome p_i, $i = 1, 2, 3$ verallgemeinern das bereits bekannte gefärbte Jonespolynom in einer Variablen.

Die Durchführung dieser Arbeit gliedert sich wie folgt:

Im ersten Kapitel geben wir zunächst einen Überblick über die zugrunde liegende Theorie, wobei wir uns an den Notationen von [Jan96], [SS11] und [FSS12] orientieren. Die Vektorräume, die wir den Strängen zuordnen, sind irreduzible Moduln der kleinsten Quantengruppe $U_q(\mathfrak{sl}_2)$; wir werden $U_q(\mathfrak{sl}_2)$ genauer betrachten und feststellen, dass diese Quantengruppe eine Hopfalgebrastruktur aufweist und dass durch das damit verbundene Koprodukt eine sinnvolle Wirkung auf Tensoren von $U_q(\mathfrak{sl}_2)$-Moduln formuliert werden kann.

Das zweite Kapitel soll das schon bekannte gefärbte Jonespolynom in einem Parameter genauer vorstellen, damit die Verallgemeinerung auf ein Zweiparameterpolynom gelingen kann. Hierbei legen wir wieder den Artikel [FSS12] zugrunde. Darüber hinaus soll der Wert des mit n gefärbten Unknotens Schritt für Schritt berechnet werden; diese Vorgehensweise ist vorteilhaft, wenn man zur Kategorifizierung des Jonespolynoms übergehen möchte, denn jeder Rechenschritt ließe sich ganz bequem in die kategorielle Ebene übertragen. Hierzu siehe beispielsweise [Str05].

In den übrigen Kapiteln werden, basierend auf Kapitel 2, schließlich die neuen Versionen in zwei Parametern vorgestellt und wir beweisen, dass es sich tatsächlich um Knoteninvarianten handelt. p_1 wird in Kapitel 3 eingeführt, das vierte Kapitel dreht sich um p_2 und in Kapitel 5 soll kurz auf p_3 eingegangen werden. Der Nutzen des zweiten Parameters wird für alle Polynome anhand dessen analysiert, wie sich der zugehörige

Wert des gefärbten Unknotens verändert. Als weiteres Beispiel werden die sogenannten Thetanetworks eingeführt und genauer betrachtet. Der Wert des allgemeinen Falls ist für das klassische gefärbte Jonespolynom bereits ermittelt worden, beispielsweise in [MV94]; hier wird der Wert induktiv bewiesen. Wir beschränken uns hier auf das kleinste Beispiel $\vartheta(2,2,2)$.

Danksagung

Zunächst sei an dieser Stelle Frau Prof. Dr. Catharina Stroppel herzlich gedankt, die mir auch über Ozeane hinweg immer wieder mit Rat und Tat zur Seite gestanden und mir dieses interessante Projekt anheim gestellt hat, dessen Bearbeitung mir eine große Freude war. Ebenso gilt mein aufrichtiger Dank Dr. Thomas Riesenweber, Dina Hess und Lukas Kahl für ihre hilfreichen grammatikalischen und orthographischen Anmerkungen. Ganz besonders danke ich meinen Eltern und meiner Großmutter für unzählige aufbauende Worte und ihre unerlässliche Unterstützung in allen Lebenslagen und nicht zuletzt meinen lieben Kommilitonen, die mich in Freud und Leid durch mein Studium begleitet und getragen haben. Ohne euch alle wäre diese Arbeit nicht möglich gewesen, ihr seid in meinem Herzen für alle Zeit!

English introduction

The study of knots has occupied mathematicians as well as physicists of different specialisations for a long time. Based on a variety of disciplines, there are already a few methods to classify knots, including the method of assigning a Laurent polynomial to each knot. The idea is to project any knot appropriately onto the plane and to read the resulting knot diagram from bottom to top. Every strand is labelled by a vector space and every crossing or loop connecting two strands from below or above is interpreted as a morphism between tensor products of those vector spaces. The combination of all morphisms leads to a morphism of the ground field of Laurent polynomials into itself which uniquely assigns a polynomial to each knot. This polynomial has to obey certain properties, i.e. no curl forming or "pulling" on strands should influence the assigned polynomial. The first polynomial of this kind was found by the American mathematician James W. Alexander in 1928 (see [Ale28]); later followed the Jones polynomial developed by Vaughan Jones (see [Jon85]) and the so-called HOMFLY polynomial[1].

Unfortunately, the assignment of polynomials to knots and links is only unique in one direction, i.e. two similar knots (up to allowed deformation) are connected with the same polynomial, the reverse, however, is not true at all. For example, the Alexander polynomial is not able to distinguish the right-handed trefoil  from its mirror , but using the Jones polynomial the distinction succeeds (see [Jon85], p. 105).

The motivation of contemporary research and also of this work is to find a new polynomial which is hoped to be more powerful than others known so far. We proceed by taking the already defined coloured Jones polynomial and adding a second parameter; then we analyse its behaviour on certain standard knot examples. The colouring of a strand is achieved by assigning a natural number n to it, which is encoded via the dimension of the vector space attached to this strand. We will describe this later in detail. The introduction of a second parameter is motivated by the algebraic categorification of the Jones polynomial (see for example [Kho00]); in the categorified case the old parameter q works like an internal grading shift, whereas the new parameter t shall describe a shift in the homological degree.

In the following we will present three different new two-parameter polynomials p_1, p_2 and p_3. The first version p_1 arises from the Jones polynomial by simply rescaling with certain powers of the parameter t. Again, the irreducible representations of $U_q(\mathfrak{sl}_2)$ are used as in the one-parameter case but now over the ground field $\mathbb{C}(q,t)$. In the second variation p_2 we introduce the new parameter more subtly so that, as we will see, it does not already cancel for small examples like in the case of p_1. This construction enforces to use a two-parameter quantum group which will be called $U_{q,t}(\mathfrak{sl}_2)$ in the following. The latter goes back to the work of Takeuchi [Tak90], and its representation theory was studied in detail by Benkart and Witherspoon (see [BW04a] and [BW04b]). Finally, the third variation is based on the article [Kho00], where we choose a different approach and modify the polynomial starting from its skein relation. Our main results can be summarised in the following theorem:

[1] The HOMFLY, HOMFLY-PT or generalised Jones polynomial is named after its discoverers Jim Hoste, Adrian Ocneanu, Kenneth Millet, Peter J. Freyd, W. B. R. Lickorish and David N. Yetter. The addition "PT" honours the additional insights by Józef H. Przytycki and Paweł Traczyk. See also [FYH$^+$85] and [PT85].

Theorem. *Let K be an arbitrary knot.*

(i) The assignments $K \to p_i(K)$, $i = 1, 2, 3$, define three invariants of oriented knots (see 3.1.9, 4.2.7 and 5.0.5).

(ii) The polynomials p_1 and p_2 can be reformulated to tangle invariants by adequate rescaling (see 3.1.5 and 4.2.6).

(iii) The polynomials p_i, $i = 1, 2, 3$, generalise the already known coloured Jones polynomial in one variable.

This diploma thesis is structured as follows.

In chapter one we give an overview of the underlying theory the notation of which is primarily based on [Jan96], [SS11] and [FSS12]. The vector spaces associated with the strands are irreducible modules of the smallest quantum group $U_q(\mathfrak{sl}_2)$; we will describe its basic structure and describe the resulting action on tensor products of representations and their duals.

The second chapter will introduce the already known Jones polynomial in one parameter building the basic framework for our two-parameter generalisations later on. Here we work mainly based on the article [FSS12]. Moreover, the value of the unknot coloured by n is calculated step by step. This approach is useful if one wishes to move on to the categorification of the Jones polynomial since every step of the calculation could be transferred to the categorial level quite easily. For further information see, for example, [Str05] and [FSS12].

In the remaining chapters, the two new polynomials in two parameters will be introduced based on the findings in chapter two. Chapter 3 will concern p_1, the fourth chapter involves p_2 and the remaining p_3 is briefly introduced in chapter 5. The use of the second parameter will be analysed by explaining to which extend the value of the coloured unknot changes. As a second example we will also consider the so-called theta networks, a basic building block arising in the diagram calculus of the Turaev-Viro 3-manifold invariant as explained for instance in [KL94]. The value of the general case has been calculated already for the so-called classical coloured Jones polynomial, for example in [MV94] where the value is determined inductively. However, we will limit our expositions to the smallest example $\vartheta(2, 2, 2)$.

Acknowledgements

First of all, I thank my supervisor Prof. Dr. Stroppel who kindly supported my work from across the pond and who provided such an interesting project which I gladly worked on. Moreover, many thanks to Dr. Thomas Riesenweber, Dina Hess and Lukas Kahl for their helpful grammatical and orthographical remarks. In particular, I would like to thank my parents and my grandmother for countless encouraging words and their patient support in any situation; also special thanks to my dear fellow students who stood at my side throughout all those years of study. This thesis would not exist without all of you, you are in my heart forever!

Praefatio latina

Studium nodorum et mathematicos et physicos variarum professionum iam diu agitat. Nam in diversis iam disciplinis complures rationes, quibus nodi classificentur, repertae sunt, velut assignatio polynomii Laurentiani, quae fit cum una vel pluribus variabilibus. Hac autem in re proceditur, cum quilibet nodus in planitiem proicitur et ab imo ad summum legitur. Tum omnibus linis spatium vectoriale attribuitur et quicumque transitus flexusque, qui duo lina vel ab imo vel a summo coniungit, functiones inter producta tensoralia illorum spatiorum factae iudicantur. Denique ex coniunctione cunctorum functionum sequitur functio, quae corpus polynomiorum Laurentianorum fundamentale in se ipsum depignat, qua re nodo lecto polynomium certum assignatur. Oportet hoc polynomium servare qualitates quasdam, id est deformationibus illius nodi, quae fiunt, cum cincinnus paratur vel linum quolibet modo trahitur, polynomium mutare nequaquam licet. Primum autem huiusmodi polynomium invenit Iacobus Alexander ille mathematicus anno p. Chr. n. 1928 (vide [Ale28]), paulo post secutum est quod Vaughanus Jones repperit polynomium (confer [Jon85]), tum illud polynomium cui nomen est HOMFLY[1].

At vero nodis attribuere polynomia in unam tantum directionem non est ambiguum; nam duobus nodis praeter quasdam dumtaxat deformationes aequalibus idem polynomium assignari potest, in directionem autem inversam polynomio quodam nodum certum assignare minime potest. Exempli causa, polynomium Alexandrinum nodum trifolium dextrae manus  ab eius imagine ut ita dicam in speculo expressa  distinguere non potest, quod quidem contingit polynomio Jonense adhibito (vide [Jon85], p. 105).

Volo autem hac dissertatiuncula aliquid conferre ad hanc rem, de qua tot iam homines rerum mathematicarum periti et quaesiverunt et nostra quoque aetate quaerunt, an inveniri possit novum polynomium potentius fortasse quam quae adhuc novimus. Qua de causa conabimur polynomio Jonensi colorato, quod quidem iam habemus, variabili altera addita exquirere, quid certis quibusdam nodorum exemplis regularibus hac in re accidat. Coloratio linorum efficitur attributione numeri naturalis n, qui in dimensionem spatii vectorialis singuli lino adhaerentis quodam modo convertitur. Sed hanc rem diligentius inspiciamus. Ideo autem addenda videtur variabilis altera, quod polynomium illud Jonense in gradu categoriarum ratione algebraica positum illam confert (vide [Kho00]). Quod ad categorias attinet, variabilis principalis q operatur ut dilatio gradus interni, cum altera variabilis, quam nunc primum inducimus, describat dilationem gradus homologici.

Hoc igitur opere tria varia polynomia nova afferamus p_1, p_2, p_3 cum variabilibus duabus, quorum primum illud p_1 e polynomio Jonensi oritur multiplicatione quadam iterata cum varabili t. Qua in re haud aliter atque in polynomio illo Jonensi, quod unam tantum variabilem respicit, utamur quibusdam gregis quantorum $U_q(\mathfrak{sl}_2)$ repraesentationibus, sed super corpore fundamentali $\mathbb{C}(q, t)$. In variationem p_2 inseramus variabilem t subtilius, ita ut in simplicioribus quoque nodis illa variabilis non iam tollatur, sicut saepe in versione prima accidit. Hic autem uti necesse est repraesentationibus gregis

[1]Polynomium quod HOMFLY vel HOMFLY-PT sive polynomium Jonense generale nomen accepit ab inventoribus Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, W. B. R. Lickorish et David N. Yetter. Litterae illae PT adduntur, ut honorentur viros Jósef H. Przytycki et Pawel Traczyk, qui harum rerum scientiam insuper auxerint. Vide etiam [FYH$^+$85] et [PT85].

quantorum, qui duas variabiles exhibet, quem $U_{q,t}(\mathfrak{sl}_2)$ appellamus. Qui quidem grex inventus est a mathematico illo Takeuchi [Tak90], sed elementa rationis huius repraesentationum excussa sunt a mulieribus doctis Benkart et Witherspoon (vide [BW04a] et [BW04b]). Quod denique ad tertium modum, qui siglo p_3 notatur, attinet, proficiscentes a dissertatiuncula illa [Kho00] polynomium a relatione quidem skein contra atque in prioribus modis mutemus.

Inventa autem nostra in hanc sententiam cogere licet:

Theorema. *Sit K nodus quidam.*

(i) Attributiones $K \to p_i(K)$, $i = 1, 2, 3$ definiunt invariantes nodorum cum regione quadam (vide 3.1.9, 4.2.7 et 5.0.5).

(ii) Polynomia p_1 et p_2 apta multiplicatione quasi invariantia glomorum scribi possunt (confer 3.1.5 et 4.2.6).

(iii) Polynomia p_i, $i = 1, 2, 3$ generaliter extendunt polynomium illud Jonense coloratum unius variabilis.

Hanc igitur dissertatiunculam ita fere disponendam esse mihi placuit:

Capite primo describamus rationem atque doctrinam quae pertinet ad rem nostram, cum notationes in [Jan96], [SS11] et [FSS12] expositas sequamur. Spatia autem vectoris, quae linis attribuamus, sunt modula irreducibilia minimi gregis quantorum $U_q(\mathfrak{sl}_2)$; quorum gregem illum $U_q(\mathfrak{sl}_2)$ diligentius intueamur eundemque structuram habere algebrae Hopfianae demonstremus, coproducto autem de illa structura deducto effectum rationabilem in tensoribus modulorum gregis $U_q(\mathfrak{sl}_2)$ definiri posse illustremus.

Capite secundo ostendamus illud quod iam novimus polynomium Jonense coloratum una variabili adhibita, ut facilius in polynomium duas variabiles exhibens divulgare possimus; qua in re iterum utamur illo tractatu [FSS12]. Praeterea rationem anodi variabili n colorati gradatim subducamus, quo commodius polynomium Jonense in doctrinam categoriarum transferri potest; nam unus quisque gradus calculationis facile converti potest in linguam categoriarum. Hac de re vide [Str05] exempli causa.

In relinquis autem capitibus nova polynomia binis variabilibus instructa exponamus, qua in re nitamur eis quae in secundo capite demonstraverimus, easdemque re vera invariantes esse nodorum concludamus. p_1 explicetur capite tertio, p_2 capite quarto, p_3 capite quinto, sed hoc brevi tantum. Fructus alterius varibilis in illis polynomiis e mutatione virium anodi colorati aestimetur. Deinde consideramus nodum qui rete thetae dicitur; illud exemplum ad calculum varietatis quidem Turaev-Viro nominatae multum valet (confer [KL94]). Quamvis ratio casus generalis, quantum quidem ad illud polynomium quasi classice coloratum Jonense attinet, iam subducta sit, exempli causa ab [MV94], tamen hic inductione adhibita denuo subducatur. Qua quidem in re nos contineamus exemplo minimo $\vartheta(2, 2, 2)$.

Gratulatio

Primum quidem magistrae meae Prof. Dr. Catharina Stroppel gratiam debeo, quae vel super oceanum latum consilia multa dabat et mihi opus paravit quod tractare mihi placuit valde. Etiam amicis Dr. Thomas Riesenweber, Dina Hess et Lukas Kahl gratia reddatur pro meritis qui grammaticam orthographiamque animadvertebant. Imprimis autem parentibus meis aviaeque gratias ago, quod ope et consilio aderant, praeterea et amicis illis Bonnae repertis meum iter comitatis. Sine vobis hoc opus numquam ortum esset, mihi semper cordi eritis!

Contents

1 Tensors of $U_q(\mathfrak{sl}_2)$-modules

The goal of this first section is to introduce the setting for working with tensor products of $U_q(\mathfrak{sl}_2)$-modules; we define $U_q(\mathfrak{sl}_2)$ and show that it affords a Hopf algebra structure. The latter is essential for working with $U_q(\mathfrak{sl}_2)$-modules because we need a coproduct to define an adequate action on tensors of $U_q(\mathfrak{sl}_2)$-modules. All tensor products are over the ground field k if not specified otherwise.

With our notations we mainly follow [Jan96], [FSS12] and [SS11] in this chapter.

1.1 The algebra $U_q(\mathfrak{sl}_2)$ – Definitions and basic facts

In the following let $k = \mathbb{C}(q)$ be the field of rational functions in one indeterminate q.

1.1.1 Definition. *Define the so-called* nth quantum number *by* $[n] := \sum_{j=0}^{n-1} q^{n-2j-1}$ *and set* $[0] := 1$.
Using this, we can define $[n]! = [n][n-1]\cdots[2][1]$*, and* $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$.

1.1.2 Remark. *Note that also* $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$. *This holds since*

$$\begin{aligned} q^n - q^{-n} &= (q-q^{-1})(\sum_{i=0}^{n-1} q^{n-2i-1}) \\ &= q^n + q^{n-2} + ... + q^{-(n-2)} - (q^{n-2} + q^{n-4} + ... + q^{-(n-2)} + q^{-n}). \end{aligned}$$

Hence our definition agrees with the more common definition of quantum numbers used in the literature.

1.1.3 Definition. *The quantum group* $U_q(\mathfrak{sl}_2)$ *is the unitary algebra over* k *with generators* E, F, K, K^{-1} *and relations*

$$KK^{-1} = K^{-1}K = 1 \tag{1.1}$$

$$KE = q^2 EK \tag{1.2}$$

$$KF = q^{-2} FK \tag{1.3}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1.4}$$

Next, we want to show that $U_q(\mathfrak{sl}_2)$ can be equipped with a Hopf algebra structure. For this purpose we need some definitions.

1.1.4 Definition. *A* coassociative k-coalgebra *is a* k*-algebra* A *with a* k*-linear multiplication* $\cdot : A \times A \to A$ *and unit* $\eta : k \to A$ *together with two* k*-linear maps* $\Delta : A \to A \otimes A$

(the comultiplication*) and* $\varepsilon : A \to k$ *(the* counit*) such that the following diagrams commute:*

$$
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\
A \otimes A & \xrightarrow[\Delta\otimes 1]{} & A \otimes A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
A & & \\
\downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle \sim} & \\
A \otimes A & \xrightarrow[1\otimes\varepsilon]{} & A \otimes k
\end{array}
\qquad
\begin{array}{ccc}
A & & \\
\downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle \sim} & \\
A \otimes A & \xrightarrow[\varepsilon\otimes 1]{} & A \otimes k
\end{array}
\quad .
$$

Here, the maps $A \to A \otimes k$ *and* $A \to k \otimes A$ *are isomorphisms via the formulae* $a \mapsto a \otimes 1$ *and* $a \mapsto 1 \otimes a$ *respectively.*

Note that the construction of coassociative coalgebras is dual to the definition of associative algebras. A coalgebra is called cocommutative if $\Delta(a) = \tau \circ \Delta(a)$ for all a, where τ is the map flipping the tensor factors.

1.1.5 Definition. *A* bialgebra *is an algebra equipped with a coalgebra structure where comultiplication and counit are algebra homomorphisms and, respectively, multiplication and unit are coalgebra homomorphisms. In other words, the algebra and coalgebra structures are compatible by the commutativity of the following diagrams:*

$$
\begin{array}{ccccc}
B \otimes B & \xrightarrow{\Delta\otimes\Delta} & B \otimes B \otimes B \otimes B & \xrightarrow{1\otimes\tau\otimes 1} & B \otimes B \otimes B \otimes B \\
\downarrow{\scriptstyle \cdot} & & & & \downarrow{\scriptstyle \cdot\otimes\cdot} \\
B & & \xrightarrow{\Delta} & & B \otimes B
\end{array}
\qquad
\begin{array}{ccc}
k & \xrightarrow{\cong} & k \otimes k \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
B & \xrightarrow[\Delta]{} & B \otimes B
\end{array}
$$

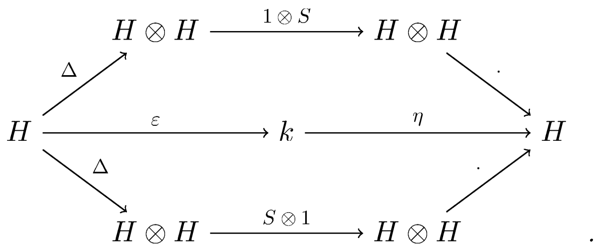
$$
\begin{array}{ccc}
B \otimes B & \xrightarrow{\cdot} & B \\
\downarrow{\scriptstyle \varepsilon\otimes\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
k \otimes k & \xrightarrow[\cong]{} & k
\end{array}
\qquad
\begin{array}{ccc}
k & \xrightarrow{\eta} & B \\
& \searrow{\scriptstyle id} & \downarrow{\scriptstyle \varepsilon} \\
& & k
\end{array}
\quad .
$$

The first diagram is also called *Hopf axiom* and it describes the compatibility of multiplication and comultiplication. Along with the second diagram it represents the algebra homomorphism structure of comultiplication and counit. On the other hand, the second two diagrams stand for the properties of multiplication and unit being coalgebra homomorphisms.

