NAZAROV-WENZL ALGEBRAS, COIDEAL SUBALGEBRAS AND CATEGORIFIED SKEW HOWE DUALITY

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Abstract. We describe how certain cyclotomic Nazarov-Wenzl algebras occur as endomorphism rings of projective modules in a parabolic version of BGG category $\mathcal{O}$ of type D. Furthermore we study a family of subalgebras of these endomorphism rings which exhibit similar behaviour to the family of Brauer algebras even when they are not semisimple. The translation functors on this parabolic category $\mathcal{O}$ are studied and proven to yield a categorification of a coideal subalgebra of the general linear Lie algebra. Finally this is put into the context of categorifying skew Howe duality for these subalgebras.

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Introduction

In [BK09] a remarkable connection between the representation theory of Hecke algebras and Lusztig's canonical bases was established by showing that cyclotomic Hecke algebras are isomorphic to cyclotomic quotients of quiver Hecke algebras introduced in [KL09], [KL08], where also a connection between the representation theory of these algebras and Lusztig's geometric construction of canonical bases was predicted. This prediction was verified,

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[VV11], and so therefore a connection between representations of cyclotomic Hecke algebras and canonical bases was established. In this way, cyclotomic Hecke algebras inherit an interesting grading which in type A can also be obtained from the graded versions of parabolic category $\mathcal{O}$'s and hence be described in terms of type A Kazhdan-Lusztig polynomials, see [BS11b], [HM11]. In other types however, Schur-Weyl duality connects the Lie algebra with a centralizer algebra (Brauer algebra) different from the group algebra of any Weyl group. In this paper we investigate relations between parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$, Brauer algebras and their degenerate affine versions $\mathbb{W}_d = \mathbb{W}_d(\Xi)$, depending on a parameter set Ξ, and their cyclotomic quotients, [AMR06]. The algebras $\mathbb{W}_d(\Xi)$ were introduced in [Naz96] based on work of Wenzl, we call them therefore *VW-algebra* an abbreviation of the German 'verallgemeinerte Wenzl algebra'.[1] These families are the Brauer algebra analogues of the cyclotomic Hecke algebras, but in contrast to them not well understood and so far slightly neglected; maybe also because of the lack of a good combinatorial description and geometric realization. The main goal of the paper is to connect these algebras to category $\mathcal{O}$ and its Kazhdan-Lusztig combinatorics and in this way obtain canonical bases of representations for coideal subalgebras in quantum groups.

We start with a type D analogue of the Arakawa-Suzuki theorem:

Theorem A. *Let M be a highest weight module in $\mathcal{O}(\mathfrak{so}_{2n})$. With an appropriate choice of Ξ there is an algebra homomorphism*

$$\Psi_M : \quad \mathbb{W}_d \longrightarrow \operatorname{End}_{\mathfrak{g}}(M \otimes (\mathbb{C}^{2n})^{\otimes d})^{\mathrm{opp}}.$$

In general this morphism is not surjective and it is impossible to describe the kernel. We study in detail the case where M is a parabolic Verma module $M^{\mathfrak{p}}(\lambda)$ for a maximal parabolic subalgebra of type A inside type D. We show that cyclotomic quotients $\mathbb{W}_d(\alpha,\beta)$ of level 2 occur as endomorphism rings if we choose $\lambda = \delta\omega_0$ as an appropriate multiple of a fundamental weight:

Theorem B (see Theorem 3.1)**.** *If $n \geq 2d$ and $\delta \in \mathbb{Z}$ then*

$$\mathbb{W}_d(\alpha,\beta) \cong \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes (\mathbb{C}^{2n})^{\otimes d})^{\mathrm{opp}}.$$

We deduce that the $\mathbb{W}_d(\alpha,\beta)$'s inherit a positive Koszul grading from the graded version $\hat{\mathcal{O}}$, of category $\mathcal{O}$, hence a geometric interpretation; in terms of first perverse sheaves on isotropic Grassmannians, [ES13b], and second topological Springer fibres, [ES12], via the Khovanov algebra of type D. This Khovanov algebra also allows us, [ES13a], to mimic (with some effort) the approach from [BS12] to construct a graded version of the Brauer algebra $\mathrm{Br}(\delta)$ for an arbitrary integral parameter δ. Since this requires passing to orthosymplectic Lie superalgebras, we just identify here a subalgebra $z_d\mathbb{W}_d(\alpha,\beta)z_d$ of the cyclotomic quotient, given by an idempotent z_d and show that this algebra shares properties of the Brauer algebra:

[1]Keeping in mind that VW can be seen as a degeneration of BMW, the affine Birman-Murakawi-Wenzl algebras.

Theorem C (see Proposition 4.4 and Theorem 4.5)**.** .

(1) *The algebra* $z_d\mathbb{W}_d(\alpha,\beta)z_d$ *has dimension* $(2d-1)!!$.
(2) *The algebra* $z_d\mathbb{W}_d(\alpha,\beta)z_d$ *is semisimple if and only if* $\delta \neq 0$ *and* $\delta \geq d-1$ *or* $\delta = 0$ *and* $d = 1,3,5$.

These connections give a conceptual explanation for the fact, [CDVM09b] and [CDVM09a], that the decomposition numbers of Brauer algebras are given by the combinatorics of Weyl groups of type D.

Theorems A and B and [ES13a] rely on a good understanding of (a graded version of) tensoring with the natural representation $\mathbb{C}^{2n}$ on category $\mathcal{O}^{\mathfrak{p}}$. Since $\mathbb{C}^{2n}$ is self-dual and hence $\_ \otimes \mathbb{C}^{2n}$ self-adjoint, we do not get an action of a quantum group action on our categories, but rather of certain *coideal subalgebras* $\mathcal{H}$ and $\mathcal{H}^{\jmath}$ of $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{Z}})$ so-called quantum symmetric pairs in [Let03], [Let02], [Kol12]. These are quantum group analogues of the fixed point subalgebra $\mathfrak{gl}_{\mathbb{N}} \times \mathfrak{gl}_{-\mathbb{N}}$ inside $\mathfrak{gl}_{\mathbb{Z}}$. Since the integral weights of $\mathfrak{so}_{2n}$ can be partitioned into integer or half-integer weights in the standard ϵ-basis, the integral part, $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$, of parabolic category $\mathcal{O}$ can be decomposed into two subcategories $\mathcal{O}^{\mathfrak{p}}_1(\mathfrak{so}_{2n})$ and $\mathcal{O}^{\mathfrak{p}}_{\jmath}(\mathfrak{so}_{2n})$ stable under tensoring with the natural representation. We obtain two categorifications

Theorem D (see Proposition 6.25)**.**

(1) *The* $\mathcal{H}$*-module* $\bigwedge^n \mathbb{C}^{\mathbb{Z}}$ *is categorified by* $\hat{\mathcal{O}}^{\mathfrak{p}}_1(\mathfrak{so}_{2n})$.
(2) *The* $\mathcal{H}^{\jmath}$*-module* $\bigwedge^n \mathbb{C}^{\mathbb{Z}+\frac{1}{2}}$ *is categorified by* $\hat{\mathcal{O}}^{\mathfrak{p}}_{\jmath}(\mathfrak{so}_{2n})$.

The classes of Verma modules correspond hereby to the standard basis. The involved categories have a contravariant duality, hence the categorification equips the modules on the left with a bar-involution, see Proposition 6.30. Mimicking Lusztig's approach for quantum groups, we define canonical bases on the above modules and show that the classes of simple modules correspond to the canonical basis. Theorem D gives a new instance of based categorifications in the context of category $\mathcal{O}$, but now connecting canonical bases of Hecke algebras with canonical bases of quantum symmetric pairs instead of quantum groups as for instance in [BS10], [FKS06], [Sar13], [Web10], see [Maz12] for an overview. The base change matrix is here given by parabolic type D Kazhdan-Lusztig polynomials; see [LS13] for explicit formulas in the Grassmannian case.

Finally we investigate generalizations of these modules from the viewpoint of skew Howe duality, [How92], and its categorification. For this we consider more general parabolic category $\mathcal{O}$'s and their block decompositions for Levi subalgebras isomorphic to products of $\mathfrak{gl}_k$'s. Denote the sum of these blocks by $\bigoplus_\Gamma \mathcal{O}_\Gamma(n)$. Considering analogous projective functors in this setup we categorify the two actions of $\mathfrak{gl}_m \times \mathfrak{gl}_m$ and $\mathfrak{gl}_r \times \mathfrak{gl}_r$ on the vector space $\bigwedge(n,m,r) := \bigwedge^n(\mathbb{C}^m \otimes \mathbb{C}^2 \otimes \mathbb{C}^r)$ separately and then show they are graded derived equivalent via Koszul duality. Under this identification we can realize **both** commuting actions on the **same** category. The projective functors turn into derived Zuckerman functors under this identification.

Theorem E (see Proposition 7.6 and Theorem 7.14)**.** *The bimodule $\bigwedge(n,m,r)$ is categorified by $\bigoplus_\Gamma \mathcal{O}_\Gamma(n)$. Furthermore, the two categorifications for the left and right action are Koszul dual to each other.*

This skew Howe duality can be squeezed (and quantized) in between two type A skew Howe dualities as follows. For an integer m let V_m be the natural $\mathfrak{gl}_m$-module. Fix integers n,m,r and consider the $\mathfrak{gl}_m \times \mathfrak{gl}_r$-module $\bigwedge^n(V_m \otimes V_r)$. By skew Howe duality, [How92], the weight spaces of the $\mathfrak{gl}_m$-action are representations of $\mathfrak{gl}_r$ and vice versa; and the decomposition is the following

$$\bigoplus_\alpha \bigwedge^{\alpha_1} V_m \otimes \cdots \otimes \bigwedge^{\alpha_r} V_m \cong \quad \bigwedge^n(V_m \otimes V_r) \quad \cong \bigoplus_\beta \bigwedge^{\beta_1} V_r \otimes \cdots \otimes \bigwedge^{\beta_m} V_r$$

where the first isomorphism is as $\mathfrak{gl}_m$-module and the second as $\mathfrak{gl}_r$-module. The sums run over all compositions (possibly with zero parts) of n with r respectively m parts. Now consider the restriction $V_{2m} = V_m \oplus V_m$ to $\mathfrak{gl}_m$, where $\mathfrak{gl}_m$ is embedded diagonally into $\mathfrak{gl}_m \times \mathfrak{gl}_m \subset \mathfrak{gl}_{2m}$. Theorem E gives a categorification of the middle piece of diagram

$$\begin{array}{ccccc}
\mathfrak{gl}_{2m} & \curvearrowright & \bigoplus_\alpha \bigwedge^{\alpha_1} V_{2m} \otimes \bigwedge^{\alpha_2} V_{2m} \otimes \cdots \otimes \bigwedge^{\alpha_r} V_{2m} & \curvearrowleft & \mathfrak{gl}_r \\
\cup & & & & \cap \\
\mathfrak{gl}_m \times \mathfrak{gl}_m & \curvearrowright & \bigwedge(n,m,r) & \curvearrowleft & \mathfrak{gl}_r \times \mathfrak{gl}_r \\
\cup & & & & \cap \\
\mathfrak{gl}_m & \curvearrowright & \bigoplus_\gamma \bigwedge^{\gamma_1} V_m \otimes \bigwedge^{\gamma_2} V_m \otimes \cdots \otimes \bigwedge^{\gamma_{2r}} V_m & \curvearrowleft & \mathfrak{gl}_{2r}
\end{array}$$

In fact, the top and bottom parts can be quantized, [CKM12], and were already categorified using parabolic-singular category $\mathcal{O}$'s in [MS09] and fit into the framework of [LQR12]. In our categorification of the middle part the commuting actions of the two Lie algebras, viewed as specializations of the coideal subalgebras mentioned above, are given by translation and derived Zuckerman functors respectively. Since this construction has a graded lift using $\hat{\mathcal{O}}$ there is a quantized version of skew Howe duality with the commuting actions of the coideal subalgebras.

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1. Basics

Throughout this paper let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{so}_{2n}$ defined over the complex numbers. We denote by $U(\mathfrak{g})$ its enveloping algebra and by $Z(U(\mathfrak{g}))$ the center of $U(\mathfrak{g})$. In Section 2 we will fix a specific presentation for $\mathfrak{so}_{2n}$.

We denote by ϵ_i the standard basis of the dual of the Cartan $\mathfrak{h}^*$ with simple roots $\alpha_i = \epsilon_{i+1} - \epsilon_i$ for $1 \leq i \leq n-1$ and $\alpha_0 = \epsilon_1 + \epsilon_2$. By abuse of notation we denote by $\mathcal{O}$ the sum of all *integral blocks* of the BGG-category $\mathcal{O}$ of $\mathfrak{g}$. We fix the maximum standard parabolic $\mathfrak{p}$ of type A corresponding to the roots $\alpha_1, \ldots, \alpha_{n-1}$ and denote the associated parabolic subcategory of $\mathcal{O}$ by $\mathcal{O}^{\mathfrak{p}}(n)$, i.e. the subcategory of $\mathcal{O}$ of all $\mathfrak{p}$-finite modules. The combinatorics of this parabolic subcategory was studied in [ES13b] and we recall some of the notations used therein.

A weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^*$ of $\mathfrak{so}_{2n}$ is integral if it is in the $\mathbb{Z}$-span or the $(\mathbb{Z}+\frac{1}{2})$-span of the ϵ_i's. In the former case we say the weight is *supported on the integers*, in the latter that it is *supported on the half-integers*.

For $\lambda \in \mathfrak{h}^*$ let $M^{\mathfrak{p}}(\lambda)$ denote the parabolic Verma module of highest weight λ, that is the maximal quotient of the ordinary Verma module $M(\lambda)$ which is locally $\mathfrak{p}$-finite. Explicitly, $M^{\mathfrak{p}}(\lambda) = 0$ if λ is not $\mathfrak{p}$-dominant and otherwise

$$M^{\mathfrak{p}}(\lambda) \quad = \quad U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E(\lambda) \tag{1}$$

where $E(\lambda)$ denotes the finite dimensional $\mathfrak{l}$-module of highest weight λ for the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ inflated trivially to a $\mathfrak{p}$-module. The set of $\mathfrak{p}$-dominant weights will be denoted by

$$\begin{aligned}
\Lambda \quad &= \quad \left\{\lambda \in \mathfrak{h}^* \text{ integral} \mid M^{\mathfrak{p}}(\lambda) \neq 0\right\} \\
&= \quad \left\{\lambda \in \mathfrak{h}^* \text{ integral} \mid \lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \text{ where } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\right\} \\
&= \quad \left\{\lambda \in \mathfrak{h}^* \text{ integral} \mid \lambda + \rho = \sum_{i=1}^{n} \lambda'_i \epsilon_i \text{ where } \lambda'_1 < \lambda'_2 < \cdots < \lambda'_n\right\}.
\end{aligned} \tag{2}$$

where ρ denotes the half-sum of positive roots, $\rho = \sum_{i=1}^{n}(i-1)\epsilon_i$ or $\rho = (0, 1, 2, \ldots, n-1)$ in the ϵ-basis. It decomposes into a disjoint union $\Lambda^{1} \cup \Lambda^{𝅗𝅥}$, the weights supported on the integers and those supported on half-integers. Since up to the last section we only work with parabolic Verma modules, we just call them by abuse of language for short Verma modules.

The action of the center $U(\mathfrak{g})$ decomposes each module into generalized eigenspaces, which induces a decomposition $\mathcal{O} = \oplus_{\chi} \mathcal{O}_{\chi}$ of the category $\mathcal{O}$ for $\mathfrak{g}$ indexed (via the Harish-Chandra isomorphism) by orbits of integral weights under the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ of the Weyl group W_n of $\mathfrak{g}$. The decomposition of $\mathcal{O}$ induces a decomposition of the subcategory $\mathcal{O}^{\mathfrak{p}}(n)$. We denote by $\mathcal{O}^{\mathfrak{p}}_{\lambda}(n)$ the summand containing $M^{\mathfrak{p}}(\lambda)$, i.e. the summand corresponding to the orbit through $\lambda + \rho$. Note that the ordinary action as well as the dot-action of the Weyl group on integral weights preserve the lattices of weights supported on integers resp. on half-integers. Hence, we have a decomposition

$$\mathcal{O}^{\mathfrak{p}}(n) = \mathcal{O}^{\mathfrak{p}}_{1}(n) \oplus \mathcal{O}^{\mathfrak{p}}_{\lrcorner}(n), \tag{3}$$

where $\mathcal{O}^{\mathfrak{p}}_{1}(n)$ (resp. $\mathcal{O}^{\mathfrak{p}}_{\lrcorner}(n)$) is the direct sum of all $\mathcal{O}^{\mathfrak{p}}_{\lambda}(n)$, such that λ is supported on the integers (resp. half-integers).

Our goal is to equip the Grothendieck groups of $\mathcal{O}^{\mathfrak{p}}_{1}(n)$ and $\mathcal{O}^{\mathfrak{p}}_{\lrcorner}(n)$ with the structure of a representation of a certain coideal subalgebra induced by the action of translation functors, see Section 6. To make this explicit it is helpful to use the combinatorics from [ES13b, Section 2.2] to identify $\mathfrak{p}$-dominant weights with diagrammatic weights, allowing us to describe the blocks of category $\mathcal{O}^{\mathfrak{p}}(n)$ combinatorially. Since we want to distinguish the combinatorics for the two subcategories in (3) we slightly modify the indexing sets for diagrammatic weights from [ES13b].