1.1.6 Remark. *In the monoidal category of* k*-vector spaces, monoids correspond to* k*-algebras because they are equipped with both a unit and a multiplication. Equally, comonoids coincide with coalgebras and bimonoids with bialgebras.*

Next, we deal with Hopf algebras. Hopf algebras differ from bialgebras by the additional structure of the so-called antipode map. In the context of category theory, the antipode turns the (bi-)monoidal structure of bialgebras into a group structure defining an inverse.

1.1.7 Definition. *A* Hopf algebra *is a bialgebra H together with a map $S : H \to H$ such that the following diagram commutes:*



Standard examples of Hopf algebras are the algebras of regular functions of complex affine algebraic groups or more easier group algebras of finite groups; see for example [Hum81]. The defining axioms here are just the dual notions of the defining axioms for a group. The associative multiplication map is turned into the comultiplication; the existence of a unit becomes the existence of the counit map and the existence of inverses gives the existence of the antipode. Aside from the second large class example, namely the universal enveloping algebras of Lie algebras, quantum groups are common examples of Hopf algebras with great impact. We want to study $U_q(\mathfrak{sl}_2)$, the smallest quantum group, in detail. Therefore, we need to define concrete formulae for comultiplication, counit and antipode to check that $U_q(\mathfrak{sl}_2)$ is indeed a Hopf algebra.

1.1.8 Theorem. *For simplicity, let $U := U_q(\mathfrak{sl}_2)$. Define a k-linear algebra morphism $\Delta : U \to U \otimes U$ by*

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \tag{1.5}$$
$$\Delta(E) = 1 \otimes E + E \otimes K^{-1}, \tag{1.6}$$
$$\Delta(F) = K \otimes F + F \otimes 1. \tag{1.7}$$

Moreover, let $\varepsilon : U \to k$ be the algebra homomorphism defined by

$$\varepsilon(K^{\pm 1}) = 1, \tag{1.8}$$
$$\varepsilon(E) = \varepsilon(F) = 0 \tag{1.9}$$

and finally let the antipode map be the map $S : U \to U$ which is a k-linear algebra antihomomorphism defined by

$$S(K^{\pm 1}) = K^{\mp 1}, \tag{1.10}$$
$$S(E) = -EK, \tag{1.11}$$
$$S(F) = -K^{-1}F. \tag{1.12}$$

Together with these maps, U is a Hopf algebra.

1.1.9 Remark. *In [Jan96], the following alternative formulae for Δ and S can be found:*

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$
$$\Delta(E) = E \otimes 1 + K \otimes E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

and

$$\begin{aligned}S(K^{\pm 1}) &= K^{\mp 1},\\ S(E) &= -K^{-1}E,\\ S(F) &= -FK.\end{aligned}$$

The proof of the theorem works equivalently with the latter definitions, but the former formulae are slightly more convenient in terms of visualisation[1]*.*

Note first that the maps defined above are indeed algebra homomorphisms. Since K, K^{-1}, E and F generate U as an algebra, it is indeed enough to specify the maps on them. It is only left to show that the maps are well-defined. This is done by direct calculations, for instance

$$\begin{aligned}\Delta(KE) &= \Delta(K)\Delta(E)\\ &= (K\otimes K)(1\otimes E + E\otimes K^{-1})\\ &= K\otimes KE + KE\otimes 1\\ &= q^2(1\otimes E)(K\otimes K) + q^2(E\otimes K^{-1})(K\otimes K)\\ &= \Delta(q^2EK).\end{aligned}$$

The rest is calculated analogously.

For the proof of the theorem, we need the following facts:

1.1.10 Lemma. *For the comultiplication, the following holds:*

$$\Delta(K^{\pm r}) = K^{\pm r}\otimes K^{\pm r}, \tag{1.13}$$

$$\Delta(E^r) = \sum_{i=0}^{r} q^{i(r-i)}\begin{bmatrix} r\\ i\end{bmatrix} E^i\otimes E^{r-i}K^{-i} \tag{1.14}$$

$$\Delta(F^r) = \sum_{i=0}^{r} q^{-i(r-i)}\begin{bmatrix} r\\ i\end{bmatrix} F^{r-i}K^{i}\otimes F^i. \tag{1.15}$$

Proof. The proof follows by induction on r.
By definition of Δ, the assertion holds for $r=1$:

$$\begin{aligned}\Delta(K^{\pm 1}) &= K^{\pm 1}\otimes K^{\pm 1},\\ 1\otimes E + E\otimes K^{\pm 1} = \Delta(E^1) &= \sum_{i=0}^{1} q^{i(1-i)}\begin{bmatrix} 1\\ i\end{bmatrix} E^i\otimes E^{1-i}K^{-i}\\ K\otimes F + F\otimes 1 = \Delta(F^1) &= \sum_{i=0}^{1} q^{-i(1-i)}\begin{bmatrix} 1\\ i\end{bmatrix} K^iF^{r-i}\otimes F^{r-i}.\end{aligned}$$

Now we want to deduce the formulae for $r+1$ assuming they hold for r. On (1.13):

$$\begin{aligned}\Delta(K^{\pm(r+1)}) &= \Delta(KK^{\pm r})\\ &= (K^{\pm 1}\otimes K^{\pm 1})(K^{\pm r}\otimes K^{\pm r})\\ &= K^{\pm(r+1)}\otimes K^{\pm(r+1)}.\end{aligned}$$

[1] See [FK97], p. 416.

On (1.14):

$$\begin{aligned}
\Delta(E^{r+1}) &= (1 \otimes E + E \otimes K^{-1})\Big(\sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^i \otimes E^{r-i}K^{-i}\Big) \\
&= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^i \otimes E^{r-i+1}K^{-i} \\
&\quad + \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^{i+1} \otimes K^{-1}E^{r-i}K^{-i} \\
&= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^i \otimes E^{r-i+1}K^{-i} \\
&\quad + \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} q^{2(r-i)} E^{i+1} \otimes E^{r-i}K^{-(i+1)} \\
&= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^i \otimes E^{r-i+1} \\
&\quad + \sum_{i=0}^{r} q^{(i+2)(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^{i+1} \otimes E^{r-i}K^{-(i+1)} \\
&= \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^i \otimes E^{r-i+1}K^{-i} \\
&\quad + \sum_{i=1}^{r+1} q^{(i+1)(r-i+1)} \begin{bmatrix} r \\ i-1 \end{bmatrix} E^i \otimes E^{r-i+1}K^{-i} \\
&= 1 \otimes E^{r+1} + E^{r+1} \otimes K^{-(r+1)} \\
&\quad + \sum_{i=1}^{r} \Big(q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} + q^{(i+1)(r-i+1)} \begin{bmatrix} r \\ i-1 \end{bmatrix}\Big) E^i \otimes E^{r-i+1}K^{-i} \\
&= 1 \otimes E^{r+1} + E^{r+1} \otimes K^{-(r+1)} \\
&\quad + \sum_{i=1}^{r} q^{i(r+1-i)} \underbrace{\Big(q^i \begin{bmatrix} r \\ i \end{bmatrix} + q^{r-i+1} \begin{bmatrix} r \\ i-1 \end{bmatrix}\Big)}_{=\left[\begin{smallmatrix} r+1 \\ i \end{smallmatrix}\right]} E^i \otimes E^{r-i+1}K^{-i} \\
&= \sum_{i=0}^{r+1} q^{i(r+1-i)} \begin{bmatrix} r+1 \\ i \end{bmatrix} E^i \otimes E^{(r+1)-i}K^{-i}.
\end{aligned}$$

The third equality results by performing $r - i$ flips of K^{-1} and E as in (1.2); the fifth equality follows by an index shift and the last equality is based on the following lemma. This concludes the proof of (1.14). The formula (1.15) follows completely analogously. □

1.1.11 Lemma. $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right] = q^{-i}\left[\begin{smallmatrix} n-1 \\ i \end{smallmatrix}\right] + q^{n-i}\left[\begin{smallmatrix} n-1 \\ i-1 \end{smallmatrix}\right]$.

Proof. Using the definition of quantum binomial coefficients we obtain

$$\begin{aligned} q^{-i}\begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix} &= q^{-i}\frac{[n-1]!}{[n-1-i]![i]!} + q^{n-i}\frac{[n-1]!}{[n-i]![i-1]!} \\ &= \frac{q^{-i}[n-i][n-1]! + q^{n-i}[i][n-1]!}{[n-i]![i]!} \\ &= \frac{[n][n-1]!}{[n-i]![i]!} \\ &= \begin{bmatrix} n \\ i \end{bmatrix}, \end{aligned}$$

where the penultimate equality holds since

$$\begin{aligned} q^{-i}[n-i] + q^{n-i}[i] &= \frac{q^{-i}(q^{n-i} - q^{-(n-i)}) + q^{n-i}(q^{i} - q^{-i})}{q - q^{-1}} \\ &= \frac{q^{n-2i} - q^{-n} + q^{n} - q^{n-2i}}{q - q^{-1}} \\ &= [n]. \end{aligned}$$

□

Proof of Theorem 1.1.8. To prove that U is a Hopf algebra, the Hopf algebra properties of U have to be verified by using the maps Δ, ε, S, $\cdot$ and η as defined above.

1. U is a vector space over $k = \mathbb{C}(q)$: This holds by definition.

2. Together with the maps $\cdot$ and η (i.e. multiplication and unit), U is a unital, associative algebra: This also follows by definition.

3. Together with the maps Δ und ε, U becomes a counital, coassociative coalgebra:

   a) For the first diagram in 1.1.4, we calculate

$$\begin{aligned} ((\Delta \otimes id) \circ \Delta)(K^{\pm 1}) &= (K^{\pm 1} \otimes K^{\pm 1}) \otimes K^{\pm 1} \\ &= K^{\pm 1} \otimes (K^{\pm 1} \otimes K^{\pm 1}) \\ &= ((id \otimes \Delta) \circ \Delta)(K^{\pm 1}), \end{aligned}$$

$$\begin{aligned} ((\Delta \otimes id) \circ \Delta)(E) &= (\Delta \otimes id)(1 \otimes E + E \otimes K^{-1}) \\ &= 1 \otimes 1 \otimes E + (1 \otimes E + E \otimes K^{-1}) \otimes K^{-1} \\ &= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K^{-1} + E \otimes K^{-1} \otimes K^{-1} \\ &= 1 \otimes (1 \otimes E + E \otimes K^{-1}) + E \otimes K^{-1} \otimes K^{-1} \\ &= (id \otimes \Delta)(1 \otimes E + E \otimes K^{-1}) \\ &= ((id \otimes \Delta) \circ \Delta)(E), \end{aligned}$$

$$\begin{aligned}((\Delta \otimes id) \circ \Delta)(F) &= (\Delta \otimes id)(K \otimes F + F \otimes 1)\\ &= K \otimes K \otimes F + (K \otimes F + F \otimes 1) \otimes 1\\ &= K \otimes K \otimes F + K \otimes F \otimes 1 + F \otimes 1 \otimes 1\\ &= K \otimes (K \otimes F + F \otimes 1) + F \otimes 1 \otimes 1\\ &= (id \otimes \Delta)(K \otimes F + F \otimes 1)\\ &= ((id \otimes \Delta) \circ \Delta)(F).\end{aligned}$$

b) For the other two diagrams of 1.1.4 we get

$$\begin{aligned}((1 \otimes \varepsilon) \circ \Delta)(K^{\pm 1}) &= (1 \otimes \varepsilon)(K^{\pm 1} \otimes K^{\pm 1})\\ &= K^{\pm 1} \otimes 1 \cong K^{\pm 1},\end{aligned}$$

$$\begin{aligned}((1 \otimes \varepsilon) \circ \Delta)(E) &= (1 \otimes \varepsilon)(1 \otimes E + E \otimes K^{-1})\\ &= 1 \otimes 0 + E \otimes 1\\ &= E \otimes 1 \cong E,\end{aligned}$$

$$\begin{aligned}((1 \otimes \varepsilon) \circ \Delta)(F) &= (1 \otimes \varepsilon)(K \otimes F + F \otimes 1)\\ &= K \otimes 0 + F \otimes 1\\ &= F \otimes 1 \cong F,\end{aligned}$$

and respectively

$$\begin{aligned}((\varepsilon \otimes 1) \circ \Delta)(K^{\pm 1}) &= (\varepsilon \otimes 1)(K^{\pm 1} \otimes K^{\pm 1})\\ &= K^{\pm 1} \otimes 1 \cong K^{\pm 1},\end{aligned}$$

$$\begin{aligned}((\varepsilon \otimes 1) \circ \Delta)(E) &= (\varepsilon \otimes 1)(1 \otimes E + E \otimes K^{-1})\\ &= 1 \otimes E + 0 \otimes K^{-1}\\ &= 1 \otimes E \cong E,\end{aligned}$$

$$\begin{aligned}((\varepsilon \otimes 1) \circ \Delta)(F) &= (\varepsilon \otimes 1)(K \otimes F + F \otimes 1)\\ &= 1 \otimes F + 0 \otimes 1\\ &= 1 \otimes F \cong F.\end{aligned}$$

4. U is a bialgebra:

a) To the Hopf axiom of 1.1.5:

$$\begin{aligned}&((\cdot \otimes \cdot) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes \Delta))(K^{\pm 1} \otimes K^{\pm 1})\\ &= ((\cdot \otimes \cdot) \circ (1 \otimes \tau \otimes 1))(K^{\pm 1} \otimes K^{\pm 1} \otimes K^{\pm 1} \otimes K^{\pm 1})\\ &= (\cdot \otimes \cdot)(K^{\pm 1} \otimes K^{\pm 1} \otimes K^{\pm 1} \otimes K^{\pm 1})\\ &= K^{\pm 2} \otimes K^{\pm 2}\\ &= \Delta(K^{\pm 2})\\ &= (\cdot \circ \Delta)(K^{\pm 1} \otimes K^{\pm 1}),\end{aligned}$$

$$\begin{aligned}
&((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1)\circ(\Delta\otimes\Delta))(E\otimes E)\\
&\quad=((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1))((1\otimes E+E\otimes K^{-1})\otimes(1\otimes E+E\otimes K^{-1}))\\
&\quad=((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1))\\
&\qquad(1\otimes E\otimes 1\otimes E+1\otimes E\otimes E\otimes K^{-1}\\
&\qquad\qquad+E\otimes K^{-1}\otimes 1\otimes E+E\otimes K^{-1}\otimes E\otimes K^{-1})\\
&\quad=(\cdot\otimes\cdot)(1\otimes 1\otimes E\otimes E+1\otimes E\otimes E\otimes K^{-1}\\
&\qquad\qquad+E\otimes 1\otimes K^{-1}\otimes E+E\otimes E\otimes K^{-1}\otimes K^{-1})\\
&\quad=1\otimes E^2+E\otimes EK^{-1}+E\otimes K^{-1}E+E^2\otimes K^{-2}\\
&\quad=1\otimes E^2+(q^2+1)E\otimes EK^{-1}+E^2\otimes K^{-2}\\
&\quad=1\otimes E^2+q\begin{bmatrix}2\\1\end{bmatrix}E\otimes EK^{-1}+E^2\otimes K^{-2}\\
&\quad=\sum_{i=0}^{2}q^{i(2-i)}\begin{bmatrix}2\\i\end{bmatrix}E^i\otimes E^{2-i}K^{-i}\\
&\quad=\Delta(E^2)\\
&\quad=(\cdot\circ\Delta)(E\otimes E),
\end{aligned}$$

$$\begin{aligned}
&((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1)\circ(\Delta\otimes\Delta))(F\otimes F)\\
&\quad=((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1))((K\otimes F+F\otimes 1)\otimes(K\otimes F+F\otimes 1))\\
&\quad=((\cdot\otimes\cdot)\circ(1\otimes\tau\otimes 1))(K\otimes F\otimes K\otimes F+K\otimes F\otimes F\otimes 1\\
&\qquad\qquad+F\otimes 1\otimes K\otimes F+F\otimes 1\otimes F\otimes 1)\\
&\quad=(\cdot\otimes\cdot)(K\otimes K\otimes F\otimes F+K\otimes F\otimes F\otimes 1\\
&\qquad\qquad+F\otimes K\otimes 1\otimes F+F\otimes F\otimes 1\otimes 1)\\
&\quad=K^2\otimes F^2+KF\otimes F+FK\otimes F+F^2\otimes 1\\
&\quad=K^2\otimes F^2+(q^{-2}+1)FK\otimes F+F^2\otimes 1\\
&\quad=K^2\otimes F^2+q^{-1}\begin{bmatrix}2\\1\end{bmatrix}FK\otimes F+F^2\otimes 1\\
&\quad=\sum_{i=0}^{2}q^{-i(2-i)}\begin{bmatrix}2\\i\end{bmatrix}F^{2-i}K^i\otimes F^i\\
&\quad=\Delta(F^2)\\
&\quad=(\cdot\circ\Delta)(F\otimes F).
\end{aligned}$$

In the last two lines, Lemma 1.1.10 was used. The mixed terms $E\otimes F$ etc. are proved similarly.

b) Regarding the other diagrams:
The commutativity of the second diagram follows directly via the following calculation:

$$(\Delta\circ\eta)(\underbrace{1}_{\in k})=\Delta(\underbrace{1}_{\in U})=1\otimes 1=(\eta\otimes\eta)(1\otimes 1)=((\eta\otimes\eta)\circ\varphi)(1),$$

where $\varphi(1)=1\otimes 1$ is the natural isomorphism between k and $k\otimes k$.

It is left to show that the penultimate diagram in 1.1.5 commutes (the final diagram is obvious).

$$(\varepsilon \circ \cdot)(E \otimes E) = \varepsilon(E^2) = 0 = \varphi(0 \otimes 0) = (\varphi \circ (\varepsilon \otimes \varepsilon))(E \otimes E).$$
$$(\varepsilon \circ \cdot)(F \otimes F) = \varepsilon(F^2) = 0 = \varphi(0 \otimes 0) = (\varphi \circ (\varepsilon \otimes \varepsilon))(F \otimes F).$$
$$(\varepsilon \circ \cdot)(K^{\pm 1} \otimes K^{\pm 1}) = \varepsilon(K^{\pm 2}) = 1 = \varphi(1 \otimes 1) = (\varphi \circ (\varepsilon \otimes \varepsilon))(K^{\pm 1} \otimes K^{\pm 1}).$$

Mixed terms like $E \otimes F$ etc. work similarly.

5. Finally, the antipode in (1.10)-(1.12) turns U into a Hopf algebra: For that we need to show that the diagram in 1.1.7 commutes.

$$(\cdot \circ (1 \otimes S) \circ \Delta)(K^{\pm 1}) = K^{\pm 1}K^{\mp 1} = 1 = \varepsilon(1) = (\varepsilon \circ \eta)(K^{\pm 1}),$$
$$(\cdot \circ (S \otimes 1) \circ \Delta)(K^{\pm 1}) = K^{\mp 1}K^{\pm 1} = 1 = \varepsilon(1) = (\varepsilon \circ \eta)(K^{\pm 1}),$$

$$\begin{aligned}(\cdot \circ (1 \otimes S) \circ \Delta)(E) &= (\cdot \circ (1 \otimes S))(1 \otimes E + E \otimes K^{-1})\\ &= -EK + EK = 0 = \varepsilon(0)(\varepsilon \circ \eta)(E),\\ (\cdot \circ (S \otimes 1) \circ \Delta)(E) &= (\cdot \circ (S \otimes 1))(1 \otimes E + E \otimes K^{-1})\\ &= E + (-EK)K^{-1} = 0 = \varepsilon(0)(\varepsilon \circ \eta)(E),\end{aligned}$$

$$\begin{aligned}(\cdot \circ (1 \otimes S) \circ \Delta)(F) &= (\cdot \circ (1 \otimes S))(K \otimes F + F \otimes 1)\\ &= K(-K^{-1}F) + F = 0 = \varepsilon(0)(\varepsilon \circ \eta)(F),\\ (\cdot \circ (S \otimes 1) \circ \Delta)(F) &= (\cdot \circ (S \otimes 1))(K \otimes F + F \otimes 1)\\ &= K^{-1}F + (-K^{-1}F) = 0 = \varepsilon(0)(\varepsilon \circ \eta)(F).\end{aligned}$$

□

1.2 Tensors of $U_q(\mathfrak{sl}_2)$-modules

The calculations of the previous section have a great impact for our purposes since the Hopf algebra structure of U enables an adequate definition of an action on tensor products of U-modules:

1.2.1 Definition. *Let $u \in U$ und V, W be two U-modules with $v \in V$, $w \in W$. We define the* action of U *on $V \otimes W$ by*

$$u.(v \otimes w) := \Delta(u).(v \otimes w). \tag{1.16}$$

Explicitly, we get the following formulae:

$$K^{\pm 1}.(v \otimes w) := \Delta(K^{\pm 1}).(v \otimes w) = K^{\pm 1}.v \otimes K^{\pm 1}.w, \tag{1.17}$$

$$E.(v \otimes w) := \Delta(E).(v \otimes w) = v \otimes E.w + E.v \otimes K^{-1}.w, \tag{1.18}$$

$$F.(v \otimes w) := \Delta(F).(v \otimes w) = K.v \otimes F.w + F.v \otimes w. \tag{1.19}$$

By using this coproduct, tensors of U-modules are not just $U^{\otimes n}$-modules but in fact U-modules. In the next chapter, this will be important as we will identify framed tangles with maps between tensors of U-modules.

1.2.2 Remark. *The coproduct shows a very characteristic feature of the quantum group in contrast to the ordinary universal enveloping algebra. For the latter, the coproduct is cocommutative and explicitly given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for any $x \in \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra. Looking at the defining formulae* (1.5)-(1.12)*, we see that the comultiplication for U is not cocommutative. This asymmetry has also consequences for the representation theory. In the case of the classical $U(\mathfrak{g})$ and two finite dimensional representations V, W we always have the canonical map $\tau : V \otimes W \cong W \otimes V$, $v \otimes w \mapsto w \otimes v$ which just flips the factors. In case of U, the noncocommutativity implies that this flipping map in general is not a homomorphism of modules. This failure will turn out to be very important in the study of knots, links and tangles. We will replace the map given above by a more complicated map which is not an involution and which then makes it possible to distinguish between over- and undercrossings in braids, links and tangles.*

2 Morphisms of tensors of $U_q(\mathfrak{sl}_2)$-modules as a presentation of oriented knots

The purpose of this chapter is to describe a formalisation of coloured knots, namely the coloured Jones polynomial. By choosing an appropriate projection of the tangle to the plane we can interpret the result as an intertwiner of tensor products of U-modules by labelling each strand of the tangle with an irreducible U-module of dimension $n+1$. The labelling should be imagined as a colouring of the strands by a natural number n, which introduces more complexity to the theory. The setup of the original Jones polynomial corresponds with the subcase where all strands are labelled by the same representation, namely the standard irreducible representation of dimension 2.

The following section aims to introduce this formalisation of coloured knots in detail; after that, we like to calculate the value of the coloured unknot which is given by the polynomial defined below.

2.1 Coloured knots – notations and definitions

2.1.1 Definition ([SS11]). *Let V_n be the unique (up to isomorphism) $(n+1)$-dimensional, irreducible U-module of type 1 with basis $v_0, ..., v_n$, where E, F and K act via*

$$\begin{aligned} K^{\pm 1}.v_i &= q^{\pm(2i-n)}v_i, \\ E.v_i &= [i+1]v_{i+1} \qquad \text{and} \\ F.v_i &= [n-i+1]v_{i-1}. \end{aligned}$$

2.1.2 Definition ([SS11]). *Define the following linear maps presenting the cup and the cap (which are easily verified as $U_q(\mathfrak{sl}_2)$-homomorphisms) by*

$$\begin{aligned} \cup : \mathbb{C}(q) \to V_1 \otimes V_1 \text{ with } & \cup(1) = v_1 \otimes v_0 - q v_0 \otimes v_1 \\ \cap : V_1 \otimes V_1 \to \mathbb{C}(q) \text{ with } & \cap(v_0 \otimes v_0) = 0 = \cap(v_1 \otimes v_1), \\ & \cap(v_0 \otimes v_1) = 1, \\ & \cap(v_1 \otimes v_0) = -q^{-1}, \end{aligned}$$

and let the overgoing and the undergoing crossings be defined via

$$\begin{aligned} \Pi : V_1 \otimes V_1 \to V_1 \otimes V_1 \quad \text{with} \quad & \Pi(v_i \otimes v_j) = -q^{-1}C(v_i \otimes v_j) - q^{-2} v_i \otimes v_j \quad \text{and} \\ \Omega : V_1 \otimes V_1 \to V_1 \otimes V_1 \quad \text{with} \quad & \Omega(v_i \otimes v_j) = -qC(v_i \otimes v_j) - q^2 v_i \otimes v_j, \end{aligned}$$

where $C = \cup \circ \cap$ and $i, j \in \{0, 1\}$.

2.1.3 Definition ([FSS12]). *Let $a = (a_1, ..., a_n)$, $a_i \in \{0, 1\}$, be an n-tuple containing i zeros and $n - i$ ones. The symmetric group S_n acts transitively on a and $S_i \times S_{n-i}$ stabilises $a_{dom} := (\underbrace{1, ..., 1}_{n-i}, \underbrace{0, ..., 0}_{i})$. Now we can identify each tuple a like above with its corresponding coset representative of minimal length in $S_n/(S_i \times S_{n-i})$.*
Having this in mind, we can define the Coxeter length $l(a)$ by taking the Coxeter length of the corresponding element in $S_n/(S_i \times S_{n-i})$. Furthermore, let $|a| := \#\{\text{ones in } a\}$. Finally, by $a \cup b$ we mean the concatenation $(a_1, ..., a_n, b_1, ..., b_m)$ of two tuples $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_m)$.

2.1.4 Definition ([FSS12],[SS11]). *Let $\pi_n : V_1^{\otimes n} \to V_n$ be given by*

$$\pi_n(v_a) = q^{-l(a)} \begin{bmatrix} n \\ |a| \end{bmatrix}^{-1} v_{|a|},$$

where a is an n-tuple as defined above. Vice versa, define $\iota_n : V_n \to V_1^{\otimes n}$ by

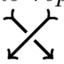
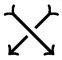
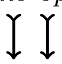
$$\iota_n(v_k) = \sum_{|a|=k} q^{|a|(n-|a|)-l(a)} v_a$$

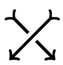
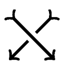
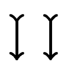
with $k \in \{0, ..., n\}$.
The composition $p_n = \iota_n \circ \pi_n$ is called Jones-Wenzl projector *or* JW-projector.

2.1.5 Remark. *The JW-projector is indeed a projector, i.e. it satisfies $p_n^2 = p_n$. It is uniquely characterised by being a projector and the fact that $\cap \circ p_n = 0$ and $p_n \circ \cup = 0$ for any cup or cap involving the same strands as the projector[1]. We will observe this in particular in 2.2.8.*

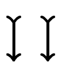
2.1.6 Proposition. *The formulae in 2.1.2 and 2.1.4 give an invariant of oriented[2], coloured knots, the so-called* coloured Jones polynomial.

We omit the proof of the previous proposition at this point. In chapters 3-5, we introduce three two-parameter versions of the Jones polynomial and the proof of 2.1.6 will be a special case of propositions 3.1.9 and 4.2.7.

2.1.7 Remark. *Every knot polynomial is uniquely linked to a certain relation, the so-called skein relation. One can ask how the value of a knot changes if one crossing in it gets replaced by its opposite or by the identity; such a triple of knots only differing by* ( ,  , ) *is called a Conway triple. The skein relation asserts the relation between the three elements of a conway triple by*

a  $+ b$  $= c$  ,

where a, b and c are elements of the ground field. In case of the Jones polynomial, the skein relation

q^{-2}  $- q^2$  $= (q - q^{-1})$ 

[1] See for example [FK97, Corollary 1.4] and [FSS12, Proposition 5].
[2] The orientation is realised by rescaling the polynomial adequately. A more detailled description is found in the following chapters.

holds[3].

The final definition of this section gives a map which flips a given sequence in $\{0,1\}^n$ and changes zeros and ones mutually. Because of the flipping we will call it *mirror* although it is technically something else.

2.1.8 Definition. *Let* $a = (a_1, ..., a_n)$ *be an n-tuple as above. Then we define the* mirror *of* a *as* $\mathrm{mir}(a) = (\tilde{a}_n, ..., \tilde{a}_1)$*, where* $\tilde{0} = 1$ *and* $\tilde{1} = 0$*. This means that* mir *flips the sequence* $(a_1, ..., a_n)$ *and swaps zeros and ones.*

2.1.9 Lemma. *Let* $a = (a_1, ..., a_n) \in \{0,1\}^n$ *and* $b = (b_1, ..., b_m) \in \{0,1\}^m$*. The map* mir *has the following properties:*

(i) $\mathrm{mir}(a \cup b) = \mathrm{mir}(b) \cup \mathrm{mir}(a)$,

(ii) $l(\mathrm{mir}(a)) = l(a)$.

Proof. On (i):

$$\begin{aligned}\mathrm{mir}(a \cup b) &= \mathrm{mir}((a_1, ..., a_n, b_1, ..., b_m)) \\ &= (\tilde{b}_m, ..., \tilde{b}_1, \tilde{a}_n, ..., \tilde{a}_1) \\ &= \mathrm{mir}(b) \cup \mathrm{mir}(a). \end{aligned} \tag{2.1}$$

On (ii): We proceed by induction on n:
base step: $a = (0)$ or $a = (1) \Rightarrow \mathrm{mir}(a) = (0)$ and $\mathrm{mir}(a) = (1)$ respectively. In each case, $l(a) = l(\mathrm{mir}(a)) = 0$.
$n \to n+1$: If we lengthen a by one digit, we get either $a \cup 0$ or $a \cup 1$. In the first case, $l(a)$ does not change, neither does $l(\mathrm{mir}(a))$ because applying *mir* changes the 0 to 1 which sits at the beginning of $\mathrm{mir}(a)$. In the second case, we get $l(a \cup 1) = l(a) + (n - |a|)$ because $n - |a|$ zeros are left from the newly added 1. Vice versa, we get $l(\mathrm{mir}(a \cup 1)) = l(0 \cup \mathrm{mir}(a)) = (n - |a|) + l(\mathrm{mir}(a)) = (n - |a|) + l(a) = l(a \cup 1)$; the first equation holds because of (i) and the penultimate uses the induction hypothesis. □

2.2 The coloured unknot

In this section, we consider the coloured unknot as the simplest example of coloured knots. It consists of n nested cups, two Jones-Wenzl projectors and n nested caps.

2.2.1 Theorem. *With the definitions in 2.1, the trivial knot coloured by* n *has the value* $(-1)^n[n+1]$.

We split the proof into several lemmata.

2.2.2 Lemma (1)**.** *Let* δ'_n *be the map nesting* n *cups* $\cup$ *one into another like*  *via* $\delta'_n = (id^{\otimes(n-1)} \otimes \cup \otimes id^{\otimes(n-1)}) \circ ... \circ (id \otimes \cup \otimes id) \circ \cup$*. Then*

$$\delta'_n(1) = \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a \in \{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)} - q v_a \otimes v_{01} \otimes v_{mir(a)}.$$

[3]See for example [KT08, p. 174]. Here the Jones polynomial is alternatively defined by replacing q by $q^{\frac{1}{2}}$ in the formulae of this section.

Proof. We show the first lemma by induction on n.
In the base step, we look at $n = 1$:

$$\begin{aligned}\delta'_1(1) = \cup(1) &= v_{10} - qv_{01}\\ &= \sum_{i=0}^{1-1=0} (-q)^i \sum_{\substack{|a|=1-1-i\\ a\in\{0,1\}^0}} v_a \otimes v_{10} \otimes v_{mir(a)} - qv_a \otimes v_{01} \otimes v_{mir(a)}.\end{aligned}$$

Now we show the induction step $n \to n+1$. Write down $\delta'_{n+1}(1)$ explicitly:

$$\begin{aligned}\delta'_{n+1}(1) &= (id^{\otimes n} \otimes \cup \otimes id^{\otimes n}) \circ \delta'_n(1)\\ &= (id^{\otimes n} \otimes \cup \otimes id^{\otimes n})\big(\sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)}\\ &\qquad -qv_a \otimes v_{01} \otimes v_{mir(a)}\big)\\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} v_a \otimes v_1 \otimes (v_{10} - qv_{01}) \otimes v_0 \otimes v_{mir(a)}\\ &\qquad -qv_a \otimes v_0 \otimes (v_{10} - qv_{01}) \otimes v_1 \otimes v_{mir(a)}\\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} (v_a \otimes v_{1100} \otimes v_{mir(a)} - qv_a \otimes v_{1010} \otimes v_{mir(a)})\\ &\qquad -q(v_a \otimes v_{0101} \otimes v_{mir(a)} - qv_a \otimes v_{0011} \otimes v_{mir(a)}).\end{aligned}$$

We split the last sum into two and get

$$\begin{aligned}\delta'_1(1) &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} (v_a \otimes v_{1100} \otimes v_{mir(a)} - qv_a \otimes v_{1010} \otimes v_{mir(a)})\\ &\qquad + \sum_{i=0}^{n-1}(-q)^{i+1} \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} (v_a \otimes v_{0101} \otimes v_{mir(a)} - qv_a \otimes v_{0011} \otimes v_{mir(a)})\\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} (v_{a\cup 1} \otimes v_{10} \otimes v_{mir(a\cup 1)} - qv_{a\cup 1} \otimes v_{01} \otimes v_{mir(a\cup 1)})\\ &\qquad + \sum_{i=0}^{n-1}(-q)^{i+1} \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} (v_{a\cup 0} \otimes v_{10} \otimes v_{mir(a\cup 0)} - qv_{a\cup 0} \otimes v_{01} \otimes v_{mir(a\cup 0)}).\end{aligned}$$

An index shift in the last term yields

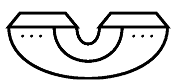
$$\begin{aligned}\delta_1'(1) = &\sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \big(v_{a\cup 1}\otimes v_{10}\otimes v_{mir(a\cup 1)} - q v_{a\cup 1}\otimes v_{01}\otimes v_{mir(a\cup 1)}\big)\\ &+\sum_{i=1}^{n}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \big(v_{a\cup 0}\otimes v_{10}\otimes v_{mir(a\cup 0)} - q v_{a\cup 0}\otimes v_{01}\otimes v_{mir(a\cup 0)}\big)\\ =&\sum_{i=0}^{n}(-q)^i \sum_{\substack{|a|=n-i\\ a\in\{0,1\}^{n}}} \big(v_{a}\otimes v_{10}\otimes v_{mir(a)} - q v_{a}\otimes v_{01}\otimes v_{mir(a)}\big).\end{aligned}$$

The last equation holds since the two big summands in the penultimate step are precisely the two possibilities extending a to an array of length n, namely by 0 and by 1. □

2.2.3 Lemma (2)**.** *Let δ_n' be as before. Then*

$$\begin{aligned}\delta_n := &(\pi_n\otimes\pi_n)\circ\delta_n'(1)\\ =&\sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2(l(a)+i)}\frac{1}{{\begin{bmatrix}n\\ i\end{bmatrix}}^2}\ v_{n-i}\otimes v_i\\ &\qquad -q^{-2l(a)+1}\frac{1}{{\begin{bmatrix}n\\ i+1\end{bmatrix}}^2}\ v_{n-(i+1)}\otimes v_{i+1}.\end{aligned}$$

δ_n would be pictured like follows (the trapezium stands for the projection π_n):



Proof.

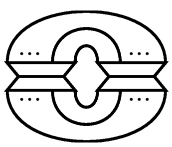
$$\begin{aligned}
\delta_n &:= (\pi_n \otimes \pi_n) \circ \delta_n'(1) \\
&= (\pi_n \otimes \pi_n)\Bigg(\sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)} - q v_a \otimes v_{01} \otimes v_{mir(a)} \Bigg) \\
&= \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-l(a\cup 1)} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \; v_{n-i} \otimes q^{-l(0\cup mir(a))} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \; v_i \\
&\quad -q\Bigg(q^{-l(mir(a))} \frac{1}{\begin{bmatrix} n \\ n-(i+1) \end{bmatrix}} \; v_{n-(i+1)} \otimes q^{-l(mir(a))} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} \; v_{n-(i+1)} \Bigg) \\
&= \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \; v_{n-i} \otimes q^{-(l(a)+i)} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \; v_i \\
&\quad -q\Bigg(q^{-l(a)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} \; v_{n-(i+1)} \otimes q^{-l(a)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} \; v_{n-(i+1)} \Bigg) \\
&= \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}^2} \; v_{n-i} \otimes v_i \\
&\quad -q^{-2l(a)+1} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}^2} \; v_{n-(i+1)} \otimes v_{n-(i+1)} \Bigg).
\end{aligned}$$

□

2.2.4 Lemma (3)**.** *Let ε_n' the map nesting n caps $\cap$ one into another like*  *via $\varepsilon_n' = \cap \circ (id \otimes \cap \otimes id) \circ (id^{\otimes n-1} \otimes \cap \otimes id^{\otimes n-1})$.*
Then $\varepsilon_n := \varepsilon_n' \circ (\iota_n \otimes \iota_n)$ and for $\varepsilon_n \circ \delta_n(1)$ we get the formula

$$\begin{aligned}
\varepsilon_n \circ \delta_n(1) = (-1)^n \sum_{i=0}^{n-1} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} &\Bigg(q^{2(i(n-i)-l(a)-i)-n+2i} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}^2} \sum_{|b|=n-i} q^{-2l(b)} \\
&+ q^{2((i+1)(n-(i+1))-l(a))-n+2(i+1)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}^2} \sum_{|b|=n-(i+1)} q^{-2l(b)} \Bigg).
\end{aligned}$$

This lemma already leads to the value of the unknot coloured by n, but in a quite lengthy form. Indeed we evaluated the picture



step by step; the trapezia pointing up with the smaller side represent the projections π_n, those pointing down with their smaller side stand for the inclusions ι_n. In particular, the picture describes the unknot involving n strands and the trapezia indicate the colouring which can be imagined as the strands being intertwined with each other. As we will see later, it suffices to apply only one Jones-Wenzl projector.

Proof of Lemma 2.2.4. First, we evaluate $(\iota_n \otimes \iota_n) \circ \delta_n(1)$.

$$\begin{aligned}
&(\iota_n \otimes \iota_n) \circ \delta_n(1) \\
&= (\iota_n \otimes \iota_n)\Bigg(\sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)} \frac{1}{{\begin{bmatrix} n \\ i \end{bmatrix}}^2} v_{n-i} \otimes v_i \\
&\quad -q^{-2l(a)+1} \frac{1}{{\begin{bmatrix} n \\ i+1 \end{bmatrix}}^2} v_{n-(i+1)} \otimes v_{i+1}\Bigg)\Bigg) \\
&= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)} \frac{1}{{\begin{bmatrix} n \\ i \end{bmatrix}}^2} \Big(\sum_{|b|=n-i} q^{i(n-i)-l(b)} v_b\Big) \otimes \Big(\sum_{|b'|=i} q^{i(n-i)-l(b')} v_{b'}\Big) \\
&\quad -q^{-2l(a)+1} \frac{1}{{\begin{bmatrix} n \\ i+1 \end{bmatrix}}^2} \Big(\sum_{|b|=n-(i+1)} q^{(i+1)(n-(i+1))-l(b)} v_b\Big) \otimes \Big(\sum_{|b'|=i+1} q^{(i+1)(n-(i+1))-l(b')} v_{b'}\Big)\Bigg) \\
&= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)} \frac{1}{{\begin{bmatrix} n \\ i \end{bmatrix}}^2} \Big(\sum_{|b|=n-i} q^{-l(b)} v_b\Big) \otimes \Big(\sum_{|b'|=i} q^{-l(b')} v_{b'}\Big) \\
&\quad -q^{-2l(a)+1+2(i+1)(n-(i+1))} \frac{1}{{\begin{bmatrix} n \\ i+1 \end{bmatrix}}^2} \Big(\sum_{|b|=n-(i+1)} q^{-l(b)} v_b\Big) \otimes \Big(\sum_{|b'|=i+1} q^{-l(b')} v_{b'}\Big)\Bigg) \\
&= (\star).
\end{aligned}$$

Applying ε'_n to the latter formula leads to

$$\begin{aligned}
\varepsilon'_n(\star) &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)} \frac{1}{{\begin{bmatrix} n \\ i \end{bmatrix}}^2} \sum_{|b|=n-i} q^{-l(b)} (-q^{-1})^{|b|} \\
&\quad -q^{-2l(a)+1+2(i+1)(n-(i+1))} \frac{1}{{\begin{bmatrix} n \\ i+1 \end{bmatrix}}^2} \sum_{|b|=n-(i+1)} q^{-l(b)} (-q^{-1})^{|b|}\Bigg) \\
&= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)} \frac{1}{{\begin{bmatrix} n \\ i \end{bmatrix}}^2} \Big(\sum_{|b|=n-i} q^{-l(b)} (-q^{-1})^{n-i}\Big) \\
&\quad -q^{-2l(a)+1+2(i+1)(n-(i+1))} \frac{1}{{\begin{bmatrix} n \\ i+1 \end{bmatrix}}^2} \sum_{|b|=n-(i+1)} q^{-l(b)} (-q^{-1})^{n-(i+1)}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)-(n-i)} \frac{1}{{\begin{bmatrix} n\\ i\end{bmatrix}}^2}(-1)^{n-i} \sum_{|b|=n-i} q^{-l(b)} \\
&\quad - q^{-2l(a)+1+2(i+1)(n-(i+1))-(n-(i+1))} \frac{1}{{\begin{bmatrix} n\\ i+1\end{bmatrix}}^2}(-1)^{n-(i+1)} \sum_{|b|=n-(i+1)} q^{-l(b)} \Bigg) \\
&= \sum_{i=0}^{n-1}(-1)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)-(n-i)+i} \frac{1}{{\begin{bmatrix} n\\ i\end{bmatrix}}^2}(-1)^{n-i} \sum_{|b|=n-i} q^{-l(b)} \\
&\quad + q^{-2l(a)+1+2(i+1)(n-(i+1))-(n-(i+1))+i} \frac{1}{{\begin{bmatrix} n\\ i+1\end{bmatrix}}^2}(-1)^{n-i} \sum_{|b|=n-(i+1)} q^{-l(b)} \Bigg) \\
&= (-1)^n \sum_{i=0}^{n-1} \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \Bigg(q^{-2(l(a)+i)+2i(n-i)-n+2i} \frac{1}{{\begin{bmatrix} n\\ i\end{bmatrix}}^2} \sum_{|b|=n-i} q^{-l(b)} \\
&\quad + q^{-2l(a)+2(i+1)(n-(i+1))-n+2(i+1)} \frac{1}{{\begin{bmatrix} n\\ i+1\end{bmatrix}}^2} \sum_{|b|=n-(i+1)} q^{-l(b)} \Bigg).
\end{aligned}
$$

Hence the asserted formula follows. □

To conclude the proof of theorem 2.2.1, we need one last proposition which is proven completely analogously to Theorem 1.2 in [Hag08].