Definition 1.1. We denote by $\mathbb{X}_n$ the set of sequences $a = (a_i)_{i\in\mathbb{Z}_{\geq 0}}$ such that $a_i \in \{\wedge, \vee, \times, \circ, \diamond\}$, $a_i \neq \diamond$ for $i \neq 0$, $a_0 \in \{\circ, \diamond\}$, and

$$\#\{a_i \mid a_i \in \{\wedge, \vee, \diamond\}\} + 2\#\{a_i \mid a_i = \times\} = n.$$

Elements of $\mathbb{X}_n$ are called *diagrammatic weights supported on the integers.*

Remark 1.2. The translation from a diagrammatic weight $a \in \mathbb{X}$ to a diagrammatic weight $\widetilde{a}$ in the sense of [ES13b] is done by setting $\widetilde{a}_i = a_{i-1}$, except for $i = 1$ and $a_0 = \diamond$ where we choose $\widetilde{a}_1 \in \{\wedge, \vee\}$ such that the total number of $\vee$'s is even. In the language of [ES13b] we have thus always fixed the even parity for these blocks where $a_0 = \diamond$.

Similarly we have a set-up for half-integers.

Definition 1.3. We denote by $\mathbb{X}^{\lrcorner}_n$ the set of sequences $a = (a_i)_{i\in\mathbb{Z}_{\geq 0}+\frac{1}{2}}$ such that $a_i \in \{\wedge, \vee, \times, \circ\}$ and $\#\{a_i \mid a_i \in \{\wedge, \vee\}\} + 2\#\{a_i \mid a_i = \times\} = n$. We call these elements *diagrammatic weights supported on the half-integers.*

Remark 1.4. Translating a diagrammatic weight $a \in \mathbb{X}^{\lrcorner}$ to a diagrammatic weight $\widetilde{a}$ in the sense of [ES13b] is done by putting $\widetilde{a}_i = a_{i-\frac{1}{2}}$.

Obviously a diagrammatic weight a is uniquely determined by the sets

$$P_{\star}(a) = \{i \mid a_i = \star\},$$

for $\star \in \{\wedge, \vee, \times, \circ, \diamond\}$. Given a $\mathfrak{p}$-dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, denote $\lambda' = (\lambda'_1, \ldots, \lambda'_n) = \lambda + \rho$ and let a_λ be the diagrammatic weight defined by

$$\begin{aligned}
P_{\vee}(\lambda) &= \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\},\\
P_{\wedge}(\lambda) &= \{-\lambda'_i \mid \lambda'_i < 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\},\\
P_{\times}(\lambda) &= \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ appears in } \lambda'\},\\
P_{\diamond}(\lambda) &= \{\lambda'_i \mid \lambda'_i = 0\},\\
P_{\circ}(\lambda) &= \begin{cases} \mathbb{Z}_{\geq 0} \setminus (P_{\vee} \cup P_{\wedge} \cup P_{\times} \cup P_{\diamond}) & \text{if } \lambda \text{ is supported on integers,}\\ \mathbb{Z}_{\geq 0} + \frac{1}{2} \setminus (P_{\vee} \cup P_{\wedge} \cup P_{\times}) & \text{if } \lambda \text{ is supported on half-integers.} \end{cases}
\end{aligned}$$

The assignment $\lambda \mapsto a_\lambda$ defines a bijection between $\mathfrak{p}$-dominant weights and $\mathbb{X}_n \cup \mathbb{X}_n^{\lrcorner}$. In the following we will not distinguish between a weight and a diagrammatic weight and denote both by λ. We use the following identifications

$$K_0\left(\mathcal{O}_1^{\mathfrak{p}}(n)\right) \cong \langle \mathbb{X}_n \rangle_{\mathbb{Q}} \quad \text{and} \quad K_0\left(\mathcal{O}_{\lrcorner}^{\mathfrak{p}}(n)\right) \cong \left\langle \mathbb{X}_n^{\lrcorner} \right\rangle_{\mathbb{Q}},$$

between the $\mathbb{Q}$-vector spaces on basis $\mathbb{X}_n$ resp. $\mathbb{X}_n^{\lrcorner}$ and the Grothendieck group scalar extended to $\mathbb{Q}$ by sending the class of a parabolic Verma module of highest weight λ to a_λ.

Our weight dictionary induces an action of the Weyl group W_n on diagrammatic weights corresponding to the dot-action on weights for $\mathfrak{g}$. Two diagrammatic weights are in the same orbit (and thus the corresponding Verma modules have the same central character) if and only if one is obtained from the other by a finite sequence of changes of the following form: swapping a $\vee$ with an $\wedge$ or replacing two $\vee$'s with two $\wedge$'s or two $\wedge$'s with two $\vee$'s (keeping all $\times$'s and $\circ$'s untouched), see [ES13b]. Orbits of diagrammatic weights, called *diagrammatic blocks* are given by fixing the positions of the $\times$'s and $\circ$'s (called block diagram in [ES13b, Section 2.2]) and the parity of $\#\vee + \#\times$ of any of its weights. Moving $\vee$'s to the left or turning two $\wedge$'s into two $\vee$'s makes the weight bigger (with respect to the standard ordering on weights). Note that diagrammatic blocks correspond precisely to blocks of $\mathcal{O}^{\mathfrak{p}}(n)$ by sending Γ to the summand $\mathcal{O}^{\mathfrak{p}}(n)_\Gamma$ containing all Verma modules with highest weight in Γ which is indeed a block of the category.

2. Brauer and VW-algebra

2.1. **VW-algebras and translated highest weight modules.** The main purpose of this section is a generalization of the Arakawa-Suzuki action, [AS98], to the Lie algebra of type D_n. The replacement of the degenerate affine Hecke algebra is the affine Nazarov-Wenzl algebra $\mathbb{W}_d(\Xi)$.

Definition 2.1. Let $d \in \mathbb{N}$. We fix a set Ξ of parameters $w_k \in \mathbb{C}$, $k \geq 0$. Then the *affine Nazarov-Wenzl algebra* $\mathbb{W}_d = \mathbb{W}_d(\Xi)$, short *VW-algebra*, is generated by

$$s_i, e_i, y_j \qquad 1 \leq i \leq d-1, 1 \leq i \leq d, k \in \mathbb{N}, \tag{4}$$

subject to the relations (for $1 \leq a, b \leq d-1$, $1 \leq c < d-1$, and $1 \leq i, j \leq d$):

(VW.1) $s_a^2 = 1$
(VW.2) (a) $s_a s_b = s_b s_a$ for $|\, a-b \,| > 1$
(b) $s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}$
(c) $s_a y_i = y_i s_a$ for $i \notin \{a, a+1\}$
(VW.3) $e_a^2 = w_0 e_a$
(VW.4) $e_1 y_1^k e_1 = w_k e_1$ for $k \in \mathbb{N}$
(VW.5) (a) $s_a e_b = e_b s_a$ and $e_a e_b = e_b e_a$ for $|\, a-b \,| > 1$
(b) $e_a y_i = y_i e_a$ for $i \notin \{a, a+1\}$
(c) $y_i y_j = y_j y_i$

(VW.6) (a) $e_a s_a = e_a = s_a e_a$
(b) $s_c e_{c+1} e_c = s_{c+1} e_c$ and $e_c e_{c+1} s_c = e_c s_{c+1}$
(c) $e_{c+1} e_c s_{c+1} = e_{c+1} s_c$ and $s_{c+1} e_c e_{c+1} = s_c e_{c+1}$
(d) $e_{c+1} e_c e_{c+1} = e_{c+1}$ and $e_c e_{c+1} e_c = e_c$
(VW.7) $s_a y_a - y_{a+1} s_a = e_a - 1$ and $y_a s_a - s_a y_{a+1} = e_a - 1$
(VW.8) (a) $e_a(y_a + y_{a+1}) = 0$
(b) $(y_a + y_{a+1})e_a = 0$

Remark 2.2. Relations (6b), (6c), (6d), (7) come in pairs and it is in fact sufficient to either require the first set of relations or the second, the other is then satisfied automatically. All relations are symmetric or come in symmetric pairs, thus we have a canonical isomorphism $\mathbb{W}_d(\Xi) \cong \mathbb{W}_d(\Xi)^{\mathrm{opp}}$.

From now on we fix a natural number $n \geq 4$ and set $N = 2n$. Let $I^+ := \{1, \dots, n\}$ and $I := I^+ \cup -I^+$. We denote by V the vector space with basis $\{v_i \mid i \in I\}$ and by $\mathfrak{gl}(I)$ its corresponding Lie algebra of endomorphisms, viewed as the matrices with respect to the chosen basis. Let J be the matrix such that $J_{kl} = \delta_{k,-l}$ for $k, l \in I$ with respect to the chosen basis. If we order columns and rows decreasing from top to bottom and left to right this is the matrix with ones on the anti-diagonal and zeros elsewhere.