2.2.5 Proposition.

$$
q^{i(n-i)} \sum_{|b|=n-i} q^{-2l(b)} = \begin{bmatrix} n\\ i\end{bmatrix}. \tag{2.2}
$$

2.2.6 Proposition (4)**.** *Based on the result of 2.2.4 we actually get* $\varepsilon_n \circ \delta_n(1) = [n+1]$.

Proof. We start with the result of 2.2.4.

$$
\begin{aligned}
\varepsilon_n \circ \delta_n(1) &= (-1)^n \sum_{i=0}^{n-1} \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \Bigg(q^{2(i(n-i)-l(a)-i)-n+2i} \frac{1}{{\begin{bmatrix} n\\ i\end{bmatrix}}^2} \sum_{|b|=n-i} q^{-2l(b)} \\
&\quad + q^{2((i+1)(n-(i+1))-l(a))-n+2(i+1)} \frac{1}{{\begin{bmatrix} n\\ i+1\end{bmatrix}}^2} \sum_{|b|=n-(i+1)} q^{-2l(b)} \Bigg) \\
&= (-1)^n \sum_{i=0}^{n-1} \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} \Bigg(q^{2i(n-i)-2l(a)-\cancel{2i}-n+\cancel{2i}} \frac{1}{{\begin{bmatrix} n\\ i\end{bmatrix}}^2} \sum_{|b|=n-i} q^{-2l(b)} \\
&\quad + q^{2(i+1)(n-(i+1))-2l(a)-n+2(i+1)} \frac{1}{{\begin{bmatrix} n\\ i+1\end{bmatrix}}^2} \sum_{|b|=n-(i+1)} q^{-2l(b)} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^n \sum_{i=0}^{n-1} q^{2i(n-i)-n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}^2} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \sum_{|b|=n-i} q^{-2l(b)} \\
&\quad + q^{2(i+1)(n-(i+1))-n+2(i+1)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}^2} \sum_{\substack{|a|=n-(i+1) \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \sum_{|b|=n-(i+1)} q^{-2l(b)}.
\end{aligned}
$$

Now we apply Proposition 2.2.5 and obtain

$$
\begin{aligned}
\varepsilon_n \circ \delta_n(1) \quad &= \quad (-1)^n \Bigg(\sum_{i=0}^{n-1} q^{i(n-i)-n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \\
&\qquad + \sum_{i=0}^{n-1} q^{(i+1)(n-(i+1))-n+2(i+1)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} \sum_{\substack{|a|=n-(i+1) \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \Bigg); \\
&\overset{\text{index shift}}{=} (-1)^n \Bigg(\sum_{i=0}^{n-1} q^{i(n-i)-n-i+i} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \\
&\qquad + \sum_{i=1}^{n} q^{i(n-i)-n+2i} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \sum_{\substack{|a|=(n-i)=(n-1)-(i-1) \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \Bigg) \\
&= \quad (-1)^n \Bigg(\sum_{i=0}^{n-1} q^{-n+i} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \underbrace{q^{i(n-1-i)} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}}_{=\begin{bmatrix} n-1 \\ i \end{bmatrix}} \\
&\qquad + \sum_{i=1}^{n} q^{i} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \underbrace{q^{i(n-i)-n+i} \sum_{\substack{|a|=(n-i)=(n-1)-(i-1) \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}}_{=\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^n \Bigg(\sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{-n+i} \begin{bmatrix} n-1 \\ i \end{bmatrix} + \sum_{i=1}^{n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \Bigg) \\
&\overset{(\maltese)}{=} (-1)^n \Bigg(q^{-n} + \sum_{i=1}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} \underbrace{(q^{-n+i} \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix})}_{=q^{-n+2i}\begin{bmatrix} n \\ i \end{bmatrix}} + q^{n} \Bigg) \\
&= (-1)^n \sum_{i=0}^{n} q^{-n+2i} \quad = \quad (-1)^n [n+1].
\end{aligned}
$$

To obtain (✠), we use lemma 1.1.11:

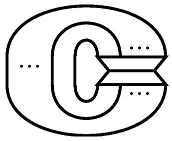
$$\begin{bmatrix} n \\ i \end{bmatrix} = q^{-i}\begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}.$$

Multiplied by q^{-n+2i}, this is equivalent to

$$\begin{aligned} q^{-n+2i}\begin{bmatrix} n \\ i \end{bmatrix} &= q^{-n+2i-i}\begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{-n+2i+n-i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \\ &= q^{-n+i}\begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{i}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}, \end{aligned} \tag{2.3}$$

which is just the equation we needed. □

The following proposition tells us that it suffices to apply the Jones-Wenzl projector just once to get the value for the coloured unknot. So we only have to calculate the value of the following diagram.


.

2.2.7 Proposition.

$$\varepsilon_n' \circ (id^{\otimes n} \otimes \iota_n) \circ (id^{\otimes n} \otimes \pi_n) \circ \delta_n'(1) = (-1)^n[n+1].$$

2.2.8 Remark. *Here we see that the Jones-Wenzl projector is indeed an idempotent as mentioned earlier in 2.1.5.*

Proof. We have already seen that

$$\delta_n'(1) = \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)} - qv_a \otimes v_{01} \otimes v_{mir(a)}.$$

Now apply $(id^{\otimes n} \otimes \pi_n)$:

$$\begin{aligned} (id^{\otimes n} \otimes \pi_n) \circ \delta_n'(1) &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_{a\cup 1} \otimes q^{-l(0\cup mir(a))}\begin{bmatrix} n \\ i \end{bmatrix}^{-1} v_i \\ &\quad -q\big(v_{a\cup 0} \otimes q^{-l(mir(a))}\begin{bmatrix} n \\ i+1 \end{bmatrix}^{-1} v_{i+1}\big) \\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)}\begin{bmatrix} n \\ i \end{bmatrix}^{-1} v_{a\cup 1} \otimes v_i \\ &\quad -q^{-l(a)+1}\begin{bmatrix} n \\ i+1 \end{bmatrix}^{-1} v_{a\cup 0} \otimes v_{i+1} \\ &=: \delta_n''. \end{aligned}$$

Next, we compute $(id^{\otimes n}\otimes\pi_n)\circ\delta_n''(1)$:

$$\begin{aligned}(id^{\otimes n}\otimes\pi_n)\circ\delta_n''(1) &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} v_{a\cup 1}\otimes\sum_{|b|=n-i} q^{i(n-i)-l(b)}v_b\\ &\quad -q^{-l(a)+1}\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)} q^{(i+1)(n-(i+1))-l(b)}v_b\\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)+i(n-i)}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} v_{a\cup 1}\otimes\sum_{|b|=n-i} q^{-l(b)}v_b\\ &\quad -q^{-l(a)+1+(i+1)(n-(i+1))}\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)} q^{-l(b)}v_b.\end{aligned}$$

We finish the proof by applying ε_n':

$$\begin{aligned}&\varepsilon_n'(id\otimes\pi_n)\circ\delta_n''(1)\\ &= \sum_{i=0}^{n-1}(-q)^i \sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)+i(n-i)}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} q^{-l(a\cup 1)}(-q^{-1})^{n-i}\\ &\quad -q^{-l(a)+1+(i+1)(n-(i+1))}\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} q^{-l(a\cup 0)}(-q^{-1})^{n-(i+1)}\\ &= (-1)^n\sum_{i=0}^{n-1}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} q^{-l(a)+i(n-i)-(l(a)+i)-(n-i)}\\ &\quad +\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} q^{-l(a)+1+(i+1)(n-(i+1))+i-l(a)-(n-(i+1))}\\ &= (-1)^n\sum_{i=0}^{n-1}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} q^{-2l(a)+i(n-i)-n}\\ &\quad +\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} q^{-2l(a)+(i+1)(n-(i+1))-n+2(i+1)}\\ &= (-1)^n\sum_{i=0}^{n-1}\begin{bmatrix}n\\ i\end{bmatrix}^{-1} q^{i(n-i)-n}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}\\ &\quad +\begin{bmatrix}n\\ i+1\end{bmatrix}^{-1} q^{(i+1)(n-(i+1))-n+2(i+1)}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}.\end{aligned}$$

We split the sum again. By an index shift and Proposition 2.2.5 it follows that $(id^{\otimes n} \otimes \pi_n) \circ \delta''_n(1)$

$$
\begin{aligned}
&= (-1)^n \Bigg(\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} q^{i(n-i)-i+i-n} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \\
&\quad + \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} q^{i(n-i)-n+2i} \sum_{\substack{|a|=n-1-i-1 \\ a\in\{0,1\}^{n-1}}} q^{-2l(a)} \Bigg) \\
&= (-1)^n \Bigg(\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} q^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \Bigg) \\
&= (-1)^n \Bigg(q^{-n} + \sum_{i=1}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} \big(q^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \big) q^{n-2i-n+2i} + q^n \Bigg) \\
&= (-1)^n \sum_{i=0}^{n} q^{-n+2i} \quad = \quad (-1)^n [n+1].
\end{aligned}
$$

□

3 The first two-parameter variation p_1 of the coloured Jones polynomial

The next three chapters are devoted to the introduction of the three two-parameter versions of the formulae from chapter 2. Those variations are new and quite unexpected since it is a priori not visible in the quantum group setup that a second parameter might be reasonable. The motivation lies in the theory around the categorification of the Jones polynomial; for further reading see for example [Str05]. One might think of the two parameters as analogues of a grading where q denotes an internal grading shift, whereas the new parameter t denotes a homological grading shift.

In this chapter, we want to introduce the first way of deforming the formulae of chapter 2 by rescaling them. The morphisms, however, stay $U_q(\mathfrak{sl}_2)$-linear.

3.1 The polynomial

From now on we change the ground field and work over $\mathbb{C}(q,t)$. The irreducible $U_q(\mathfrak{sl}_2)$-modules are defined naturally by extending the scalars.

3.1.1 Definition. *Again we abbreviate $v_{ij} := v_i \otimes v_j$ as before. Define the new cup morphism $\cup_t : \mathbb{C} \to V_1 \otimes V_1$ by*

$$\cup_t(1) = -t(v_{10} - qv_{01}) = -tv_{10} + tqv_{01}. \tag{3.1}$$

Furthermore, let $\cap_t : V_1 \otimes V_1 \to \mathbb{C}(q,t)$ be the $U_q(\mathfrak{sl}_2)$-linear morphism defined by

$$\begin{aligned} \cap_t(v_{10}) &= t^{-1}q^{-1}, \\ \cap_t(v_{01}) &= -t^{-1}, \\ \cap_t(v_{00}) &= 0 \quad = \quad \cap(v_{11}). \end{aligned} \tag{3.2}$$

Moreover, we define the deformation of the crossings via the $U_q(\mathfrak{sl}_2)$-linear maps

$$\Pi_t : V_1 \otimes V_1 \to V_1 \otimes V_1 \tag{3.3}$$
$$\Pi_t(v_{ij}) = (tq^{-1}C + tq^{-2}id)(v_{ij});$$
$$\Omega_t : V_1 \otimes V_1 \to V_1 \otimes V_1 \tag{3.4}$$
$$\Omega_t(v_{ij}) = (t^{-1}qC + t^{-1}q^2 id)(v_{ij}),$$

where $i, j \in 0, 1$.

Note that for $t = -1$, we regain the formulae from chapter 2. In this chapter, we will use the formulae (3.1) - (3.4) without the index t for clarity.

3.1.2 Lemma. *The trivial knot coloured by one has the value* $-[2] = -(q^{-1} + q)$*. Moreover,* $C^2 = -[2]C$*, where*

$C :=$  $= \cup \circ \cap.$

Proof. First, we compute the value of the unknot:

$$(\cap \circ \cup)(1) = \cap(-tv_{10} + tqv_{01}) = -t(tq)^{-1} + tq(-t^{-1}) = -q^{-1} - q.$$

Now we get for C:

$$\begin{aligned}
C(v_{10}) = (tq)^{-1}(-t(v_{10} - qv_{01})) &= -q^{-1}v_{10} + v_{01}, \\
C(v_{01}) = -t^{-1}(-t(v_{10} - qv_{01})) &= v_{10} - qv_{01}, \\
C(v_{00}) = 0 = C(v_{11}). &
\end{aligned}$$

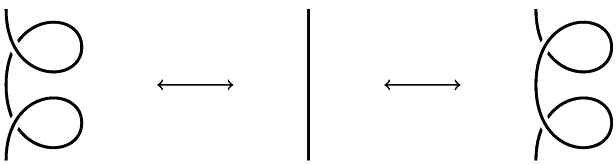
So for C^2, we obtain

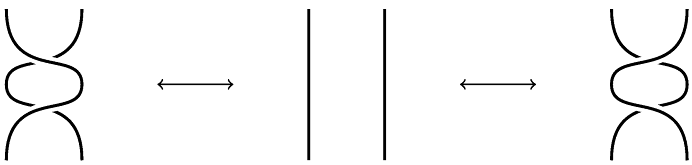
$$\begin{aligned}
C^2(v_{10}) &= C(v_{10} - qv_{01}) \\
&= (-q^{-1}v_{10} + v_{01}) - q(v_{10} - qv_{01}) \\
&= (q^{-2} + 1)v_{10} + (-q^{-1} - q)v_{01} \\
&= -(q^{-1} + q)C(v_{10}); \\
C^2(v_{01}) &= C(v_{10} - qv_{01}) \\
&= (-q^{-1}v_{10} + v_{01}) - q(v_{10} - qv_{01}) \\
&= -(q^{-1} + q)C(v_{01}).
\end{aligned}$$

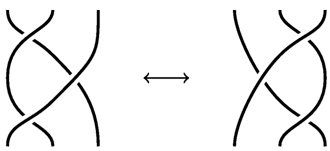
Thus $C^2 = -(q^{-1} + q)C$. □

Note that the ts vanish within the proof of 3.1.2.

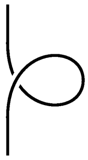
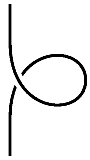
3.1.3 Proposition. *The formulae* (3.1)-(3.4) *respect the Reidemeister moves [R1'], [R2] and [R3], where*

$[R1'] =$  , (3.5)

$[R2] =$  , (3.6)

$[R3] =$  . (3.7)

3.1.4 Remark. *A priori, the original first Reidemeister move*

$$[R1] = $$  $\longleftrightarrow$ $\longleftrightarrow$  (3.8)

is not fulfilled because we work in a framed tangle setting. This is due to the fact that adding a curl in a framed strand causes a twist in the strand, and the formulae above are built such that this is respected. (Just imagine building a curl with a strip of paper.) So we need to check at first that adding a curl and its opposite afterwards is equivalent to the identity. Later we will reformulate the theory by adding an orientation on the strands so that [R1] *will be respected.*

Proof of Proposition 3.1.3. First we calculate the values of the two curls of (3.8) and see that they cause parameter shifts which are inverse to each other. For v_0 we get

$$\begin{aligned} v_0 &\overset{id\otimes\cup}{\longmapsto} -tv_{010} + tqv_{001} \\ &\overset{\Pi\otimes id}{\longmapsto} -t(tq^{-1}C(v_{01}) + tq^{-2}v_{01}) \otimes v_0 + tq(tq^{-2}v_{001}) \\ &= -t^2q^{-1}(v_{100} - qv_{010}) - t^2q^{-2}v_{010} + t^2q^{-1}v_{001} \\ &= -t^2q^{-1}v_{100} + (t^2 - t^2q^{-2})v_{010} + t^2q^{-1}v_{001} \\ &\overset{id\otimes\cap}{\longmapsto} 0 + (t^2 - t^2q^{-2})v_0(tq)^{-1} + t^2q^{-1}v_0(-t^{-1}) = -tq^{-3}v_0 \end{aligned}$$

for the first curl and

$$\begin{aligned} v_0 &\overset{id\otimes\cup}{\longmapsto} -tv_{010} + tqv_{001} \\ &\overset{\Omega\otimes id}{\longmapsto} -t(t^{-1}qC(v_{01}) + t^{-1}q^2v_{01}) \otimes v_0 + tq(t^{-1}q^2v_{001}) \\ &= -q(v_{100} - qv_{010}) - q^2v_{010} + q^3v_{001} \\ &= -qv_{100} + (q^2 - q^2)v_{010} + q^3v_{001} = -qv_{100} + q^3v_{001} \\ &\overset{id\otimes\cap}{\longmapsto} 0 + q^3v_0(-t^{-1}) = -t^{-1}q^3v_0 \end{aligned}$$

for the second curl.

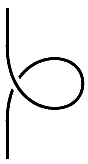
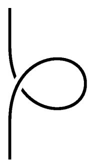
By similar calculations, we also obtain

$$(id \otimes \cap) \circ (\Pi \otimes id) \circ (id \otimes \cup)(v_1) = -tq^{-3}v_1$$

and

$$(id \otimes \cap) \circ (\Omega \otimes id) \circ (id \otimes \cup)(v_1) = -t^{-1}q^3v_1,$$

so we see that

$-t^{-1}q^3$  $=$ $=$ $-tq^{-3}$  . (3.9)

This gives us [R1'].

The move [R2] follows by

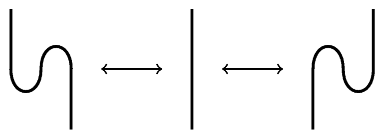
$$\begin{aligned} \Pi \circ \Omega &= tq^{-1}(C + q^{-1}\ id) \circ t^{-1}q(C + q\ id) \\ &= C^2 + q^{-1}C + qC + id \\ &\overset{3.1.2}{=} -(q + q^{-1})C + (q + q^{-1})C + id = id. \end{aligned}$$

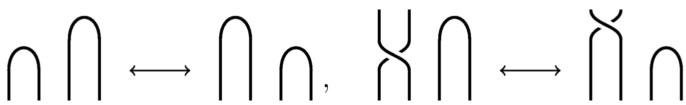
To show [R3], we proceed by evaluating $(\Pi \otimes id) \circ (id \otimes \Pi) \circ (\Pi \otimes id)(v_{100})$ and $(id \otimes \Pi) \circ (\Pi \otimes id) \circ (id \otimes \Pi)(v_{100})$ to see that they are the same:

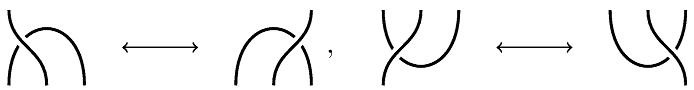
$$\begin{aligned} v_{100} &\overset{\Pi\otimes id}{\longmapsto} (tq^{-1}C(v_{10}) + tq^{-2}v_{1}0) \otimes v_0 \\ &\quad = tq^{-1}(-q^{-1}v_{100} + v010) + tq^{-2}v_{100} = tq^{-1}v_{010} \\ &\overset{id\otimes\Pi}{\longmapsto} tq^{-1}v_0 \otimes (tq^{-1}C(v_{10}) + tq^{-2}v_{10}) \\ &\quad = t^2q^{-2}(-q^{-1}v_{010} + v_{001}) + t^2q^{-3}v_{010} = t^2q^{-2}v_{001} \\ &\overset{\Pi\otimes id}{\longmapsto} t^2q^{-2}(tq^{-2}v_{001}) = t^3q^{-4}v_{001}, \\ v_{100} &\overset{id\otimes\Pi}{\longmapsto} tq^{-2}v_{100} \\ &\overset{\Pi\otimes id}{\longmapsto} tq^{-2}(tq^{-1}C(v_{10}) + tq^{-2}v_{10}) \otimes v_0 \\ &\quad = t^2q^{-3}(-q^{-1}v_{100} + v_{010}) + t^2q^{-4}v_{100} = t^2q^{-3}v_{010} \\ &\overset{id\otimes\Pi}{\longmapsto} t^2q^{-3}v_0 \otimes (tq^{-1}C(v_{10}) + tq^{-2}v_{10}) \\ &\quad = t^3q^{-4}(-q^{-1}v_{010} + v_{001}) + t^3q^{-5}v_{010} = t^3q^{-4}v_{001}. \end{aligned}$$

Via a similar calculation we get $(\Omega \otimes id) \circ (id \otimes \Omega) \circ (\Omega \otimes id)(v_{100}) = t^{-3}q^4v_{001} = (id \otimes \Omega) \circ (\Omega \otimes id) \circ (id \otimes \Omega)(v_{100})$; we omit the similar proof for the other basis vectors $v_{000}, v_{010}, v_{001}, v_{110}, v_{101}, v_{011}$ and v_{111}. This concludes the proof. □

The Reidemeister moves alone, however, are not enough. If we want to achieve an isotopy of framed tangles by our formulae, we have to ensure that the following tangle isotopies are fulfilled (see [Kas95]):

$[T1] =$  , (3.10)

$[T2] =$  *etc.*, (3.11)

$[T3] =$  *etc.* (3.12)

In the following we will call [T1] the *S-move* and [T2] the *height moves*. The height moves are of course valid since the occurring crossings, cups or caps involve different strands.