Definition 2.3. The Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}$ is the Lie subalgebra of $\mathfrak{gl}(I)$ of all matrices A satisfying $JA + A^t J = 0$; that is all matrices which are skew-symmetric with respect to the anti-diagonal, $A_{i,j} = -A_{-j,-i}$. In terms of the bilinear form $\langle -, - \rangle$ on V defined by J we thus have $\langle Xv, w \rangle + \langle v, Xw \rangle = 0$.

Fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ given by all diagonal matrices and a basis $\{\epsilon_i \mid i \in I^+\}$ for $\mathfrak{h}^*$ such that the weight of v_i is ϵ_i if $i \in I^+$ and the weight of v_i is $-\epsilon_i$ if $i \in I^-$. For any $\alpha \in R_n = R(\mathfrak{so}_{2n}) = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\}$ fix a root vector X_α of weight α and for $i \in I^+$ let X_i be the element in $\mathfrak{h}$ dual to ϵ_i. Then $\{X_\gamma \mid \gamma \in B_n\}$ with $B_n = R_n \cup I^+$ form a basis of $\mathfrak{so}_{2n}$. We set $\mathfrak{n}^+ = \langle X_{\epsilon_i \pm \epsilon_j} \mid i > j \rangle$, and $\mathfrak{n}^- = \langle X_{-(\epsilon_i \pm \epsilon_j)} \mid i > j \rangle$ and fix the Borel subalgebra $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. In this notation the natural representation V is the irreducible representation $L(\epsilon_n)$ with highest weight ϵ_n, the fundamental weight corresponding to $\alpha_{n-1} = \epsilon_n - \epsilon_{n-1}$. Furthermore, the $X_{\pm(\epsilon_i - \epsilon_j)}$'s for $i > j$ together with $\mathfrak{h}$ form a Levi subalgebra $\mathfrak{l}$ isomorphic to $\mathfrak{gl}_n$ with corresponding standard parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ from Section 1.

For $i \in I$ denote by $v_i^* = v_{-i}$ the basis element dual to v_i with respect to $\langle -, - \rangle$ and for X_γ denote by X_γ^* the element dual to X_γ with respect to the Killing form of $\mathfrak{so}_{2n}$.

Definition 2.4. Let M be a $\mathfrak{g}$-module. For $d \geq 0$ consider $M \otimes V^{\otimes d}$. The linear endomorphisms $\tau, \sigma : V \otimes V \longrightarrow V \otimes V$ defined as

$$\tau: \quad v \otimes w \quad \mapsto \quad \langle v, w \rangle \sum_{i \in I} v_i \otimes v_i^*, \tag{5}$$

$$\sigma: \quad v \otimes w \quad \mapsto \quad w \otimes v. \tag{6}$$

induce the following endomorphisms s_i, e_i of $M \otimes V^{\otimes d}$ for $1 \le i \le d-1$

$$s_i = \mathrm{Id} \otimes \mathrm{Id}^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{Id}^{\otimes(d-i-3)}, \tag{7}$$

$$e_i = \mathrm{Id} \otimes \mathrm{Id}^{\otimes(i-1)} \otimes \tau \otimes \mathrm{Id}^{\otimes(d-i-3)}. \tag{8}$$

By definition of the comultiplication it is obvious that s_i is a $\mathfrak{g}$-homomorphism and using the compatibility of $\mathfrak{g}$ and the bilinear form it immediately follows that e_i is as well (see also Remark 2.6), hence both are in $\mathrm{End}_{\mathfrak{g}}\left(M \otimes V^{\otimes d}\right)$.

Definition 2.5. The *pseudo Casimir element* in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined as

$$\Omega = \sum_{\gamma \in B_n} X_\gamma \otimes X_\gamma^*. \tag{9}$$

It is connected to the ordinary *Casimir element* $C = \sum_{\gamma \in B_n} X_\gamma X_\gamma^*$ by the formula

$$\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C),$$

where Δ denotes the comultiplication of $\mathfrak{g}$. Denote by c_λ the value by which C acts on a module of highest weight λ. For $0 \le i < j \le d$ we define

$$\Omega_{ij} = \sum_{\gamma \in B_n} 1 \otimes \ldots \otimes X_\gamma \otimes 1 \otimes \ldots \otimes 1 \otimes X_\gamma^* \otimes 1 \otimes \ldots \otimes 1, \tag{10}$$

where X_γ is at position i and x_γ^* is at position j. Multiplication with $\Omega_{i,j}$ defines an element $\Omega_{i,j} \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$ and we finally set for $i \in I$

$$y_i = \sum_{0 \le k < i} \Omega_{ki} + \left(\frac{2n-1}{2}\,\mathrm{Id}\right), \tag{11}$$

By definition of Ω it is clear that y_i is a $\mathfrak{g}$-endomorphism on $M \otimes V^{\otimes d}$.

Remark 2.6. The representation $V \otimes V$ decomposes as a $\mathfrak{g}$-module into the irreducible representations $L(0)$, $L(2\epsilon_n)$, and $L(\epsilon_n + \epsilon_{n-1})$. A small computation shows that on $V \otimes V$ the following equation holds

$$\Omega = -\mathrm{pr}_{L(0)}(c_{\epsilon_n}\mathrm{id} + \sigma) + \sigma.$$

Note that τ is a quasi-projection from $V \otimes V$ onto the copy of the trivial representation $L(0)$ inside $V \otimes V$. A small computation using the explicit form of Ω given above shows that multiplication with Ω on $V \otimes V$ is equal to the morphism $\sigma - \tau$.

Recall that a *highest weight module* for $\mathfrak{g}$ is a $\mathfrak{g}$-module M which is generated by a non-zero vector $m \in M$ satisfying $\mathfrak{n}^+ m = 0$ and $\mathfrak{h} m \subseteq \mathbb{C}m$. Note that it satisfies $\mathrm{End}_{\mathfrak{g}}(M) = \mathbb{C}$. Associated with $\lambda \in \mathfrak{h}^*$ and the corresponding 1-dimensional module $\mathbb{C}_\lambda$ we have the (ordinary) Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ of highest weight λ and its irreducible quotient $L(\lambda)$.

Lemma 2.7. *If M is a highest weight module then for each $k \in \mathbb{N}$, there exist $a_k(M) \in \mathbb{C}$ such that $e_1 y_1^k e_1 = a_k(M) e_1$ as elements in* $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$ *with $a_0(M) = N$.*

Proof. The first equality follows directly from the definitions. Recalling Remark 2.6, we first let $d = 2$ and consider the composition

$$f : M = M \otimes L(0) \longrightarrow M \otimes V \otimes V \xrightarrow{y_1^k} M \otimes V \otimes V \xrightarrow{e_1} M \otimes L(0) = M,$$

where the first map is the canonical inclusion. Now f is an endomorphism of M, hence must be a multiple, say $a_k(M)$, of the identity. By pre-composing with e_1 we obtain $e_1 y_1^k e_1 = a_k(M) e_1$. This identity also holds for $d > 2$ since we just add a couple of identities on the following tensor factors. □

Theorem 2.8. *Let M be a highest weight module for $\mathfrak{g} = \mathfrak{so}_{2n}$ and $\Xi_M = \{a_k(M) \mid k \geq 0\}$ as in Lemma 2.7. Then there is a well-defined right action of $\mathbb{W}_d(2n)$ on $M \otimes V^{\otimes d}$ defined by*

$$p.s_i = s_i(p), \quad p.e_i = e_i(p), \quad p.y_j = y_j(p), \quad p.w_k = a_k(M)p$$

for $p \in M \otimes V^{\otimes d}$, $1 \leq i \leq r-1$, $1 \leq j \leq r$ and $k \in \mathbb{Z}$. In particular, we get an algebra homomorphism

$$\Psi_M = \Psi_M^{d,n} : \quad \mathbb{W}_d(\Xi_M) \quad \longrightarrow \quad \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})^{\mathrm{opp}}.$$

Proof. To prove the statement we need to show that the assignment respects the relations of the degenerate affine Wenzl algebra. This will be done in separate Lemmas in the Appendix. Relation (1) is obvious, as are relations (2a) and (2b), while (2c) follows from Lemma 8.1. Relation (3) is obvious as well and relation (4) follows from Lemma 2.7. The relations (6a)-(6d) follow from Lemma 8.2, Relation (5a) is trivial as well, while (5b) follows from Lemma 8.4 and (5c) from Lemma 8.5. Finally relation (7) follows from Lemma 8.3 and relations (8a)-(8b) from Lemma 8.6. □

As one can see from the proof of Theorem 2.8 the parameters of the degenerate affine Wenzl algebra depend on n and the highest weight of M. This is different from the type A situation. There, the degenerate affine Hecke algebra H_d acts on the endofunctor $\mathcal{E}^r := \_ \otimes V^{\otimes r}$ of $\mathcal{O}(\mathfrak{gl}_n)$ for any n, with V being the natural representation of $\mathfrak{gl}_n$. This property plays an important role in the context of categorification of modules over quantum groups, [KL09], [Rou08], [BK08], [BS11b]. To achieve a similar situation one could enlarge the algebra to the algebra generated by e_i, s_i $1 \leq i \leq d-1$ plus an extra central generator w_0 subject to the same relations as before except that we have to drop relation (4). If one works with the Birman-Murakawi-Wenzl algebra instead a similar action is given in [OR07].

Remark 2.9. The action defined here can also be modified to give an action of $\mathbb{W}_d(\Xi_M)$ for a highest weight module M for a Lie algebra of types B or C and the respective defining representation as V.

2.2. **Brauer algebras.** Let $r \in \mathbb{Z}_{\geq 0}$. A *Brauer diagram* on $2r$ vertices is a partitioning b of the set $\{1, 2, \ldots, r, 1^*, 2^*, \ldots r^*\}$ into r subsets of cardinality 2. Let $\mathcal{B}[r]$ be the set of such Brauer diagrams. A Brauer diagram can be displayed graphically by arranging $2r$ vertices in two rows

$1, 2, \dots, r$ and $1, 2, \dots, r^*$, such that each vertex is linked to precisely one other vertex. Two such diagrams are considered to be the same if they link the same r pairs of points. Special Brauer diagrams are the "unit" $1 = \{\{1, 1^*\}, \{2, 2^*\}, \cdots, \{r, r^*\}\}$ connecting always j with j^* for all $1 \leq j \leq r$, and for $1 \leq i \leq r-1$ the $\tilde{s}_i$ (respectively $\tilde{e}_i$) which connects j with j^* except of the new pairs $\{i, (i+1)^*\}, \{i+1, i^*\}$ (respectively $\{i, i+1\}, \{i^*, (i+1)^*\}$) involving the numbers i and $i+1$.

Example 2.10.

(12)

Given two Brauer diagrams b and b', their concatenation $b \circ b'$ is obtained by identifying vertex i^* in b with vertex i in b' and removing all the internal loops. Let $c(b, b')$ be the number of loops removed. Brauer introduced the following algebra [Bra37].

Definition 2.11. Let $d \in \mathbb{Z}_{\geq 0}$ and $\delta \in \mathbb{C}$. The *Brauer algebra* $\mathrm{Br}_d(\delta)$ is the $\mathbb{C}$-algebra with basis b, $b \in \mathcal{B}[d]$ and multiplication $bb' = \delta^{c(b,b')} b \circ b'$.

The algebra is associative with unit 1 and generated by the elements $\tilde{s}_i$, $\tilde{e}_i$, $1 \leq i \leq d-1$ modulo the relations from Lemma 8.2 together with $\tilde{e}_i^2 = \delta$ for $1 \leq i \leq n-1$, hence the defining relations are

$$\begin{gathered}\tilde{s}_a^2 = 1, \quad \tilde{s}_a\tilde{s}_b = \tilde{s}_b\tilde{s}_a, \quad \tilde{s}_c\tilde{s}_{c+1}\tilde{s}_c = \tilde{s}_{c+1}\tilde{s}_c\tilde{s}_{c+1}, \\ (\tilde{e}_a)^2 = \delta\tilde{e}_a \quad \tilde{e}_c\tilde{e}_{c+1}\tilde{e}_c = \tilde{e}_{c+1} \quad \tilde{e}_{c+1}\tilde{e}_c\tilde{e}_{c+1} = \tilde{e}_c \\ \tilde{s}_a\tilde{e}_a = \tilde{e}_a = \tilde{e}_a\tilde{s}_a, \quad \tilde{s}_c\tilde{e}_{c+1}\tilde{e}_c = \tilde{s}_{c+1}\tilde{e}_c \quad \tilde{s}_{c+1}\tilde{e}_c\tilde{e}_{c+1} = \tilde{s}_c\tilde{e}_{c+1}\end{gathered} \tag{13}$$

Note that $\mathrm{Br}_d(\delta)$ is of dimension $d!!$, where for any natural number m we set $m!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot m'$ where $m' = m$ if m odd and $m' = m-1$ if m even.

Remark 2.12. If $M = L(0)$ is the trivial representation, then the action from Theorem 2.8 factors through the quotient $B_d(N) = \mathbb{W}_d(\Xi_{L(0)})/(y_1 - \frac{N-1}{2})$ which is canonically isomorphic to the Brauer algebra $\mathrm{Br}_d(N)$, see [Naz96, (2.2)]. In this case $w_a = N(\frac{N-1}{2})^a$ for $a \geq 0$. In case $N \geq r$, Theorem 2.8 turns then into the classical action of the Brauer algebra on tensor space, see e.g. [GW09], in particular $\mathrm{Br}_d(N)$ is semisimple.

The algebra $\mathrm{Br}_d(\delta)$ is generically semisimple, [Wen88]; for $\delta \geq 0$ it is semisimple in precisely the following cases, see [Bro56] or [Rui05]:

$$\delta \neq 0, \text{ and } \delta \geq d-1 \quad \text{ or } \quad \delta = 0, \text{ and } d = 1, 3, 5. \tag{14}$$

We are interested to relate the non-semisimple Brauer algebras $\mathrm{Br}_d(\delta)$ for $\delta \geq 0$ to VW-algebras.

2.3. **Cyclotomic quotients and admissibility.** Recall from [AMR06, Definition 2.10] that the parameters w_a, $a \geq 0$ are *admissible* if they satisfy the following admissibility condition

$$w_{2a+1} + \frac{1}{2}w_{2a} - \frac{1}{2}\sum_{b=1}^{2a}(-1)^{b-1}w_{b-1}w_{2a-b+1} = 0 \tag{15}$$

Example 2.13. The values $w_a := N(\frac{N-1}{2})^a$ for $a \geq 0$ from Remark 2.12 form an admissible sequence.

Admissibility ensures the existence of a nice basis of $\mathbb{W}_d$ as follows: Fix a Brauer diagram $b \in \mathcal{B}[d]$. By Remark 2.12 we can write it as a product of generators s_i, e_i. Fix such an expression and consider the corresponding expression B in the VW-algebra. More generally, given $\gamma, \eta \in \mathbb{Z}^r_{\geq 0}$ and $b \in \mathcal{B}[d]$ we have the monomial $y_1^{\gamma_1}y_2^{\gamma_2}\cdots y_d^{\gamma_d}By_1^{\eta_1}y_2^{\eta_2}\cdots y_d^{\eta_d} \in \mathbb{W}_d$. A monomial of this form is *regular* if $\gamma_i \neq 0$ implies i is the left endpoint of a horizontal arc in b, and $\eta_i = 0$ if i^* is the left endpoint of a horizontal arc in b, see (27). These monomials form a basis for $\mathbb{W}_d$ by [Naz96, Theorem 4.6] in case the parameters are admissible.

Given $\mathbf{u} = (u_1, u_2, \ldots, u_l) \in \mathbb{C}^l$ we denote by $\mathbb{W}_d(\Xi; \mathbf{u})$ the quotient

$$\mathbb{W}_d(\Xi, \mathbf{u}) = \mathbb{W}_d(\Xi)/\prod_{i=1}^{l}(y_1 - u_i) \tag{16}$$

and call it the *cyclotomic VW-algebra of level* l with parameters $\mathbf{u}$. Under some additional admissibility condition, [AMR06, Theorem A, Prop. 2.15], the above basis is compatible with the cyclotomic quotients:

Proposition 2.14. *If the* w_a*,* $a \geq 0$ *are* $\mathbf{u}$*-admissible then* $\mathbb{W}_d(\mathbf{u})$ *has dimension* $l^d(2d-1)!!$*. The regular monomials with* $0 \leq \gamma_i, \eta_i < l$ *for* $1 \leq i \leq r$ *form a basis* $\mathbb{B}(\mathbb{W}_d(\mathbf{u}))$*.*

For the definition of $\mathbf{u}$-admissibility see [AMR06, Def. 3.6]. We only need the special example from [AMR06, Lemma 3.5]:

Example 2.15. Assume the entries of $\mathbf{u}$ are pairwise distinct and non-zero. Then the $w_a = \sum_{i=1}^{l}(2u_i - (-1)^l)u_i^a \prod_{1 \leq j \neq i \leq l} \frac{u_i + u_j}{u_i - u_j}$ for $a \geq 0$ form a $\mathbf{u}$-admissible sequence.

2.4. **The special case** $M = M^{\mathfrak{p}}(\underline{\delta})$**.** In the following we study Theorem 2.8 in detail in the special case where M is a specific type of parabolic Verma module. Recall our choice of parabolic $\mathfrak{p}$ from Section 1.