We show the S-move explicitly on the basis vectors v_0 and v_1:

$$\begin{aligned}
v_0 &\overset{id\otimes\cup}{\longmapsto} -tv_{010} + tqv_{001} \overset{\cap\otimes id}{\longmapsto} -t(-t^{-1})v_0 = v_0,\\
v_0 &\overset{\cup\otimes id}{\longmapsto} -tv_{100} + tqv_{010} \overset{id\otimes\cap}{\longmapsto} tqv_0(tq)^{-1} = v_0;\\
v_1 &\overset{id\otimes\cup}{\longmapsto} -tv_{110} + tqv_{101} \overset{\cap\otimes id}{\longmapsto} tq(tq)^{-1}v_1 = v_1,\\
v_1 &\overset{\cup\otimes id}{\longmapsto} -tv_{101} + tqv_{011} \overset{id\otimes\cap}{\longmapsto} -tv_1(-t^{-1}) = v_1.
\end{aligned}$$

The isotopy [T3] requires a bit more care as we will see in the following. We check the moves explicitly on all basis vectors:

$$\begin{aligned}
&v_{000} \overset{\Pi\otimes id}{\longmapsto} tq^{-2}v_{000} \overset{id\otimes\cap}{\longmapsto} 0;\\
&v_{111} \overset{\Pi\otimes id}{\longmapsto} tq^{-2}v_{111} \overset{id\otimes\cap}{\longmapsto} 0;\\
&v_{100} \overset{\Pi\otimes id}{\longmapsto} (tq^{-1}C(v_{10}) + tq^{-2}v_{10}) \otimes v_0 = tq^{-1}(-q^{-1}v_{100} + tqv010) + tq^{-2}v_{100}\\
&\qquad = (-tq^{-2} + tq^{-2})v_{100} + tq^{-1}v_{010} = tq^{-1}v_{010}\\
&\qquad \overset{id\otimes\cap}{\longmapsto} q^{-2}v_0;\\
&v_{010} \overset{\Pi\otimes id}{\longmapsto} (tq^{-1}C(v_{01}) + tq^{-2}v_{01}) \otimes v_0 = tq^{-1}(v_{100} - qv_{010}) + tq^{-2}v_{010}\\
&\qquad = tq^{-1}v_{100} + (tq^{-2} - t)v_{010}\\
&\qquad \overset{id\otimes\cap}{\longmapsto} (q^{-3} - q^{-1})v_0;\\
&v_{001} \overset{\Pi\otimes id}{\longmapsto} tq^{-2}v_001 \overset{id\otimes\cap}{\longmapsto} -q^{-2}v_0;\\
&v_{110} \overset{\Pi\otimes id}{\longmapsto} tq^{-2}v_{110} \overset{id\otimes\cap}{\longmapsto} q^{-3}v_1;\\
&v_{101} \overset{\Pi\otimes id}{\longmapsto} (tq^{-1}C(v_{10}) + tq^{-2}v_{10}) \otimes v_1 = tq^{-1}v_{011} \overset{id\otimes\cap}{\longmapsto} 0;\\
&v_{011} \overset{\Pi\otimes id}{\longmapsto} (tq^{-1}C(v_{01}) + tq^{-2}v_{01}) \otimes v_0 = tq^{-1}v_{101} + (tq^{-2} - t)v_{011} \overset{id\otimes\cap}{\longmapsto} -q^{-1}v_1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&v_{000} \overset{id\otimes\Omega}{\longmapsto} t^{-1}q^2v_{000} \overset{\cap\otimes id}{\longmapsto} 0;\\
&v_{111} \overset{id\otimes\Omega}{\longmapsto} t^{-1}q^2v_{111} \overset{\cap\otimes id}{\longmapsto} 0;\\
&v_{100} \overset{id\otimes\Omega}{\longmapsto} t^{-1}q^2v_{100} \overset{\cap\otimes id}{\longmapsto} t^{-2}qv_0;\\
&v_{010} \overset{id\otimes\Omega}{\longmapsto} v_0 \otimes (t^{-1}q(-q^{-1}v_{10} + v_{01}) + t^{-1}q^2v_{10}) = -t^{-1}v_{010} + t^{-1}q^{v}_{001} + t^{-1}q^2v_{010}\\
&\qquad \overset{\cap\otimes id}{\longmapsto} t^{-2}(1 - q^2)v_0;\\
&v_{001} \overset{id\otimes\Omega}{\longmapsto} v_0 \otimes (t^{-1}q(v_{10} - qv_{01}) + t^{-1}q^2v_{01}) = t^{-1}qv_{010} + (-t^{-1}q^2 + t^{-1}q^2)v_{001}\\
&\qquad \overset{\cap\otimes id}{\longmapsto} -t^{-2}qv_0;\\
&v_{110} \overset{id\otimes\Omega}{\longmapsto} v_1(t^{-1}q(-q^{-1}v_{10} + v_{01}) + t^{-1}q^2v_{10}) = (t^{-1}q^2 - t^{-1})v_{110} + t^{-1}qv_{101}\\
&\qquad \overset{\cap\otimes id}{\longmapsto} (t^{-1}q)(t^{-1}q^{-1})v_1 = t^{-2}v_1;\\
&v_{101} \overset{id\otimes\Omega}{\longmapsto} v_1 \otimes (t^{-1}q(v_{10} - qv_{01}) + t^{-1}q^2v_{01}) = t^{-1}qv_{110} \overset{\cap\otimes id}{\longmapsto} 0;\\
&v_{011} \overset{id\otimes\Omega}{\longmapsto} t^{-1}q^2v_{011} \overset{\cap\otimes id}{\longmapsto} -t^{-2}q^2v_1.
\end{aligned}$$

Comparison of the two results for each basis vector shows that all solutions are shifted by the factor $t^{-2}q^3$. The same phenomenon can be seen if we calculate the second

[T3]-move:

$$\begin{aligned} v_0 &\overset{id\otimes\cup}{\longmapsto} -tv_{010} + tqv_{001} \\ &\overset{\Pi\otimes id}{\longmapsto} -t(tq^{-1}(v_{100} - qv_{010}) + tq^{-2}v_{010}) + tq(tq^{-2}v_{001}) \\ &\quad = -t^2q^{-1}v_{100} + t^2q^{-1}(q - q^{-1})v_{010} + t^2q^{-1}v_{001}; \\ v_1 &\overset{id\otimes\cup}{\longmapsto} -tv_{110} + tqv_{101} \\ &\overset{\Pi\otimes id}{\longmapsto} -t(tq^{-1}v_{110}) + tq(tq^{-1}(-q^{-1}v_{101} + v_{011}) + tq^{-2}v_{101}) \\ &\quad = -t^2q^{-2}v_{110} + t^2v_{011} \end{aligned}$$

and

$$\begin{aligned} v_0 &\overset{\cup\otimes id}{\longmapsto} -tv_{100} + tqv_{010} \\ &\overset{id\otimes\Omega}{\longmapsto} -t(t^{-1}q^2v_{100}) + tqv_0 \otimes (t^{-1}q(-q^{-1}v_{10} + v_{01}) + t^{-1}q^2v_{10}) \\ &\quad = -q^2v_{100} - q^2(q^{-1} - q)v_{010} + q^2v_{001}; \\ v_1 &\overset{\cup\otimes id}{\longmapsto} -tv_{101} + tqv_{011} \\ &\overset{id\otimes\Omega}{\longmapsto} -t(t^{-1}q(v_{110} - qv_{101}) + t^{-1}q^2v_{101}) + tq(t^{-1}q^2v_{011}) \\ &\quad = -qv_{110} + q^3v_{011}. \end{aligned}$$

As a consequence, we rescale the crossings Π and Ω as follows:

$$\widetilde{\Pi} = tq^{-\frac{3}{2}}\ \Pi, \tag{3.13}$$
$$\widetilde{\Omega} = t^{-1}q^{\frac{3}{2}}\ \Omega. \tag{3.14}$$

The factors are chosen like this since

$t^{-2}q^3$  $=$  $\Leftrightarrow$ $t^{-1}q^{\frac{3}{2}}$  $= tq^{-\frac{3}{2}}$  ,

so if the crossings get rescaled, we obtain [T3].

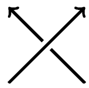
Of course the Reidemeister moves and the other tangle isotopies have to be reviewed to ensure that the rescaling of the crossings does not interfere. There is nothing left to show for both the height moves and the S-move as the latter does not contain any crossings. In [R2], both $\widetilde{\Pi}$ and $\widetilde{\Omega}$ are involved once, so the factors cancel. In [R3], both sides get multiplied three times with the rescaling factor, so there is no change either. If we look at [R1'], there are also both crossings involved, so [R1'] is also fulfilled. This proves the following:

3.1.5 Proposition. *The formulae* (3.1),(3.2),(3.13) *and* (3.14) *yield an invariant of framed tangles via the assignment $L \to p_2'(L)$ where L denotes an arbitrary framed tangle and p_2' represents the polynomial associated to L via the morphisms defined in this chapter.*

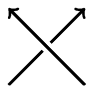
3.1.6 Remark. *The proof of Proposition 3.1.5 also proves Proposition 2.1.6 since the one-parameter Jones polynomial is just the special case $t = -1$ of 3.1.5. (Only rescale the crossings in a similar manner.)*

Our next goal is to achieve a genuine and not just framed link invariant that also respects the original first Reidemeister move [R1] and that does not depend on a framing[1]. For this we will introduce an orientation as follows:

3.1.7 Definition. *Assume that we give each strand of a link an orientation. Then we call a crossing of the form*



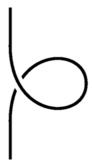
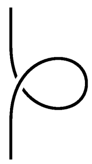
a positive crossing *and the other type*



a negative crossing. *Furthermore, we define the* linking number *of an oriented link L via*

$$\lambda = \lambda(L) := \#\{\textit{positive crossings}\} - \#\{\textit{negative crossings}\}. \tag{3.15}$$

Now consider the curl again. From 3.1.5 and (3.9) one gets that

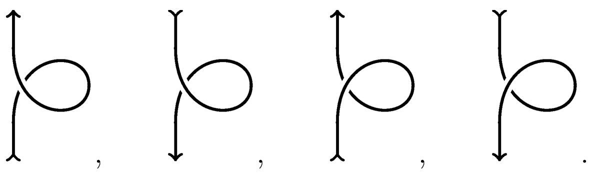
$-q^{\frac{3}{2}}$  $=$ $=$ $-q^{-\frac{3}{2}}$  .

We need the value $\beta := -q^{-\frac{3}{2}}$ for the following definition.

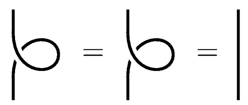
3.1.8 Definition. *Let β be as above and let L be an oriented tangle diagram. Then define its invariant as the framed tangle invariant multiplied with the value $\beta^{\lambda(L)}$.*

3.1.9 Proposition. *The invariant p_2' yields an invariant of oriented tangles via the assignment $L \to p_2(L) = \beta^{\lambda(L)} p_2'(L)$ which satisfies [R1] in particular.*

Proof. We have to recheck the Reidemeister moves and the tangle isotopies. For the first Reidemeister move, look at the four possible settings:



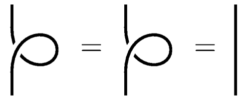
The first two diagrams have a negative crossing, so we get

$$\underbrace{\beta^1}_{\text{from the unoriented case}} \cdot \underbrace{\beta^{-1}}_{\text{from orientation}}$$  .

[1] The same can be done in case of the categorified Jones polynomial as in [Str05].

Similarily, since the last two curls contain a positive crossing, we obtain

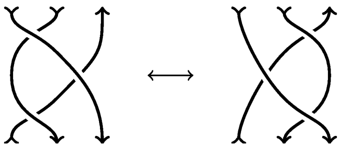
$$\underbrace{\beta^{-1}}_{\text{from the unoriented case}} \cdot \underbrace{\beta^{1}}_{\text{from orientation}}$$



and hence [R1] is fulfilled.

The second Reidemeister move [R2] holds since, regardless of the orientation, one positive and one negative crossing are involved, so the factors cancel.

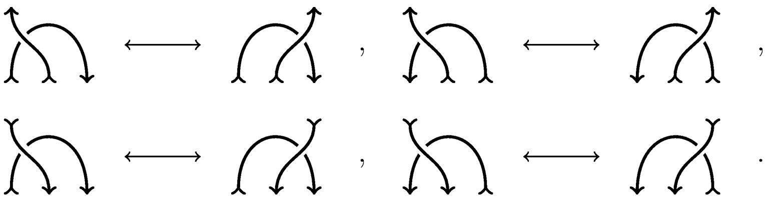
Considering [R3], we see that the number of positive and negative crossings does not change by executing this Reidemeister move. For example,



has one positive and two negative crossings that are merely rearranged. All other possible orientations work similarily. Thus we see that the factor β^{λ} only rescales this move.

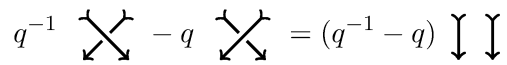
The S-move contains no crossing, so there is nothing to show; the height moves also obviously go through. So it remains to check the tangle invariant [T3]:

Consider the four possible orientations:

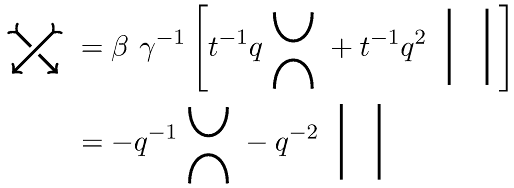


We see that in all cases the orientation of the involved crossing stays unchanged under the move. Thus [T3] follows. □

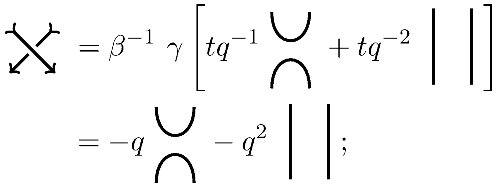
3.1.10 Remark. *The invariant in proposition 3.1.9 satisfies the skein relation*



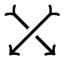
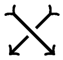
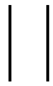
since we can write (recall $\beta = -q^{-\frac{3}{2}}$ and set $\gamma = t^{-1}q^{\frac{3}{2}}$)



and



Hence, it follows that

$-q$  $+ q^{-1}$  $= (q^{-1} - q)$  .

Note that this corresponds to the skein relation of the one-parameter Jones polynomial.

3.2 Two examples

3.2.1 The unknot

First we show the impact of the new formulae on the coloured unknot; the calculations will be carried out similarily to chapter 2.

3.2.1 Proposition. *Using the two-parameter formulae from chapter 3, the unknot coloured by n has value $(-1)^n[n+1]$.*

Proof. We proceed as in proposition 2.2.7, replacing the one-parameter formulae by the two-parameter ones from chapter 3.
Step 1:

$$\hat{\delta}'_n(1) = (-t)^n \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)} - q v_a \otimes v_{01} \otimes v_{mir(a)}.$$

This is proven just as in lemma 2.2.2; the cup formula is only rescaled.
Step 2:

$$\begin{aligned}(id^{\otimes n} \otimes \pi_n) \circ \hat{\delta}'_n(1) &= (-t)^n \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_{a\cup 1} \otimes q^{-l(0\cup mir(a))} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} v_i \\ &\quad - q v_{a\cup 0} \otimes q^{-l(1\cup mir(a))} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} v_{i+1} \\ &= (-t)^n \sum_{i=0}^{n-1} (-q)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} v_{a\cup 1} \otimes v_i \\ &\quad - q^{1-l(a)} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} v_{a\cup 0} \otimes v_{i+1} \\ &=: \hat{\delta}_n(1).\end{aligned}$$

Nothing special happens here either, the new factor is just taken along (just like in the next step).

Step 3:

$$\begin{aligned}(id^{\otimes n}\otimes\iota_n)\circ\hat{\delta}_n(1) &= (-t)^n\sum_{i=0}^{n-1}(-q)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-(l(a)+i)}\frac{1}{\left[{n\atop i}\right]}\, v_{a\cup 1}\otimes\sum_{|b|=n-i} q^{i(n-i)-l(b)}v_b\\ &\quad -q^{1-l(a)}\frac{1}{\left[{n\atop i+1}\right]}\, v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)} q^{(i+1)(n-(i+1))-l(b)}v_b\\ &= (-t)^n\sum_{i=0}^{n-1}(-q)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-l(a)-i+i(n-i)}\frac{1}{\left[{n\atop i}\right]}\, v_{a\cup 1}\otimes\sum_{|b|=n-i} q^{-l(b)}v_b\\ &\quad -q^{1-l(a)+(i+1)(n-(i+1))}\frac{1}{\left[{n\atop i+1}\right]}\, v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)} q^{-l(b)}v_b\\ &=: \star\star\,.\end{aligned}$$

In the last step, the factor t vanishes and we regain the old formula as given in chapter 2:

$$\begin{aligned}\varepsilon'_n(\star\star) &= (-t)^n\sum_{i=0}^{n-1}(-q)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-l(a)-i+i(n-i)}\frac{1}{\left[{n\atop i}\right]}\, q^{-l(a\cup 1)}((tq)^{-1})^{n-i}(-t^{-1})^i\\ &\quad -q^{1-l(a)+(i+1)(n-(i+1))}\frac{1}{\left[{n\atop i+1}\right]}\, q^{-l(a)}((tq)^{-1})^{n-(i+1)}(-t^{-1})^{i+1}\\ &= (-t)^n\sum_{i=0}^{n-1}(-1)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}\frac{1}{\left[{n\atop i}\right]}\, q^{-l(a)-i+i(n-i)-l(a)+i-i-(n-i)}t^{-n+i-i}(-1)^i\\ &\quad -\frac{1}{\left[{n\atop i+1}\right]}\, q^{1-l(a)+(i+1)(n-(i+1))-l(a)-(n-(i+1))+i}t^{-n+(i+1)-(i+1)}(-1)^{i+1}\\ &= (-1)^n\Big(\sum_{i=0}^{n-1}\underbrace{(-1)^{2i}}_{=1}\frac{1}{\left[{n\atop i}\right]}q^{i(n-i)-n}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}\\ &\quad +\sum_{i=0}^{n-1}\underbrace{(-1)^{2i}}_{=1}\frac{1}{\left[{n\atop i+1}\right]}\, q^{(i+1)(n-(i+1))-n+2(i+1)}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}\Big)\\ &= (-1)^n\Big(\sum_{i=0}^{n-1}\frac{1}{\left[{n\atop i}\right]}\, q^{i(n-i)-i+i-n}\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}\\ &\quad +\sum_{i=1}^{n}\frac{1}{\left[{n\atop i}\right]}\, q^{i(n-i)-n+2i}\sum_{\substack{|a|=n-1-(i-1)\\ a\in\{0,1\}^{n-1}}} q^{-2l(a)}\Big)\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n (\sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + \sum_{i=1}^{n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}) \\
&= (-1)^n (q^{-n} + \sum_{i=1}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{-n+2i+n-2i} (q^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}) + q^n) \\
&= (-1)^n (q^{-n} + \sum_{i=1}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} q^{2i-n} \begin{bmatrix} n \\ i \end{bmatrix} + q^n) \\
&= (-1)^n [n+1].
\end{aligned}
$$

□

3.2.2 Remark. *We see that the addition of the parameter t has no effect on the value of the unknot coloured by n if we only rescale the formulae. This will change with the definitions in the next chapter.*

We conclude this chapter by looking at a second example, namely the so-called theta networks.