To $\delta \in \mathbb{Z}$ we associate the weight $\underline{\delta} = \delta\omega_0 = \frac{\delta}{2}\sum_{i=1}^{n}\epsilon_i$, where $\omega_0 = \sum_{i-1}^{n}\epsilon_i$ denotes the fundamental weight of $\mathfrak{g}$ corresponding to α_0. In particular, $\underline{\delta} \in \Lambda$ and we have the Verma module $M^{\mathfrak{p}}(\underline{\delta})$.

Lemma 2.16. *For* $\lambda \in \Lambda$*,* $M^{\mathfrak{p}}(\lambda) \otimes V$ *has a filtration with sections isomorphic to* $M^{\mathfrak{p}}(\lambda \pm \epsilon_j)$ *for all* $j \in I^+$ *such that* $\lambda \pm \epsilon_j \in \Lambda$ *and each of these Verma modules appearing exactly once.*

Proof. This is a standard consequence of the definition (1) and the tensor identity; see e.g. [Hum08, Theorem 3.6]. □

Applying Lemma 2.16 iteratively, we obtain a bijection between Verma modules $M^{\mathfrak{p}}(\mu)$ appearing as subquotients in a Verma filtration of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ and d-admissible weight sequences ending at μ where the latter is defined as follows: For fixed $d \geq 1$ and $\delta \geq 0$ a weight $\mu \in \Lambda$ is called *d-admissible for δ* if there is a sequence

$$\underline{\delta} = \lambda^1 \to \lambda^2 \to \cdots \to \lambda^d = \mu \tag{17}$$

of length d, starting at $\underline{\delta}$ and ending at μ, of weights in Λ such that λ^{i+1} differs from λ^i by adding precisely one weight of V, i.e. there exists $j \in I^+$ such that $\lambda^{i+1} = \lambda^i \pm \epsilon_j$. For instance, there are eight 2-admissible weight sequences for $\underline{\delta}$.

$$\begin{array}{ll}
\underline{\delta} \to \underline{\delta} - \epsilon_1 \to \underline{\delta} - 2\epsilon_1, & \underline{\delta} \to \underline{\delta} - \epsilon_1 \to \underline{\delta} - \epsilon_1 - \epsilon_2, \\
\underline{\delta} \to \underline{\delta} - \epsilon_1 \to \underline{\delta}, & \underline{\delta} \to \underline{\delta} - \epsilon_1 \to \underline{\delta} - \epsilon_1 + \epsilon_n, \\
\underline{\delta} \to \underline{\delta} + \epsilon_n \to \underline{\delta} + 2\epsilon_n, & \underline{\delta} \to \underline{\delta} + \epsilon_n \to \underline{\delta} + \epsilon_n + \epsilon_{n-1}, \\
\underline{\delta} \to \underline{\delta} + \epsilon_n \to \underline{\delta}, & \underline{\delta} \to \underline{\delta} + \epsilon_n \to \underline{\delta} + \epsilon_n - \epsilon_1.
\end{array}$$

Proposition 2.17. *There is an isomorphism of $\mathfrak{g}$-modules $M(\underline{\delta}) \otimes V \cong M(\underline{\delta} - \epsilon_1) \oplus M(\underline{\delta} + \epsilon_n)$. This is an eigenspace decomposition for the action of y_1. The eigenvalues are $\alpha = \frac{1}{2}(1 - \delta)$ and $\beta = \frac{1}{2}(\delta + N - 1)$.*

Proof. By Lemma 2.16, $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$ has a Verma flag of length two with the asserted Verma modules appearing. The filtration obviously splits since they have different central character. The Casimir $C = \sum_{\gamma \in B_n} X_\gamma X_\gamma^*$ acts on a highest weight module with highest weight λ by $c_\lambda = \langle \lambda, \lambda + 2\rho \rangle$, see e.g. [Mus12, Lemma 8.5.3] and on the tensor product $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$ as $\Delta(C) = C \otimes 1 + 1 \otimes C + \Omega_{0,1}$. Hence $y_1 - \frac{N-1}{2}$ acts on the summands $M^{\mathfrak{p}}(\underline{\delta} + \nu)$, $\nu = -\epsilon_1, \epsilon_n$ of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V$ by

$$\frac{1}{2}\left(\langle \delta + \nu, \delta + \nu + 2\rho \rangle - \langle \delta, \delta + 2\rho \rangle - \langle \epsilon_n, \epsilon_n + 2\rho \rangle\right) = \begin{cases} \frac{-\delta}{2} - (n-1) & \text{if } \nu = -\epsilon_1 \\ \frac{\delta}{2} & \text{if } \nu = \epsilon_n \end{cases}$$

The statement follows now from the definition of α and β. □

Corollary 2.18. *For $M = M^{\mathfrak{p}}(\underline{\delta})$, the action from Theorem 2.8 factors through the cyclotomic quotient with parameters (α, β) inducing an algebra homomorphism*

$$\Psi^{d,n}_{M(\underline{\delta})}: \quad \mathbb{W}_d(\Xi_{M(\underline{\delta})}; \alpha, \beta) \quad \longrightarrow \quad \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}}. \tag{18}$$

Definition 2.19. From now on set $\alpha = \frac{1}{2}(1 - \delta)$ and $\beta = \frac{1}{2}(\delta + N - 1)$ and abbreviate $\mathbb{W}_d = \mathbb{W}_d(\Xi_{M(\underline{\delta})})$ and $\mathbb{W}_d(\alpha, \beta) = \mathbb{W}_d(\Xi_{M(\underline{\delta})}; \alpha, \beta)$.

Lemma 2.20. *The elements w_a, $a \geq 0$ in $\mathbb{W}_d$ satisfy the recursion formula $w_0 = N$, $w_1 = N\frac{N-1}{2}$, and for $a \geq 2$*

$$w_a = (\alpha + \beta) w_{a-1} - \alpha\beta w_{a-2}. \tag{19}$$

Proof. By definition we have $e_1y_1^0e_1 = e_1^2 = Ne_1$, hence $w_0 = N$. On the other hand for any $j \in I^+$ we have

$$e_1y_1e_1(m \otimes v_j \otimes v_j^*) \quad = \quad \sum_{k \in I} e_1y_1(m \otimes v_k \otimes v_k^*)$$

Recalling (9), (11) this is equal to

$$\begin{aligned} & e_1 \sum_{k\in I}\left(\sum_{i\in I^+} X_i^* m \otimes X_i v_k \otimes v_k^* + \sum_{\alpha\in R_n} X_\alpha m \otimes X_\alpha^* v_k \otimes v_k^*\right) \\ + \quad & \frac{N(N-1)}{2} e_1 m \otimes v_j \otimes v_j^* = \\ & e_1 \sum_{i\in I^+}\left(\sum_{k\in I^+} X_i^* m \otimes \epsilon_k(X_i) v_k \otimes v_k^* - \sum_{k\in I^-} X_i^* m \otimes \epsilon_k(X_i) v_k \otimes v_k^*\right) \\ + \quad & \frac{N(N-1)}{2} e_1 m \otimes v_j \otimes v_j^*. \end{aligned}$$

We obtain $w_1 = \frac{1}{2}(N-1)N$. Finally, $y_1^2 = (\alpha+\beta)y_1 - \alpha\beta$ by Proposition 2.17, and hence $e_1y_1^ne_1 = (\alpha+\beta)e_1y_1^{n-1}e_1 - \alpha\beta e_1y_1^{n-2}e_1$. □

Lemma 2.21. *The w_a from Lemma 2.20 are explicitly given as*

$$w_a \quad = \quad N\sum_{k=0}^{a} \alpha^{a-k}\left(\frac{N}{2}-\alpha\right)^k - \frac{N}{2}\sum_{k=0}^{a-1}\alpha^{a-1-k}\left(\frac{N}{2}-\alpha\right)^k \tag{20}$$

Proof. For $a = 0, 1$ the formula is obviously correct. Note that the recursion formula (19) has the general solution $w_a = A\alpha^a + B\beta^a$ with boundary conditions $A + B = N$ and $A\alpha + B\beta = \frac{N}{2}(N-1)$, see e.g. [LP98, Theorem 33.10]. Hence

$$A \quad = \quad \frac{1}{\alpha-\beta}\left(\frac{N}{2}(N-1) - N\beta\right) = \frac{1}{\alpha-\beta}\left(N\alpha - \frac{N}{2}\right)$$

and therefore

$$w_a = N(\alpha - \frac{1}{2})\frac{\alpha^a - \beta^a}{\alpha-\beta} + N\beta^a = N(\alpha-\frac{1}{2})\sum_{k=0}^{a-1}\alpha^{a-1-k}\beta^k + N\beta^a. \tag{21}$$

The Lemma follows then by plugging in $\beta = \frac{N}{2} - \alpha$. □

For convenience we give a direct proof of the following result (which could alternatively be deduced from Lemma 2.24 using [AMR06, Corollary 3.9]).