3.2.2 Theta networks – some useful notation

In the beginning, we introduce some notation which is useful when talking about theta networks. Again, we follow [FSS12] and [FK97], but we refer to [KL94] for more detailed background information.

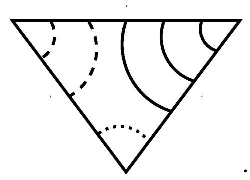
3.2.3 Definition. *A triple (a, b, c) with $a, b, c \in \mathbb{N}$ is called admissible if the following triangle identities hold:*

$$a + b + c = 0 \ mod\ 2, \tag{3.16}$$

$$
\begin{aligned}
&a + b - c \geq 0, \\
&a + c - b \geq 0, \\
&b + c - a \geq 0.
\end{aligned} \tag{3.17}
$$

Furthermore, define an (a, b, c)-*triangle* $\Delta_{a,b,c}$ to be a triangle with a marked points on the first side, b on the second and c marked points on the third respectively. A *line arrangement* L on a triangle $\Delta_{a,b,c}$ is an isotopy class of a collection of arcs which do not intersect each other and connect exactly two points that lie on different sides of the triangle. Denote the number of arcs which connect points from the a-side with points from the c-side with $x(L)$, respectively those between b-side and c-side points with $y(L)$ and those between a-side and b-side points with $z(L)$.

3.2.4 Example. *Consider a (3,4,5)-triangle. Then its line arrangement is the following:*

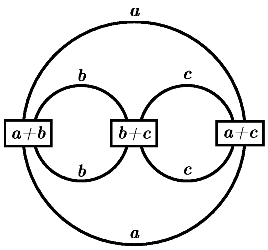


The resulting values for x,y and z are: $x(L) = 2$ (the dashed lines), $y(L) = 3$ (the solid lines) and $z(L) = 1$ (the dotted line).

The following result is proved as Lemma 6 in [FSS12].

3.2.5 Lemma ([FSS12])**.** *For any given (a,b,c)-triangle there is a line arrangement if and only if the tuple (a,b,c) is admissible in the above sense. Furthermore, if a line arrangement exists, then it is unique up to isotopy.*

3.2.6 Definition. *For any admissible triple (a,b,c) we define its theta-network or theta-curve $\vartheta_{a,b,c}$ via the diagram in the following figure:*



(3.18)

The labels a, b and c say that the strands consist of a, b or c parallel small strands coloured by 1, while the labelled boxes stand for the corresponding Jones-Wenzl projectors $\iota_{a+b} \circ \pi_{a+b}$, $\iota_{b+c} \circ \pi_{b+c}$ and $\iota_{a+c} \circ \pi_{a+c}$. Denote the value of this diagram by $\vartheta_{a,b,c}(1)$.

3.2.7 Example. *In the following, we are interested in the smallest theta network $\vartheta_{2,2,2}$. It is defined by*

$$\vartheta_{2,2,2} = \cap \circ (id \otimes \cap \otimes \cap \otimes id) \circ (\iota_2 \otimes \iota_2 \otimes \iota_2) \circ (\pi_2 \otimes \pi_2 \otimes \pi_2) \circ (id \otimes \cup \otimes \cup \otimes id) \circ \cup.$$

3.2.3 The smallest example for theta networks: $\vartheta_{2,2,2}$

We will now calculate the smallest network $\vartheta_{2,2,2}$ with the definitions of this chapter.

3.2.8 Proposition. *With the definitions from chapter 3 it follows that*

$$\vartheta_{2,2,2}(1) = -\frac{1}{[2]}(q^{-4} + 4 + q^4) - \frac{1}{[2]^3}(q^{-4} - 2 + q^4).$$

3.2.9 Remark. *Note that, just like in the unknot case, all ts vanish. Indeed, the first two-parameter version leads to the same value for $\vartheta_{2,2,2}$ as the one-parameter Jones polynomial.*

Proof. We proceed by calculating

$$\cap \circ (id \otimes \cap \otimes \cap \otimes id) \circ (\iota_2 \otimes \iota_2 \otimes \iota_2) \circ (\pi_2 \otimes \pi_2 \otimes \pi_2) \circ (id \otimes \cup \otimes \cup \otimes id) \circ \cup(1)$$

successively.

$$
\begin{array}{rcl}
1 & \overset{\cup}{\longmapsto} & -tv_{10} + tqv_{01} \\
& \overset{id\otimes\cup\otimes\cup\otimes id}{\longmapsto} & -tv_1 \otimes (-tv_{10} + tqv_{01}) \otimes (-tv_{10} + tqv_{01}) \otimes v_0 \\
& & +tqv_0 \otimes (-tv_{10} + tqv_{01}) \otimes (-tv_{10} + tqv_{01}) \otimes v_1 \\
& = & -t^3 v_{110100} + t^3 qv_{110010} + t^3 qv_{101100} - t^3 q^2 v_{101010} \\
& & +t^3 qv_{010101} - t^3 q^2 v_{010011} - t^3 q^2 v_{001101} + t^3 q^3 v_{001011} \\
& \overset{\pi_2\otimes\pi_2\otimes\pi_2}{\longmapsto} & -t^3 \frac{1}{[2]} q^{-1} v_{210} + t^3 q \frac{1}{[2]} v_{201} + t^3 q \frac{1}{[2]} v_{120} - t^3 q^2 \frac{1}{[2]^3} v_{111} \\
& & +t^3 q \frac{1}{[2]^3} q^{-3} v_{111} - t^3 q^2 \frac{1}{[2]} q^{-1} v_{102} - t^3 q^2 \frac{1}{[2]} q^{-1} v_{021} + t^3 q^3 \frac{1}{[2]} v_{012} \\
& = & -\frac{1}{[2]} t^3 q^{-1} v_{210} + \frac{1}{[2]} t^3 qv_{201} + \frac{1}{[2]} t^3 qv_{120} + \frac{1}{[2]^3} (t^3 q^{-2} - t^3 q^2) v_{111} \\
& & -\frac{1}{[2]} t^3 qv_{102} - \frac{1}{[2]} t^3 qv_{021} + \frac{1}{[2]} t^3 q^3 v_{012} \\
& \overset{\iota_2\otimes\iota_2\otimes\iota_2}{\longmapsto} & -\frac{1}{[2]} t^3 q^{-1} v_{11} \otimes (qv_{10} + v_{01}) \otimes v_{00} \\
& & +\frac{1}{[2]} t^3 qv_{11} \otimes v_{00} \otimes (qv_{10} + v_{01}) \\
& & +\frac{1}{[2]} t^3 q (qv_{10} + v_{01}) \otimes v_{11} \otimes v_{00} \\
& & +\frac{1}{[2]^3} (t^3 q^{-2} - t^3 q^2)(qv_{10} + v_{01}) \otimes (qv_{10} + v_{01}) \otimes (qv_{10} + v_{01}) \\
& & -\frac{1}{[2]} t^3 q (qv_{10} + v_{01}) \otimes v_{00} \otimes v_{11} \\
& & -\frac{1}{[2]} t^3 qv_{00} \otimes v_{11} \otimes (qv_{10} + v_{01}) \\
& & +\frac{1}{[2]} t^3 q^3 v_{00} \otimes (qv_{10} + v_{01}) \otimes v_{11} \\
& = & -\frac{1}{[2]} t^3 v_{111000} - \frac{1}{[2]} t^3 q^{-1} v_{110100} \\
& & +\frac{1}{[2]} t^3 q^2 v_{110010} + \frac{1}{[2]} t^3 qv_{110001} \\
& & +\frac{1}{[2]} t^3 q^2 v_{101100} + \frac{1}{[2]} t^3 qv_{011100} \\
& & +\frac{1}{[2]^3} (t^3 q^{-2} - t^3 q^2)(q^3 v_{101010} + q^2 v_{100110} + q^2 v_{011010} \\
& & \quad +qv_{010110} + q^2 v_{101001} + qv_{100101} + qv_{011001} + v_{010101}) \\
& & -\frac{1}{[2]} t^3 q^2 v_{100011} - \frac{1}{[2]} t^3 qv_{010011} \\
& & -\frac{1}{[2]} t^3 q^2 v_{001110} - \frac{1}{[2]} t^3 qv_{001101} \\
& & +\frac{1}{[2]} t^3 q^4 v_{001011} + \frac{1}{[2]} t^3 q^3 v_{000111}
\end{array}
$$

$$
\begin{aligned}
&\overset{id\otimes\cap\otimes\cap\otimes id}{\longmapsto} -\frac{1}{[2]}t^3q^{-1}v_{10}(tq)^{-2}+\frac{1}{[2]}t^3q^2v_{10}(tq)^{-1}(-t^{-1})+\frac{1}{[2]}t^3q^2v_{10}(tq)^{-1}(-t^{-1})\\
&\quad +\frac{1}{[2]^3}(t^3q^{-2}-t^3q^2)(q^3v_{10}(-t^{-1})^2+v_{01}(tq)^{-2})\\
&\quad -\frac{1}{[2]}t^3qv_{01}(tq)^{-1}(-t^{-1})-\frac{1}{[2]}t^3qv_{01}(tq)^{-1}(-t^{-1})+\frac{1}{[2]}t^3q^4v_{01}(-t^{-1})^2\\
&\overset{\cap}{\longmapsto} -\frac{1}{[2]}t^3q^{-1}(tq)^{-3}+\frac{1}{[2]}t^3q^2(tq)^{-2}(-t^{-1})+\frac{1}{[2]}t^3q^2(tq)^{-2}(-t^{-1})\\
&\quad +\frac{1}{[2]^3}t^3(q^{-2}-q^2)(q^3(tq)^{-1}(-t^{-1})^2+(tq)^{-2}(-t^{-1}))\\
&\quad -\frac{1}{[2]}t^3q(tq)^{-1}(-t^{-1})^2-\frac{1}{[2]}t^3q(tq)^{-1}(-t^{-1})^2+\frac{1}{[2]}t^3q^4(-t^{-1})^3\\
&= \frac{1}{[2]}(-q^{-4}-1-1-1-1-q^4)+\frac{1}{[2]^3}(1-q^{-4}-q^4+1)\\
&= -\frac{1}{[2]}(q^{-4}+4+q^4)-\frac{1}{[2]^3}(q^{-4}-2+q^4).
\end{aligned}
$$

□

4 The second variation p_2

In the following, we will formulate a second two-parameter variation of the setup from chapter 2. In contrast to the definitions of the last chapter, the formulae are not just scaled by a factor $\pm t^{\pm 1}$. Instead, the new variable is built in more subtly.

4.1 A two-parameter quantum group and its modules

Again, we change the ground field to $k = \mathbb{C}(q,t)$. First of all, we have to replace the quantum number $[n]$ by a two-parameter version:

4.1.1 Definition. *Let*

$$[n]_{tq} := \sum_{j=0}^{n-1} (tq)^{n-2j-1} = \frac{(tq)^n - (tq)^{-n}}{tq - (tq)^{-1}}$$

and set $[0]_{tq} := 1$*. Based on this, define the two-parameter* n*th quantum number by*

$$[n] := [n]_{-tq} = (-1)^{n+1}[n]_{tq}.$$

Finally, define $[n]!$ *and* $\begin{bmatrix} n \\ k \end{bmatrix}$ *as usual.*

Secondly, the algebra $U_q(\mathfrak{sl}_2)$ has to be replaced by a two-parameter quantum group as follows:

4.1.2 Definition. *Define the two-parameter quantum group* $U_{q,t}(\mathfrak{sl}_2)$ *to be the unitary algebra over* k *with generators* E, F, K *and* K^{-1} *subject to the relations*

$$KK^{-1} = K^{-1}K = 1 \tag{4.1}$$

$$KE = q^2t^2EK \tag{4.2}$$

$$KF = q^{-2}t^{-2}FK \tag{4.3}$$

$$EF - FE = \frac{K - K^{-1}}{q^{-1}t^{-1} - qt}. \tag{4.4}$$

Note that we obtain $U_q(\mathfrak{sl}_2)$ from the algebra $U_{q,t}(\mathfrak{sl}_2)$ by specifying $t = -1$. We first verify the existence of a Hopf algebra structure on $U_{q,t}(\mathfrak{sl}_2)$. The formulae for the comultiplication, counit, unit and antipode are as before, except that the maps are all $\mathbb{C}(q,t)$-linear now:

4.1.3 Lemma. *Define the comultiplication* $\Delta : U_{q,t}(\mathfrak{sl}_2) \to U_{q,t}(\mathfrak{sl}_2) \otimes U_{q,t}(\mathfrak{sl}_2)$ *and the antipode* $S : U_{q,t}(\mathfrak{sl}_2) \to U_{q,t}(\mathfrak{sl}_2)$ *again by the formulas in* (1.5) - (1.7) *and* (1.10) - (1.12) *respectively, but now* $\mathbb{C}(q,t)$*-linear. Moreover, define the unit* $\eta : \mathbb{C}(t,q) \to U_{q,t}(\mathfrak{sl}_2)$ *via natural embedding along with a counit* $\varepsilon : U_{q,t}(\mathfrak{sl}_2) \to \mathbb{C}(t,q)$ *as in* (1.8) *and* (1.9)*. Then* Δ*,* S*,* η *and* ε *are* $U_{q,t}(\mathfrak{sl}_2)$*-homomorphisms and induce a Hopf algebra structure on* $U_{q,t}(\mathfrak{sl}_2)$*.*

Proof. To see that Δ, S, η and ε are $U_{q,t}(\mathfrak{sl}_2)$-homomorphisms we check the compatibility with the relations (4.1)-(4.4).

On relation (4.1):

$$\Delta(KK^{-1}) = \Delta(K)\Delta(K^{-1}) = KK^{-1} \otimes KK^{-1} = 1 \otimes 1 = \Delta(1),$$
$$\Delta(K^{-1}K) = \Delta(K^{-1})\Delta(K) = K^{-1}K \otimes K^{-1}K = 1 \otimes 1 = \Delta(1);$$

on relation (4.2):

$$\begin{aligned} \Delta(KE) &= \Delta(K)\Delta(E) = K \otimes KE + KE \otimes 1, \\ \Delta(EK) &= \Delta(E)\Delta(K) = K \otimes EK + EK \otimes 1; \\ \Rightarrow \Delta(KE) &\overset{(4.2)}{=} q^2t^2\Delta(EK); \end{aligned}$$

on relation (4.3):

$$\begin{aligned} \Delta(KF) &= \Delta(K)\Delta(F) = K^2 \otimes KF + KF \otimes K, \\ \Delta(FK) &= \Delta(F)\Delta(K) = K^2 \otimes FK + FK \otimes K; \\ \Rightarrow \Delta(KF) &\overset{(4.3)}{=} q^{-2}t^{-2}\Delta(FK); \end{aligned}$$

and finally on relation (4.4):

$$\begin{aligned} \Delta(EF - FE) &= \Delta(EF) - \Delta(FE) \\ &= (1 \otimes E + E \otimes K^{-1})(K \otimes F + F \otimes 1) \\ &\quad -(K \otimes F + F \otimes 1)(1 \otimes E + E \otimes K^{-1}) \\ &= K \otimes EF + F \otimes E + EK \otimes K^{-1}F + EF \otimes K^{-1} \\ &\quad -K \otimes FE - KE \otimes FK^{-1} - F \otimes E - FE \otimes K^{-1} \\ &= K \otimes (EF - FE) + (EF - FE) \otimes K^{-1} \\ &\overset{(4.4)}{=} K \otimes \frac{K - K^{-1}}{q^{-1}t^{-1} - qt} + \frac{K - K^{-1}}{q^{-1}t^{-1} - qt} \otimes K^{-1} \\ &= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q^{-1}t^{-1} - qt} \\ &= \Delta(\frac{K - K^{-1}}{q^{-1}t^{-1} - qt}). \end{aligned}$$

The properties for the antipode, unit and counit follow precisely as in the proof of Theorem 1.1.8. This gives the required Hopf algebra structure on $U_{q,t}(\mathfrak{sl}_2)$. □

4.1.4 Remark. *If we set $r := -qt$ and $s := -q^{-1}t^{-1}$, we get $U_{r,s}(\mathfrak{sl}_2)$ as in the paper [BW04b] of Benkart and Witherspoon, but with a different coproduct to make it consistent with our previous considerations. Originally, two-parameter quantum groups were introduced by Takeuchi, [Tak90].*

Finally, we have to lift the irreducible finite dimensional $U_q(\mathfrak{sl}_2)$ -modules to our new setting:

4.1.5 Definition. *Define the trivial representation of* $U_{q,t}(\mathfrak{sl}_2)$ *as the vector space* $\mathbb{C}(q,t)$ *with the action*

$$\begin{aligned} E.1 &= 0, \\ F.1 &= 0, \\ K.1 &= K^{-1}.1 = 1. \end{aligned}$$

Furthermore, define V_n *to be the* $(n+1)$*-dimensional* $U_{q,t}(\mathfrak{sl}_2)$*-module* V_n *with basis* $\{v_0, ..., v_n\}$ *where* E*,* F *and* $K^{\pm 1}$ *act via*

$$\begin{aligned} K^{\pm 1}.v_i &= (-tq)^{\pm(2i-n)} v_i, \\ E.v_i &= [i+1] v_{i+1}, \quad \textit{and} \\ F.v_i &= [n-i+1] v_{i-1}. \end{aligned}$$

We verify now that the representation V_n as defined above is indeed a $U_{q,t}(\mathfrak{sl}_2)$-module by checking the relations (4.1)-(4.4) again. As in [BW04b] it follows then that V_n is the unique (up to isomorphism) irreducible $(n+1)$-dimensional $U_{q,t}(\mathfrak{sl}_2)$ -module of type 1.

Relation (4.1):

$$\begin{aligned} KK^{-1}.v_i &= ((-tq)^{2i-n})((-tq)^{-(2i-n)})v_i = v_i, \\ K^{-1}K.v_i &= ((-tq)^{-(2i-n)})((-tq)^{2i-n})v_i = v_i; \end{aligned}$$

relation (4.2):

$$\begin{aligned} KE.v_i &= [i+1]K.v_{i+1} = [i+1](-tq)^{2i+2-n} v_{i+1}, \\ EK.v_i &= (-tq)^{2i-n} E.v_i = [i+1](-tq)^{2i-n} v_{i+1}; \\ \Rightarrow KE.v_0 &= (-tq)^2 EK.v_0 = (tq)^2 EK.v_0; \end{aligned}$$

relation (4.3):

$$\begin{aligned} KF.v_i &= [n-i+1]K.v_{i-1} = [n-i+1](-tq)^{2i-2-n} v_{i-1}, \\ FK.v_i &= (-tq)^{2i-n} F.v_i = [n-i+1](-tq)^{2i-n} v_{i-1}; \\ \Rightarrow KF.v_i &= (-tq)^{-2} FK.v_i = (tq)^{-2} FK.v_i; \end{aligned}$$

and relation (4.4):

$$\begin{aligned} \left(\frac{K-K^{-1}}{(tq)^{-1}-tq}\right).v_i \quad &= \quad \frac{(-tq)^{2i-n} + (-tq)^{-(2i-n)}}{-(tq-(tq)^{-1})} v_i \\ &\overset{(-1)^k=(-1)^{-k}}{=} \quad (-1)^{2i-n+1} \frac{(tq)^{2i-n} + (tq)^{-(2i-n)}}{tq-(tq)^{-1}} v_i \\ &= \quad [2i-n] v_i, \end{aligned}$$

$$\begin{aligned}
(EF-FE).v_i &= [n-i+1]E.v_{i-1}-[i+1]F.v_{i-1},\\
&= ([i][n-i+1]-[i+1][n-i])v_i\\
&= \Bigg((-1)^i\frac{(tq)^i-(tq)^{-i}}{tq-(tq)^{-1}}(-1)^{n-i+1}\frac{(tq)^{n-i+1}-(tq)^{-(n-i+1)}}{tq-(tq)^{-1}}\\
&\quad -(-1)^{i+1}\frac{(tq)^{i+1}-(tq)^{-(i+1)}}{tq-(tq)^{-1}}(-1)^{n-i}\frac{(tq)^{n-i}-(tq)^{-(n-i)}}{tq-(tq)^{-1}}\Bigg)v_i\\
&= \Bigg((-1)^{n+1}\frac{(tq)^{n+1}-(tq)^{-n+2i-1}-(tq)^{n-2i+1}+(tq)^{-n-1}}{(tq-(tq)^{-1})^2}\\
&\quad -(-1)^{n+1}\frac{(tq)^{n+1}-(tq)^{-n+2i+1}-(tq)^{n-2i-1}+(tq)^{-n-1}}{(tq-(tq)^{-1})^2}\Bigg)v_i\\
&= (-1)^{n+1}\frac{(tq)^{-n+2i}(tq-(tq)^{-1})-(tq)^{n-2i}(tq-(tq)^{-1})}{(tq-(tq)^{-1})^2}v_i\\
&\overset{(-1)^{2i}=1}{=} (-1)^{n-2i+1}\frac{(tq)^{2i-n}-(tq)^{-(2i-n)}}{tq-(tq)^{-1}}v_i\\
&= [2i-n]v_i.
\end{aligned}$$

Thus, V_n is indeed a $U_{q,t}(\mathfrak{sl}_2)$-module.

4.2 The new polynomial

We can also lift the morphisms $\cup_t$, $\cap_t$, Π_t and Ω_t to our new setup. Our task is now to introduce a consistent two-parameter version which finally will provide a new invariant. If specified appropriately, it gives the previously defined Jones polynomial back. The following definition of a lift of the cup and cap morphisms are motivated by [FSS12, Lemma 69]:

4.2.1 Definition. *Let $\cup_t$ be the map defined by*

$$\begin{aligned}
&\cup_t : \mathbb{C}(q,t) \to V_1 \otimes V_1 \\
&\qquad \cup_t(1) = v_1 \otimes v_0 + tqv_0 \otimes v_1.
\end{aligned} \tag{4.5}$$

Moreover define $\cap_t$ to be the map defined on the basis vectors as follows:

$$\begin{aligned}
&\cap_t : V_1 \otimes V_1 \to \mathbb{C}(q,t) \\
&\cap_t(v_0 \otimes v_0) = 0 = \cap_t(v_1 \otimes v_1),\\
&\cap_t(v_1 \otimes v_0) = t^{-1}q^{-1}\\
&\cap_t(v_0 \otimes v_1) = 1;
\end{aligned} \tag{4.6}$$

furthermore, define the maps

$$\begin{aligned}
&\Pi_t : V_1 \otimes V_1 \to V_1 \otimes V_1 \\
&\Pi_t(v_i \otimes v_j) = (-q^{-1}C + t^{-1}q^{-2}id)(v_i \otimes v_j);
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\Omega_t : V_1 \otimes V_1 \to V_1 \otimes V_1 \\
&\Omega_t(v_i \otimes v_j) = (-qC + tq^2 id)(v_i \otimes v_j).
\end{aligned} \tag{4.8}$$

From now on, we will use the formulae (4.5) - (4.8) for all calculations omitting the index t for convenience.

4.2.2 Lemma. *The maps in* (4.5) - (4.8) *are* $U_{q,t}(\mathfrak{sl}_2)$*-homomorphisms.*

Proof. Consider first the compatibility of the cup morphism with the action of E, F and $K^{\pm 1}$. For the action of E (recall $\Delta(E) = 1 \otimes E + E \otimes K^{-1}$) we have

$$\begin{array}{ccc} 1 & \overset{\cup}{\longmapsto} & v_{10} + tqv_{01} \\ {\scriptstyle E}\big\downarrow & & \big\downarrow{\scriptstyle E} \\ 0 & \underset{\cup}{\longmapsto} & v_{11} + tq(-tq)^{-1}v_{11} = 0 \end{array} \quad .$$

Similarly, we get for F (recall $\Delta(F) = K \otimes F + F \otimes 1$):

$$\begin{array}{ccc} 1 & \overset{\cup}{\longmapsto} & v_{10} + tqv_{01} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ 0 & \underset{\cup}{\longmapsto} & v_{00} + tq(-tq)^{-1}v_{00} = 0 \end{array} \quad .$$

Finally, we check the action of $K^{\pm 1}$; note that under the action of K on the right-hand-side of the following diagram, the added coefficients $-tq$ and $-t^{-1}q^{-1}$ cancel (recall $\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$):

$$\begin{array}{ccc} 1 & \overset{\cup}{\longmapsto} & v_{10} + tqv_{01} \\ {\scriptstyle K^{\pm 1}}\big\downarrow & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ 1 & \underset{\cup}{\longmapsto} & v_{10} + tqv_{01} \end{array} \quad .$$

The compatibility of the cap morphism can be checked in a similar way (we only give the details for the vector v_{10} and omit the other cases):

$$\begin{array}{ccc} v_{10} & \overset{\cap}{\longmapsto} & (tq)^{-1} \\ {\scriptstyle E/F}\big\downarrow & & \big\downarrow{\scriptstyle E/F} \\ v_{11}/v_{00} & \underset{\cap}{\longmapsto} & 0 \end{array} \qquad \begin{array}{ccc} v_{10} & \overset{\cap}{\longmapsto} & (tq)^{-1} \\ {\scriptstyle K^{\pm 1}}\big\downarrow & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ v_{10} & \underset{\cap}{\longmapsto} & (tq)^{-1} \end{array} \quad .$$

In the end, we look at the crossings (again, we only check v_{10} as an example):

$$\begin{array}{ccc} v_{10} & \overset{\text{II}}{\longmapsto} & -q^{-1}v_{01} \\ {\scriptstyle E/F}\big\downarrow & & \big\downarrow{\scriptstyle E/F} \\ v_{11}/v_{00} & \underset{\Pi}{\longmapsto} & t^{-1}q^{-2}v_{11}/t^{-1}q^{-2}v_{00} \end{array} \qquad \begin{array}{ccc} v_{10} & \overset{\text{II}}{\longmapsto} & -q^{-1}v_{01} \\ {\scriptstyle K^{\pm 1}}\big\downarrow & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ v_{10} & \underset{\Pi}{\longmapsto} & -q^{-1}v_{01} \end{array} \quad .$$

The calculations for Ω and the other basis vectors are done analogously. $\square$

Having constructed lifts for all the basic morphisms, we aim to have a two-parameter analogue of the Jones-Wenzl projector as well. It needs to be generalised to a two-parameter setting consistent with the theory introduced above.

4.2.3 Definition. *Recall the definitions in 2.1.3 and let $a \in \{0,1\}^n$. Define the projection $\pi_n := \pi_n^{t,q} : V_1^{\otimes n} \to V_n$ by*

$$\pi_n^{t,q}(v_a) = (-tq)^{-l(a)} \begin{bmatrix} n \\ |a| \end{bmatrix}^{-1} v_{|a|}.$$

Furthermore, define the inclusion $\iota_n := \iota_n^{t,q} : V_n \to V_1^{\otimes n}$ via

$$\iota_n^{t,q}(v_k) = \sum_{|a|=k} (-tq)^{|a|(n-|a|)-l(a)} v_{|a|},$$

where $k \in \{0,...,n\}$. The two-parameter Jones-Wenzl projector *is the composition $p_n := p_n^{t,q} = \iota_n^{t,q} \circ \pi_n^{t,q}$.*

Clearly, it has to be checked if the morphisms π_n and ι_n are indeed $U_{q,t}(\mathfrak{sl}_2)$-morphisms. We only check it for $n = 2$, the general case can be shown in a similar manner.

4.2.4 Lemma. *The $\mathbb{C}(q,t)$-linear maps π_2 and ι_2 are $U_{q,t}(\mathfrak{sl}_2)$-homomorphisms.*

Proof. First consider the projection π_2 and compute the corresponding values on basis vectors:

$$\begin{array}{ccc} v_{10} & \xmapsto{\pi_2} & \frac{1}{[2]}v_1 \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ v_{11} & \xmapsto[\pi_2]{} & v_2 \end{array} \qquad \begin{array}{ccc} v_{10} & \xmapsto{\pi_2} & \frac{1}{[2]}v_1 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ v_{00} & \xmapsto[\pi_2]{} & v_0 \end{array} \qquad \begin{array}{ccc} v_{10} & \xmapsto{\pi_2} & \frac{1}{[2]}v_1 \\ {\scriptstyle K^{\pm 1}}\downarrow & & \downarrow{\scriptstyle K^{\pm 1}} \\ v_{10} & \xmapsto[\pi_2]{} & \frac{1}{[2]}v_1 \end{array}$$

$$\begin{array}{ccc} v_{01} & \xmapsto{\pi_2} & (-tq)^{-1}\frac{1}{[2]}v_1 \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ (-tq)^{-1}v_{11} & \xmapsto[\pi_2]{} & (-tq)^{-1}v_2 \end{array} \qquad \begin{array}{ccc} v_{01} & \xmapsto{\pi_2} & (-tq)^{-1}\frac{1}{[2]}v_1 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ (-tq)^{-1}v_{00} & \xmapsto[\pi_2]{} & (-tq)^{-1}v_0 \end{array}$$

$$\begin{array}{ccc} v_{01} & \xmapsto{\pi_2} & (-tq)^{-1}\frac{1}{[2]}v_1 \\ {\scriptstyle K^{\pm 1}}\downarrow & & \downarrow{\scriptstyle K^{\pm 1}} \\ v_{01} & \xmapsto[\pi_2]{} & (-tq)^{-1}\frac{1}{[2]}v_1 \end{array}$$

$$\begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ (-tq)v_{10} + v_{01} & \xmapsto[\pi_2]{} & \frac{1}{[2]}((-tq)^{-1} - tq)v_1 = v_1 \end{array} \qquad \begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ 0 & \xmapsto[\pi_2]{} & 0 \end{array}$$

$$\begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ \big\downarrow{\scriptstyle K^{\pm 1}} & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ ((-tq)^{\mp 1})^2 v_{00} & \xmapsto[\pi_2]{} & (-tq)^{\mp 2} v_0 \end{array}$$

$$\begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ \big\downarrow{\scriptstyle E} & & \big\downarrow{\scriptstyle E} \\ 0 & \xmapsto[\pi_2]{} & 0 \end{array} \qquad \begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} \\ (-tq)v_{10} + v_{01} & \xmapsto[\pi_2]{} & \frac{1}{[2]}((-tq)^{-1} - tq)v_1 = v_1 \end{array}$$

$$\begin{array}{ccc} v_{00} & \xmapsto{\pi_2} & v_0 \\ \big\downarrow{\scriptstyle K^{\pm 1}} & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ ((-tq)^{\pm 1})^2 v_{00} & \xmapsto[\pi_2]{} & (-tq)^{\pm 2} v_0 \end{array} \quad .$$

Obviously, all the diagrams commute and hence π_2 is a $U_{q,t}(\mathfrak{sl}_2)$ -morphism. The corresponding considerations for ι_2 yield the following diagrams:

$$\begin{array}{ccc} v_0 & \xmapsto{\iota_2} & v_{00} \\ \big\downarrow{\scriptstyle E} & & \big\downarrow{\scriptstyle E} \\ v_1 & \xmapsto[\iota_2]{} & -tqv_{10} + v_{01} \end{array} \qquad \begin{array}{ccc} v_0 & \xmapsto{\iota_2} & v_{00} \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} \\ 0 & \xmapsto[\iota_2]{} & 0 \end{array} \qquad \begin{array}{ccc} v_0 & \xmapsto{\iota_2} & v_{00} \\ \big\downarrow{\scriptstyle K^{\pm 1}} & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ (-tq)^{\mp 2} v_0 & \xmapsto[\iota_2]{} & ((-tq)^{\mp 1})^2 v_{00} \end{array}$$

$$\begin{array}{ccc} v_1 & \xmapsto{\iota_2} & -tqv_{10} + v_{01} \\ \big\downarrow{\scriptstyle E} & & \big\downarrow{\scriptstyle E} \\ [2]v_2 & \xmapsto[\iota_2]{} & [2]v_{11} = (-tq + (-tq)^{-1})v_{11} \end{array}$$

$$\begin{array}{ccc} v_1 & \xmapsto{\iota_2} & -tqv_{10} + v_{01} \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} \\ [2]v_2 & \xmapsto[\iota_2]{} & [2]v_{00} = (-tq + (-tq)^{-1})v_{00} \end{array} \qquad \begin{array}{ccc} v_1 & \xmapsto{\iota_2} & -tqv_{10} + v_{01} \\ \big\downarrow{\scriptstyle K^{\pm 1}} & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ v_1 & \xmapsto[\iota_2]{} & -tqv_{10} + v_{01} \end{array}$$

$$\begin{array}{ccc} v_2 & \xmapsto{\iota_2} & v_{11} \\ \big\downarrow{\scriptstyle E} & & \big\downarrow{\scriptstyle E} \\ 0 & \xmapsto[\iota_2]{} & 0 \end{array} \qquad \begin{array}{ccc} v_2 & \xmapsto{\iota_2} & v_{11} \\ \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle F} \\ v_1 & \xmapsto[\iota_2]{} & -tqv_{10} + v_{01} \end{array} \qquad \begin{array}{ccc} v_2 & \xmapsto{\iota_2} & v_{11} \\ \big\downarrow{\scriptstyle K^{\pm 1}} & & \big\downarrow{\scriptstyle K^{\pm 1}} \\ (-tq)^{\pm 2} v_2 & \xmapsto[\iota_2]{} & ((-tq)^{\pm 1})^2 v_{11} \end{array} \quad .$$

Again, the diagrams commute and the lemma follows. □

Now we state some properties of the newly defined morphisms.

4.2.5 Lemma. *The formulae from definition 4.2.1 lead to the value* $-[2] = (tq)^{-1} + tq$ *for the unknot coloured by one. Moreover,* $C^2 = ((tq)^{-1} + tq)C$*, where*

$$C = \genfrac{}{}{0pt}{}{\bigcup}{\bigcap} = \cup \circ \cap.$$

Proof. First, we calculate the unknot:

$$(\cap \circ \cup)(1) = \cap(v_{10} + tqv_{01}) = t^{-1}q^{-1} + tq.$$

Now, consider C:

$$\begin{aligned} C(v_{10}) &= (tq)^{-1}(v_{10} + tqv_{01}) = (tq)^{-1}v_{10} + v_{01}; \\ C(v_{01}) &= 1 \cdot (v_{10} + tqv_{01}) = v_{10} + tqv_{01}. \end{aligned}$$

This leads to

$$\begin{aligned} C^2(v_{10}) &= C(t^{-1}q^{-1}v_{10} + v_{01}) \\ &= t^{-2}q^{-2}(v_{10} + tqv_{01}) + (v_{10} + tqv_{01}) \\ &= (t^{-1}q^{-1} + tq)t^{-1}q^{-1}(v_{10} + tqv_{01}) \\ &= (t^{-1}q^{-1} + tq)C(v_{10}); \\ C^2(v_{01}) &= C(v_{10} + tqv_{01}) \\ &= t^{-1}q^{-1}(v_{10} + tqv_{01}) + tq(v_{10} + tqv_{01}) \\ &= (t^{-1}q^{-1} + tq)C(v_{01}) \end{aligned}$$

which concludes the proof. □

4.2.6 Proposition. *The formulae in definition 4.2.1 give another invariant of framed tangles which generalises the coloured Jones polynomial from definition 2.1.2, if the crossings are adjusted by* (4.9) *and* (4.10)*. In the following we will refer to it by* p_2'.

Proof. We begin by checking on the S-move which will be useful in the following.

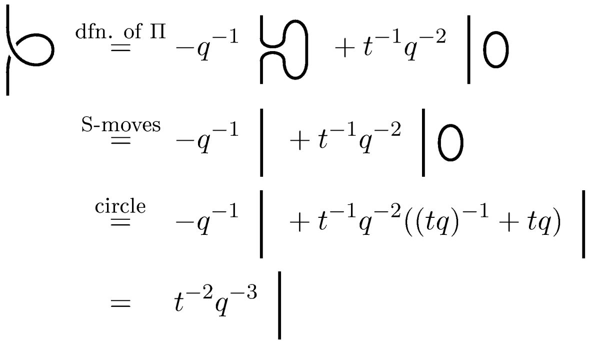
$$\begin{aligned} v_0 &\overset{id\otimes\cup}{\longmapsto} v_{010} + tqv_{001} \overset{\cap\otimes id}{\longmapsto} v_0; \\ v_1 &\overset{id\otimes\cup}{\longmapsto} v_{110} + tqv_{101} \overset{\cap\otimes id}{\longmapsto} tq(t^{-1}q^{-1})v_1 = v_1; \end{aligned}$$

similarily,

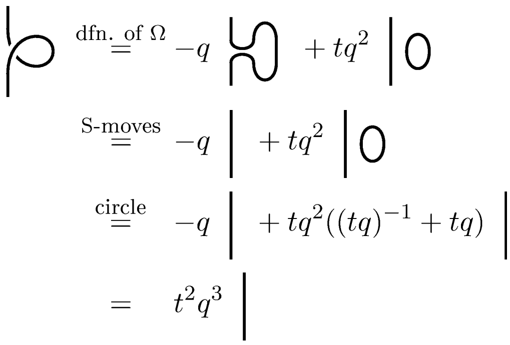
$$\begin{aligned} v_0 &\overset{\cup\otimes id}{\longmapsto} v_{100} + tqv_{010} \overset{id\otimes\cap}{\longmapsto} tqv_0(t^{-1}q^{-1}) = v_0; \\ v_1 &\overset{\cup\otimes id}{\longmapsto} v_{101} + tqv_{011} \overset{id\otimes\cap}{\longmapsto} v_1. \end{aligned}$$

We check [R1'], [R2] and [R3] slightly differently than in the previous section by working more abstractly rather than calculating on the basis vectors.

To [R1']: We obtain

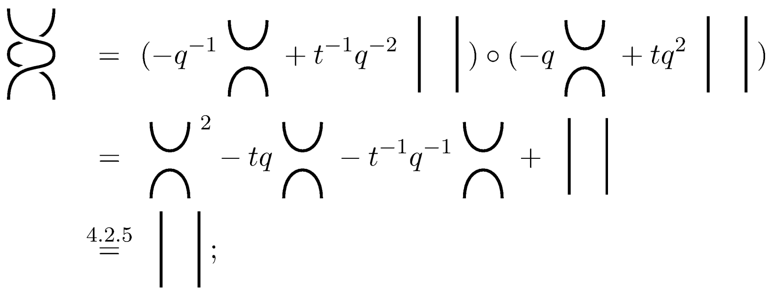


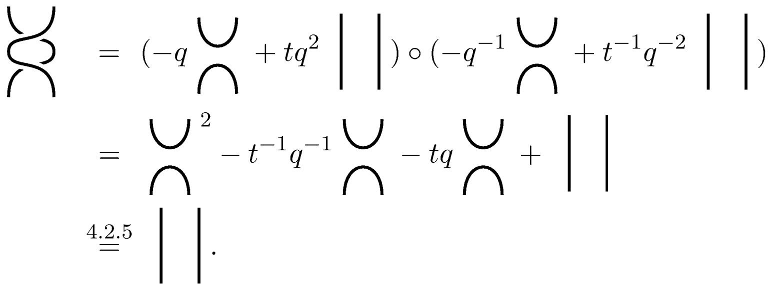
for one curl and



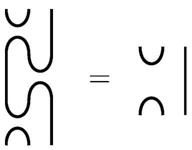
for the other. Again we see that both curls are inverse to each other, but not equal to the identity. Hence we are in a framed tangle setting and [R1'] is satisfied.