Lemma 2.22. *The w_a from Lemma 2.21 are admissible, i.e. satisfy* (15)*.*

Proof. Set $Q(a) := N\sum_{k=0}^{a}\alpha^{a-k}\left(\frac{N}{2}-\alpha\right)^k$. Then $w_a = Q(a) - \frac{1}{2}Q(a-1)$ and the admissibility condition (15) is for $m = 2a+1$ equivalent to

$$\begin{aligned} 0 \quad = \quad & 2Q(m) - \frac{1}{2}Q(m-2) + R \\ & - \sum_{b=1}^{m-1}(-1)^{b-1}\left(Q(b-1)Q(m-b) + \frac{1}{4}Q(b-2)Q(m-b-1)\right) \end{aligned} \tag{22}$$

where $R = \sum_{b=1}^{m-1}(-1)^{b-1}(Q(b-2)Q(m-b) + Q(b-1)Q(m-b-1)) = 0$, since $Q(-1) = 0$ by definition and then

$$\begin{aligned} R &= \sum_{b=0}^{m-1}(-1)^b Q(b-1)Q(m-b-1) + \sum_{b=1}^{m}(-1)^{b-1}Q(b-1)Q(m-b-1) \\ &= (-1)^{m-1}Q(m-1)Q(-1) + Q(-1)Q(m-1) = 0. \end{aligned}$$

By the right hand side of (22), it is enough to show that $S(t) := 2Q(t) - \sum_{b=1}^{t}(-1)^{b-1}Q(b-1)Q(t-b) = 0$ for $t = m, m-1$. For this we consider $S(t)$ as a polynomial in N and show that all the coefficients vanish. First note that

$$2Q(t) = 2N\sum_{k=0}^{t}\alpha^{t-k}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{2^r}N^r(-\alpha)^{k-r} = \sum_{k=0}^{t}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{2^{r-1}}(-1)^{k-r}\alpha^{t-r}N^{r+1}$$

Hence its coefficient in front of N^{s+1} equals

$$c_{s+1} = \frac{1}{2^{s-1}}(-1)^s\alpha^{t-s}\sum_{k=0}^{t}(-1)^k\binom{k}{s}. \tag{23}$$

On the other hand

$$\sum_{b=1}^{t}(-1)^{b-1}Q(b-1)Q(t-b) \quad = \quad N^2\sum_{b=1}^{t}(-1)^{b-1}\sum_{r=0}^{b-1}\sum_{j=0}^{t-b}\alpha^{b-1-r+t-b-j}\left(\frac{N}{2}-\alpha\right)^{r+j}$$

hence its coefficient in front of N^{s+1} equals

$$d_{s+1} = \frac{1}{2^{s-1}}(-1)^s\alpha^{t-s}\sum_{b=1}^{t}\sum_{r=0}^{b-1}\sum_{j=0}^{t-b}(-1)^{b+r+j}\binom{r+j}{s-1}. \tag{24}$$

Clearly $c_0 = 0 = d_0$ and then $c_{s+1} = d_{s+1}$ for all $s \geq 0$ by the Lemma 2.23 below. $\square$

Lemma 2.23. *Let $m \geq 0$ be odd and $s \geq 1$. Then*

$$\sum_{k=0}^{m}(-1)^k\binom{k}{s} = \sum_{b=1}^{m}\sum_{r=0}^{b-1}\sum_{j=0}^{m-b}(-1)^{b+r+j}\binom{r+j}{s-1}. \tag{25}$$

Proof. For $m = 1$ the statement is clear. Let $L(m)$ and $R(m)$ be the left and right side of (25). We assume $L(m) = R(m)$ and want to deduce $L(m+2) = R(m+2)$ for which it is enough to show $R(m+2) - R(m) = L(m+2) - L(m)$. Since m is odd, the latter is equivalent to verifying

$$\binom{m+1}{s} - \binom{m+2}{s} \quad = \quad R(m+2) - R(m) \tag{26}$$

Now by definition $R(m+2) - R(m)$ equals

$$\begin{aligned} &\sum_{r=0}^{m+1}\sum_{j=0}^{0}(-1)^{m+r+j}\binom{r+j}{s-1} + \sum_{r=0}^{m}\sum_{j=0}^{1}(-1)^{m+r+j+1}\binom{r+j}{s-1} \\ &+ \sum_{b=1}^{m}\sum_{r=0}^{b-1}(-1)^{m+r+1}\binom{r+m+1-b}{s-1} + \sum_{b=1}^{m}\sum_{r=0}^{b-1}(-1)^{m+r}\binom{r+m+2-b}{s-1} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{2m+1}\binom{m+1}{s-1} + \sum_{r=0}^{m}(-1)^{m+r+2}\binom{r+1}{s-1} + \sum_{r=0}^{m-1}(-1)^{m+r+1}\binom{r+1}{s-1} \\
&\quad + \sum_{r=0}^{0}(-1)^{m}\binom{m+1}{s-1} + \sum_{b=1}^{m-1}(-1)^{m+b}\binom{m+1}{s-1} \\
&= -\binom{m+1}{s-1} + \binom{m+1}{s-1} - \binom{m+1}{s-1} - \binom{m+1}{s-1}\underbrace{\sum_{b=1}^{m-1}(-1)^{b}}_{=0} = -\binom{m+1}{s-1} \\
&= \binom{m+1}{s} - \binom{m+2}{s}.
\end{aligned}$$

□

Lemma 2.24. *The sequence w_a, $a \geq 0$ from Lemma 2.21 is* $\mathbf{u}$*-admissible for* $\mathbf{u} = (\alpha, \beta)$.

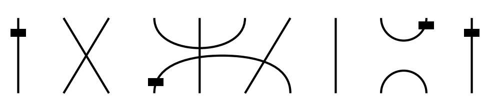
Proof. Substituting N in formula (21) we obtain

$$w_a = \frac{\alpha+\beta}{\alpha-\beta}(2\alpha^{a+1} - 2\beta^{a+1} - \alpha^a + \beta^a)$$

which is easy to see to agree with the formula for w_a in Example 2.15. □

Corollary 2.25. *The level $l = 2$ cyclotomic quotients $\mathbb{W}_d(\alpha,\beta)$ are of dimension $2^d(2d-1)!!$ with basis $\mathbb{B}_d = \mathbb{B}(\mathbb{W}_d(\alpha,\beta))$ given by the regular monomials from Proposition 2.14.*

We display basis elements diagrammatically by drawing the Brauer diagram B with small decorations indicating the y_i's. For instance



(27)

stands for $y_4 B y_1 y_{10} y_{11} \in \mathbb{B}_d$ where b is the Brauer algebra element without the decorations. Sliding a decoration through an arc produces a linear combination of basis vectors according to Definition 2.1, (VW.7), (VW.8).

3. Isomorphism theorem

Theorem 3.1. *If $n \geq 2d$ and $\delta \in \mathbb{Z}$ then the map $\Psi^{d,n}_{M(\underline{\delta})}$ from Theorem 2.8 induces an isomorphism of algebras*

$$\Psi(\underline{\delta}): \quad \mathbb{W}_d(\alpha,\beta) \longrightarrow \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} \tag{28}$$

Proof. By definition it is an algebra homomorphism. It is injective by Proposition 3.2 below and surjective since the dimensions agree, Corollary 3.6. □

3.1. **Injectivity.** We start with the proof for the injectivity.

Proposition 3.2. *If $n \geq 2d$ and $\delta \in \mathbb{Z}$ then $\Psi^{d,n}_{M(\underline{\delta})}$ from Theorem 2.8 induces an injective map of algebras*

$$\Psi(\underline{\delta}): \quad \mathbb{W}_d(\alpha,\beta) \longrightarrow \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} \tag{29}$$

Proof. By Corollary 2.25 it is enough to show that the regular monomials $y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_d^{\gamma_d} B y_1^{\eta_1} y_2^{\eta_2} \cdots y_d^{\eta_d}$ for $b \in \mathcal{B}_d$ and $1 \leq \gamma_i, \eta_i < l$ are mapped to linearly independent morphisms. By Remark 2.6 and (9) we have $y_i = \Omega_{0,i} + x$, where x is some Brauer algebra element. Let now $m \in M^{\mathfrak{p}}(\underline{\delta})$ be a highest weight vector. Then $X_\gamma m = 0$ for $X_\gamma \in \mathfrak{n}^+$ and furthermore $X_{-(\epsilon_i+\epsilon_j)} m = 0$ for all $i > j$ because of our specific choice of $\underline{\delta}$. (Note that $E(\underline{\delta})$ is one dimensional in the description of $M^{\mathfrak{p}}(\underline{\delta})$ from (1).) Applying (9) we obtain for any $a \in I^+$

$$\Omega_{0,1} m \otimes v_a = \frac{\delta}{2} m \otimes v_a, \qquad \Omega_{0,1} m \otimes v_{-a} = \frac{\delta}{2} m \otimes v_a + \sum_{i \neq a} a_i X_{-(\epsilon_i+\epsilon_j)} m \otimes v_i$$

for some $a_i \in \mathbb{C}^*$. The $pm \otimes v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_{d-1}} \otimes v_{i_d}$, where p runs through all monomials in the variables $X_{-(\epsilon_i+\epsilon_j)}$ for $i,j \in I^+$ with $i > j$ form a basis $\mathbb{B}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$ of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$. (Note that the $X_{-(\epsilon_i+\epsilon_j)}$'s commute due to the structure of the root system). In particular,

$$y_r m \otimes v_a = \begin{cases} \sum_{i \neq i_r} a_{i_r} X_{-(\epsilon_i+\epsilon_{i_r})} v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes v_i \otimes v_{i_{r+1}} \otimes v_{i_d} + (\dagger) & \text{if } -a \in I^+ \\ (\dagger) & \text{if } a \in I^+ \end{cases}$$

where $(\dagger)$ stands for some linear combination of basis vectors where the above mentioned monomials have degree zero.