To [R2]:

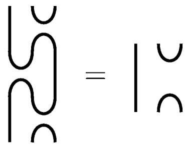




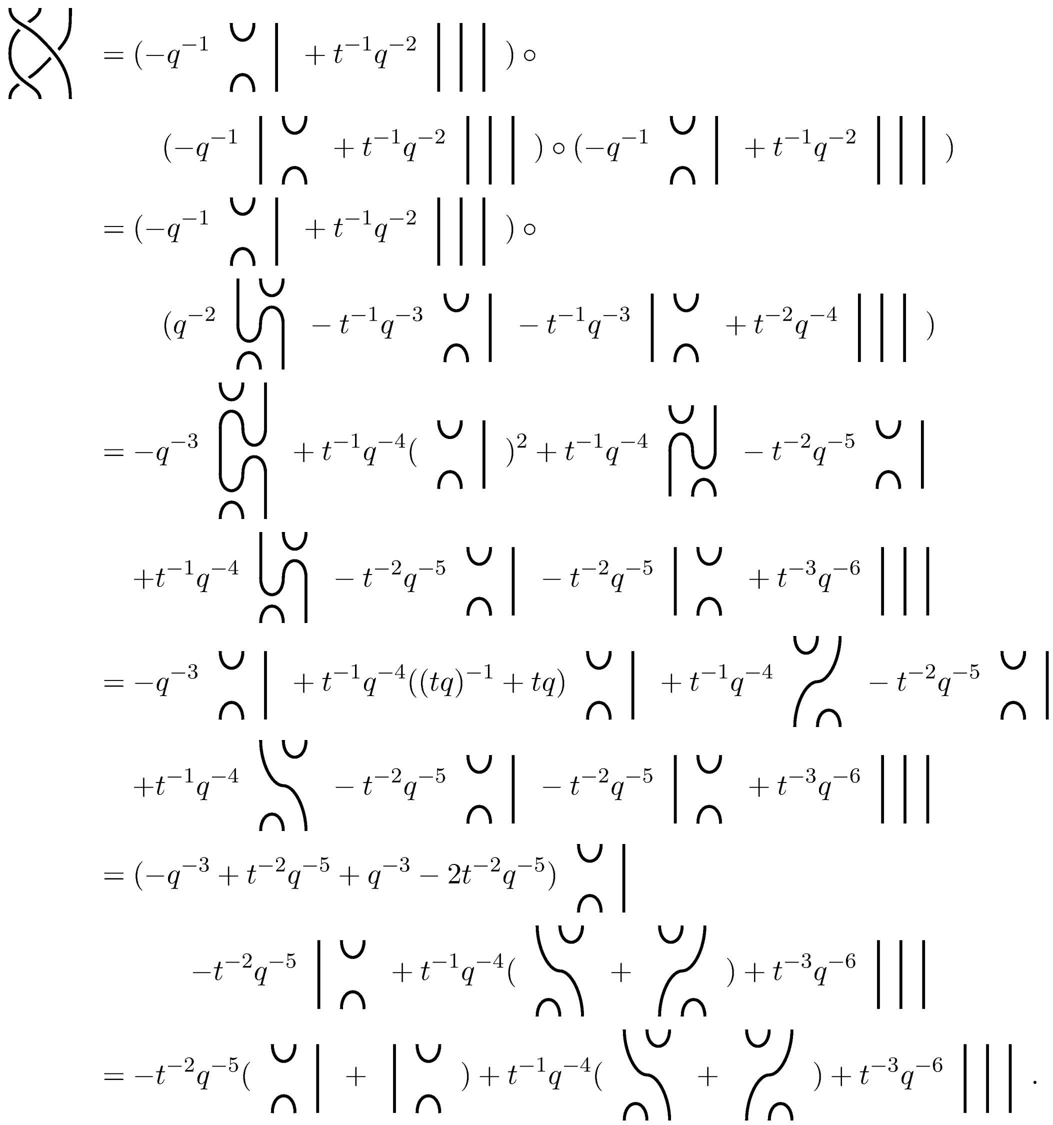
Now show [R3] in a similar manner. Note first that

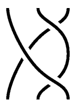


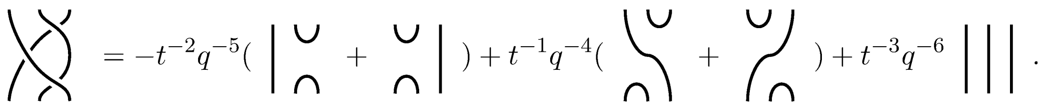
and



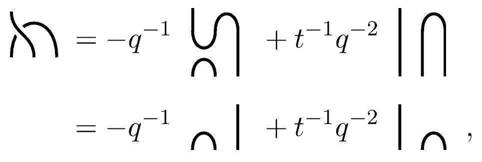
because the involved S-moves can be smoothed.

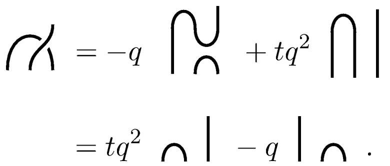


A similar calculation for  also leads to

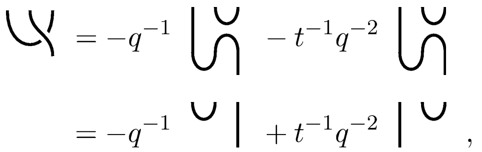


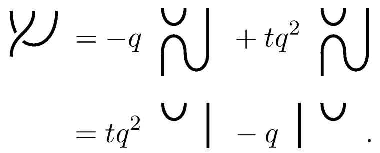
The height moves are obvious, it remains to check the third tangle isotopy [T3].





Observe that, again, the two solutions differ by a factor $-tq^3$. The same is seen if we calculate the upside-down version:



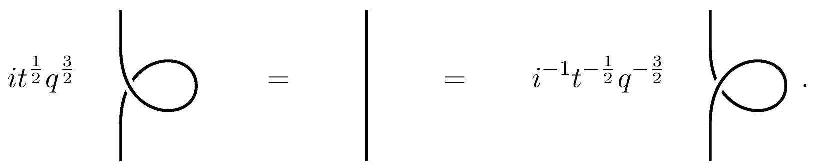


To make them coincide we rescale the crossings again as follows

$$\widetilde{\Pi} = i^{-1}t^{\frac{1}{2}}q^{\frac{3}{2}}\Omega, \tag{4.9}$$
$$\widetilde{\Omega} = it^{-\frac{1}{2}}q^{-\frac{3}{2}}\Pi, \tag{4.10}$$

where i denotes the imaginary unit. For the same reasons as in proposition 3.1.5, this does not affect the validity of the moves shown above. Thus, this yields [T3]. ☐

With the same arguments as in the previous chapter, it is possible now to produce an invariant of oriented tangles satisfying [R1] from our definitions once we define the orientation value β correctly. With (4.9)/(4.10) and the factors from [R1'] it follows that

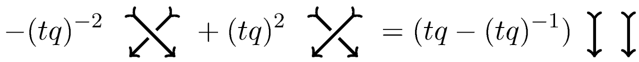


Therefore, we set $\beta := i^{-1}t^{-\frac{1}{2}}q^{-\frac{3}{2}}$.

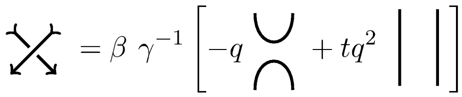
4.2.7 Theorem. *Equip each tangle diagram L with an orientation on each strand. With the definition of β as above and the linking number $\lambda(L)$ as in* (3.15)*, the framed tangle invariant from proposition 4.2.6 yields an invariant of oriented knots and tangles which also satisfies [R1] via the assignment $L \mapsto \beta^{\lambda(L)}p_2'(L) =: p_2(L)$.*

Proof. The assumption follows from the above with the same arguments as in the proof of 3.1.9. □

4.2.8 Remark. *The oriented tangle invariant p_2 satisfies the skein relation*

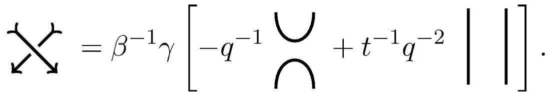


which uniquely determines the invariant together with the value of the unknot given by Lemma 4.2.5. Indeed, the relation can be deduced as follows (with the abbreviation $\gamma = (-tq^3)^{\frac{1}{2}}$)



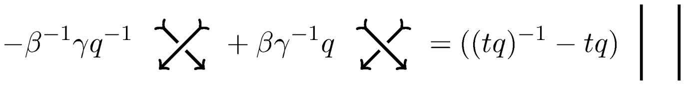
(4.11)

and

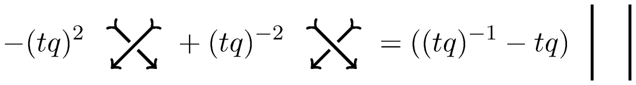


(4.12)

Therefore,



or equivalently, when plugging in the values of β and γ,



which implies the claim. Note that by setting $v = (tq)^{-1}$ and using Lemma 4.2.5 we obtain the skein relations of the (normalised) Jones polynomial from [Kho00]. Hence, our tangle invariant p_2 is a 2-parameter version of the Jones polynomial.

4.3 Our two examples

As in the previous chapter, we analyse how the values of the unknot and the small theta-network change by applying the new definitions.

4.3.1 Proposition. *The unknot coloured by n has value $(-1)^n[n+1] = \sum_{i=0}^{n}(tq)^{2i-n}$ if the formulae from chapter 4 are used.*

Proof. Again, we work analogously to proposition 2.2.7, but now with the definitions from chapter 4.
Step 1:

$$\hat{\delta}'_n(1) = \sum_{i=0}^{n-1}(tq)^i \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} v_a \otimes v_{10} \otimes v_{mir(a)} + tqv_a \otimes v_{01} \otimes v_{mir(a)}.$$

This is proven just as in Lemma 2.2.2.
Step 2:

$$\begin{aligned}(id^{\otimes n}\otimes\pi_n)\circ\hat{\delta}'_n(1) &= \sum_{i=0}^{n-1}(tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}} v_{a\cup 1}\otimes(-tq)^{-l(0\cup mir(a))}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ v_i\\ &\quad +tqv_{a\cup 0}\otimes(-tq)^{-l(1\cup mir(a))}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ v_{i+1}\\ &= \sum_{i=0}^{n-1}(tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}(-tq)^{-(l(a)+i)}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ v_{a\cup 1}\otimes v_i\\ &\quad -(-tq)^{1-l(a)}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ v_{a\cup 0}\otimes v_{i+1}\\ &=: \hat{\delta}_n(1).\end{aligned}$$

Step 3:

$$\begin{aligned}&(id^{\otimes n}\otimes\iota_n)\circ\hat{\delta}_n(1)\\ &= \sum_{i=0}^{n-1}(tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}(-tq)^{-(l(a)+i)}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ v_{a\cup 1}\otimes\sum_{|b|=n-i}(-tq)^{i(n-i)-l(b)}v_b\\ &\quad -(-tq)^{1-l(a)}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)}(-tq)^{(i+1)(n-(i+1))-l(b)}v_b\\ &= \sum_{i=0}^{n-1}(tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}(-tq)^{-l(a)-i+i(n-i)}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ v_{a\cup 1}\otimes\sum_{|b|=n-i}(-tq)^{-l(b)}v_b\\ &\quad -(-tq)^{1-l(a)+(i+1)(n-(i+1))}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ v_{a\cup 0}\otimes\sum_{|b|=n-(i+1)}(-tq)^{-l(b)}v_b\\ &=: \star\star\,.\end{aligned}$$

Step 4:

$$\begin{aligned}\varepsilon'_n(\star\star) &= \sum_{i=0}^{n-1}(tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}(-tq)^{-l(a)-i+i(n-i)}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ (-tq)^{-l(a\cup 1)}((tq)^{-1})^{n-i}1^i\\ &\quad -(-tq)^{1-l(a)+(i+1)(n-(i+1))}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ (-tq)^{-l(a)}((tq)^{-1})^{n-(i+1)}1^{i+1}\\ &= \sum_{i=0}^{n-1}(-1)^i(-tq)^i\sum_{\substack{|a|=n-1-i\\ a\in\{0,1\}^{n-1}}}(-tq)^{-l(a)-i+i(n-i)-l(a\cup 1)}\frac{1}{\begin{bmatrix}n\\ i\end{bmatrix}}\ (-1)^{n-i}(-tq)^{-(n-i)}\\ &\quad -(-tq)^{1-l(a)+(i+1)(n-(i+1))-l(a)}\frac{1}{\begin{bmatrix}n\\ i+1\end{bmatrix}}\ (-1)^{n-(i+1)}(-tq)^{-(n-(i+1))}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \sum_{i=0}^{n-1} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{-l(a)-i+i(n-i)-l(a)+i-i-(n-i)} \\
&\quad + \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} (-tq)^{1-l(a)+(i+1)(n-(i+1))-l(a)-(n-(i+1))+i} \\
&= (-1)^n \sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{i(n-i)-n} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} (-tq)^{-2l(a)} \\
&\quad + \sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i+1 \end{bmatrix}} (-tq)^{(i+1)(n-(i+1))-n+2(i+1)} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} (-tq)^{-2l(a)} \\
&= (-1)^n \sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{i(n-i)-i+i-n} \sum_{\substack{|a|=n-1-i \\ a\in\{0,1\}^{n-1}}} (-tq)^{-2l(a)} \\
&\quad + (-1)^n \sum_{i=1}^{n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{i(n-i)-n+2i} \sum_{\substack{|a|=n-1-(i-1) \\ a\in\{0,1\}^{n-1}}} (-tq)^{-2l(a)} \\
&\overset{\spadesuit}{=} (-1)^n \left(\sum_{i=0}^{n-1} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + \sum_{i=1}^{n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \right) \\
&= (-1)^n \left(\sum_{i=0}^{n} \frac{1}{\begin{bmatrix} n \\ i \end{bmatrix}} (-tq)^{-n+2i+n-2i} ((-tq)^{i-n} \begin{bmatrix} n-1 \\ i \end{bmatrix} + (-tq)^{i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}) \right) \\
&\overset{\clubsuit}{=} (-1)^n \sum_{i=0}^{n} (-tq)^{2i-n} \begin{bmatrix} n \\ i \end{bmatrix}^{1-1} \\
&= \sum_{i=0}^{n} (tq)^{2i-n}.
\end{aligned}
$$

For ($\spadesuit$) we used the formula

$$
(-tq)^{i(n-i)} \sum_{|b|=n-i} (-tq)^{-2l(b)} = \begin{bmatrix} n \\ i \end{bmatrix}
$$

which can be achieved from proposition 2.2.5 by exchanging q by $(-tq)$. Furthermore, we get $\clubsuit$ via the observations of (2.3) and a two-parameter version of Lemma 1.1.11 where we also replace q by $(-tq)$. This completes the proof. $\square$

Finally, we look at the small theta network introduced in section 3.2.

4.3.2 Proposition. *With the definitions from section 4 we get*

$$
\vartheta_{2,2,2}(1) = -\frac{1}{[2]}((tq)^{-4} + 4 + (tq)^4) - \frac{1}{[2]^3}((tq)^4 - 2 + (tq)^{-4}).
$$

Proof. We calculate $\vartheta_{2,2,2}$ as in proposition 3.2.8, carefully exchanging the formulae for $\cup$ and $\cap$. Since the calculations are otherwise completely analogous to the latter proposition, we will omit some of the details in each step.

$$
\begin{aligned}
1 &\overset{\cup}{\longmapsto} v_{10} + tqv_{01} \\
&\overset{id\otimes\cup\otimes\cup\otimes id}{\longmapsto} v_{110100} + tqv_{110010} + tqv_{101100} + t^2q^2v_{101010} \\
&\qquad +tqv_{010101} + t^2q^2v_{010011} + t^2q^2v_{001101} + t^3q^3v_{001011} \\
&\overset{\pi_2\otimes\pi_2\otimes\pi_2}{\longmapsto} -\frac{1}{[2]}(tq)^{-1}v_{210} + \frac{1}{[2]}tqv_{201} + \frac{1}{[2]}tqv_{120} + \frac{1}{[2]^3}((tq)^2 - (tq)^{-2})v_{111} \\
&\qquad -\frac{1}{[2]}tqv_{102} - \frac{1}{[2]}tqv_{021} + \frac{1}{[2]}(tq)^3v_{012} \\
&\overset{\iota_2\otimes\iota_2\otimes\iota_2}{\longmapsto} \frac{1}{[2]}v_{111000} - \frac{1}{[2]}(tq)^{-1}v_{110100} - \frac{1}{[2]}(tq)^2v_{110010} \\
&\qquad +\frac{1}{[2]}tqv_{110001} - \frac{1}{[2]}(tq)^2v_{101100} + \frac{1}{[2]}tqv_{011100} \\
&\qquad +\frac{1}{[2]^3}((tq)^2 - (tq)^{-2})(-(tq)^3v_{101010} + (tq)^2v_{100110} + (tq)^2v_{011010} - tqv_{010110} \\
&\qquad\qquad +(tq)^2v_{101001} - tqv_{100101} - tqv_{011001} + v_{010101}) \\
&\qquad +\frac{1}{[2]}(tq)^2v_{100011} - \frac{1}{[2]}tqv_{010011} + \frac{1}{[2]}(tq)^2v_{001110} \\
&\qquad -\frac{1}{[2]}tqv_{001101} - \frac{1}{[2]}(tq)^4v_{001011} + (tq)^3\frac{1}{[2]}v_{000111} \\
&\overset{id\otimes\cap\otimes\cap\otimes id}{\longmapsto} -\frac{1}{[2]}(tq)^{-1}v_{10}(tq)^{-2} - \frac{1}{[2]}(tq)^2v_{10}(tq)^{-1} - \frac{1}{[2]}(tq)^2v_{10}(tq)^{-1} \\
&\qquad +\frac{1}{[2]^3}((tq)^2 - (tq)^{-2})(-(tq)^3v_{10} + v_{01}(tq)^{-2}) \\
&\qquad -\frac{1}{[2]}tqv_{01}(tq)^{-1} - \frac{1}{[2]}tqv_{01}(tq)^{-1} - \frac{1}{[2]}(tq)^4v_{01} \\
&\overset{\cap}{\longmapsto} -\frac{1}{[2]}((tq)^{-4} + 4 + (tq)^4) - \frac{1}{[2]^3}((tq)^4 - 2 + (tq)^{-4}).
\end{aligned}
$$

□

4.3.3 Remark. *We see that for $t = -1$, we get the formula from 3.2.8 which is just the formula in the one-parameter case.*

5 The third variation p_3

In the end, we consider the setting that was introduced in [Kho00] to construct a last two-parameter polynomial. In fact, we now generalise a version of Kauffman's bracket polynomial first introduced in 1987 (see [Kau87]).

5.0.4 Definition ([Kho00], section 2.4)**.** *Let D be an arbitrary oriented link diagram. We assign a Laurent polynomial $\langle L\rangle \in \mathbb{Z}[q, q^{-1}]$ to D via*

1. $\left\langle \bigcirc \right\rangle = v + v^{-1}$.
2. *Each crossing is a linear combination of two simple resolutions of*

$$\left\langle \times \right\rangle = \left\langle \genfrac{}{}{0pt}{}{\smile}{\frown} \right\rangle - v \left\langle \,|\;|\, \right\rangle ,$$

   hence,

$$\left\langle \times \right\rangle = -v \left(\left\langle \genfrac{}{}{0pt}{}{\smile}{\frown} \right\rangle - v^{-1} \left\langle \,|\;|\, \right\rangle \right).$$

3. *For the disjoint union $D_1 \sqcup D_2$ of two diagrams D_1 and D_2 it follows that*

$$\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle .$$

This definition implies the validity of the three Reidemeister moves. The orientation is included as follows.

Let $x(D)$ and $y(D)$ be the number of positive and negative crossings in the sense of chapter 3. Then define

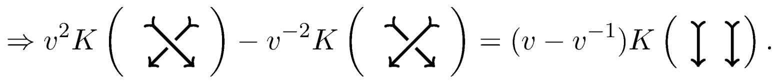
$$K(D) = (-1)^{x(D)} v^{y(D)-2x(D)} \langle D \rangle .$$

Khovanov states that this quantity is an invariant of oriented links and calls it *scaled Kauffman bracket* because a normalisation leads to the Kauffman bracket as introduced in [Kau87]. We obtain the following skein relation:

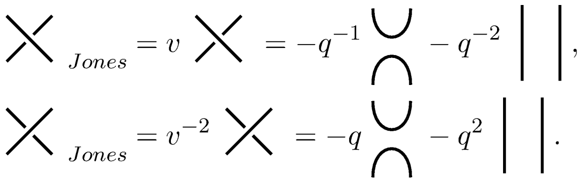
$$K\left(\times \right) = -v^{-2} \left(\left\langle \genfrac{}{}{0pt}{}{\smile}{\frown} \right\rangle - v \left\langle \,|\;|\, \right\rangle \right),$$

$$K\left(\times \right) = -v \left(v(\left\langle \genfrac{}{}{0pt}{}{\smile}{\frown} \right\rangle - v^{-1} \left\langle \,|\;|\, \right\rangle) \right)$$

$$= -v^{2} \left(\left\langle \genfrac{}{}{0pt}{}{\smile}{\frown} \right\rangle - v^{-1} \left\langle \,|\;|\, \right\rangle \right)$$

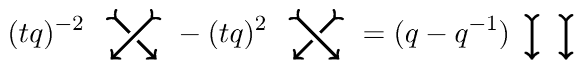


If $v = -q^{-1}$, we get the skein relation of the Jones polynomial and we regain its crossings via



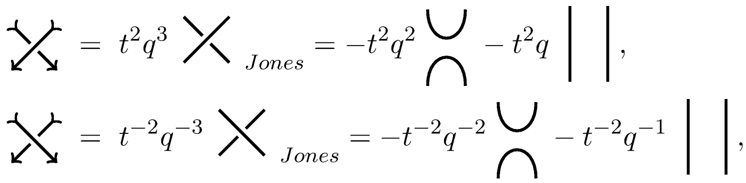
The orientation is realised by multiplying with $\beta^{\lambda(D)}$, where $\beta = q^{-3}$. If we want to introduce the second parameter now, we just set $\beta_{q,t} = t^{-2}q^{-3}$. We conclude these considerations by stating the following proposition.

5.0.5 Proposition. *The above defined scaled Kauffman bracket yields a two-parameter invariant p_3 of oriented knots as described above. Its skein relation is*

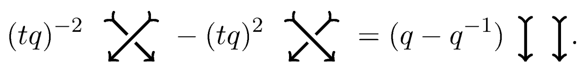


which specialises to the skein relation of the Jones polynomial for $t = -1$.

Proof. We show the validity of the skein relation.



thus yielding



□

Note that the polynomial p_3 only provides a second parameter if at least one of the two crossings is involved. Thus, we obtain the following:

5.0.6 Proposition. *Let the quantum number $[n]$ be defined as in 1.1.1.*

(i) Using p_3, the unknot coloured by n has the value $(-1)^n[n+1]$.

(ii) Moreover, we obtain $\vartheta_{2,2,2}(1) = -\frac{1}{[2]}(q^{-4}+4+q^4) - \frac{1}{[2]^3}(q^{-4}-2+q^4)$ if p_3 is used.

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