Now consider a standard basis vector $\gamma \in \mathbb{B}_d$. Take its diagram (27) and label the vertices with numbers from $\{i, -i \mid 1 \leq i \leq 2d\}$ as follows: Label all vertices at the top from left to right by $d+1$ to $2d$ and at the bottom from left to right by -1 to $-d$. In case there is a vertical arc without decoration connecting two vertices replace the label at the top endpoint with the label at the bottom endpoint. If there is a horizontal arc without decoration change the label at the right endpoint to the negative of the left endpoint. In the resulting diagram the labels, say a and b, at the endpoints of arcs satisfy the following: $|a| \neq |b|$ and negative at the bottom and positive at the top if the arc is decorated; $a = b < 0$ if the arc is undecorated and vertical; and $a = -b$ if the arc is undecorated and horizontal. Moreover, no number appears more than twice. In the example (27) we get $12, -3, -2, 15, -5, -15, -6, -8, 20, 21, 22$ at the top and $-1, -2, -3, -4, -5, -6, -7, -8, -9, 9, -11$ at the bottom.

Let now $S := \sum_{\gamma \in \mathbb{B}_d} c_\gamma \Psi(\underline{\delta})(\gamma) = 0$ with $c_\gamma \in \mathbb{C}$. Pick $\gamma \in \mathbb{B}_d$ corresponding to a diagram with all arcs decorated. and let $(-a_i, b_i)$, $1 \leq i \leq d$, $a_i, b_i > 0$ be the pairs of labels attached to each arc. Then, by (30) the coefficient of $X_{-(\epsilon_{a_1}+\epsilon_{b_1})} \cdots X_{-(\epsilon_{a_d}+\epsilon_{b_d})} m \otimes v_1 \otimes \cdots \otimes v_d$ when expressing $Sm \otimes v_{-1} \otimes \cdots \otimes$

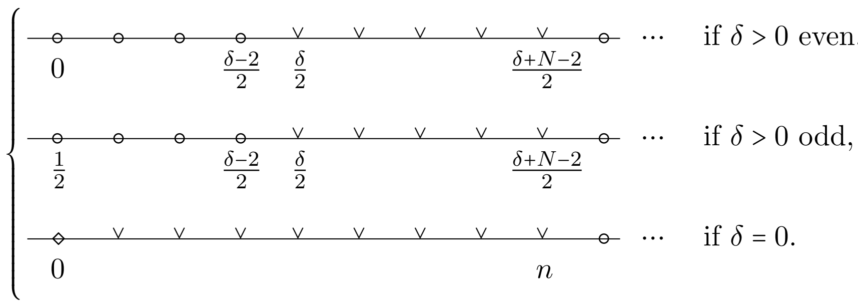
v_{-d} in our basis is precisely c_γ, hence $c_\gamma = 0$. Repeating this argument gives $c_\gamma = 0$ for all diagrams with all strands decorated. Next pick $b \in \mathbb{B}_d$ which corresponds to a diagram with all arcs except one decorated and let $j_1, \dots j_d$ and $j'_1, \dots j'_d$ be the associated labels at the bottom respectively top of the diagram read from left to right. Let $(-a_i, b_i)$, $1 \le i \le d-1$, $a_i, b_i > 0$ be the pairs of labels attached to each arc. By (30), the coefficient of $X_{-(\epsilon_{a_1}+\epsilon_{b_1})} \cdots X_{-(\epsilon_{a_{d-1}}+\epsilon_{b_{d-1}})} m \otimes v_{i_1} \otimes \cdots v_{i_d}$ when expressing $Sm \otimes v_{j_1} \otimes \cdots v_{j_d}$ in our basis is then precisely c_γ. Hence $c_\gamma = 0$ for all diagrams with only one undecorated arc. Proceeding like this gives finally $c_\gamma = 0$ for all γ. Hence the linearly independence follows. □

3.2. **Weight diagrams and bipartitions.** Based on [AMR06] we introduce a labelling set for a basis in the cyclotomic quotients and connect it with the counting of Verma modules from Lemma 2.16 via the diagrammatic weights.

Fix integers d and t with $0 \le t \le \lfloor \frac{d}{2} \rfloor$. A *partition* of d is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of weakly decreasing non-negative integers λ_i which sum up to d, i.e. $|\lambda| := \sum_{i \ge 1} \lambda_i = d$. As usual we identify a partition with its Ferres or Young diagram containing λ_i boxes in the i-th row. Its top left vertex is called the origin. A *bipartition* is an ordered pair $\lambda = (\lambda^1, \lambda^2)$ of partitions λ^1, λ^2. We denote by $\mathcal{P}^2$ (resp. $\mathcal{P}^2(d)$) the set of bipartitions (of d) and by $\mathcal{P}^1$ (resp. $\mathcal{P}^1(d)$) the set of partitions (of d). An *up-down-bitableau* is a sequence $(\varnothing, \varnothing) = \lambda(0), \lambda(1), \cdots, \lambda(d)$ of bipartitions starting from the empty bipartition and such that $\lambda(i+1)$ differs from $\lambda(i)$ by removing or adding exactly one box. If the length of the sequence is $d+1$ we call it also an up-down-d-bitableau. The set of all up-down-bitableaux is denoted $\mathcal{T}^2$ with the subset $\mathcal{T}^2_d$ of all up-down-d-bitableaux and the subset $\mathcal{T}^1_d(\lambda)$ of all λ-up-down-tableaux. i.e. all those up-down-d-bitableau ending at a fixed bipartition λ. Similarly we define the sets $\mathcal{T}^1$, $\mathcal{T}^1_d$, $\mathcal{T}^1_d(\lambda)$ of all up-down-tableaux, all up-down-d-tableaux and λ-up-down-d-tableaux for a partition λ. From [AMR06] it follows in particular that for $l = 1, 2$

$$\sum_{\lambda} |\mathcal{T}^l_d(\lambda)|^2 \quad = \quad l^d (2d-1)!! \tag{30}$$

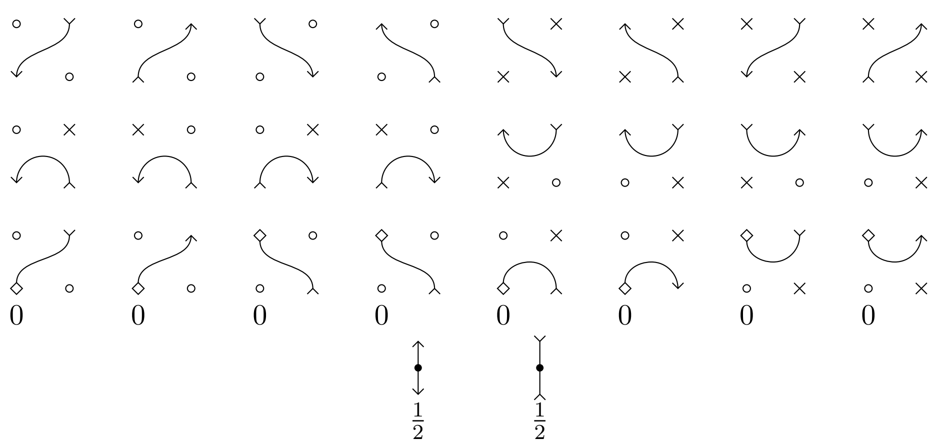
As defined in Section 1 we associate a diagrammatic weight to each weight in Λ. For simplicity we will denote this by the same letter and in light of Corollary 3.6 stick here to the case $\delta \ge 0$. Then the corresponding diagrammatic weights are the following. (The first and third are diagrammatic weights in $\mathbb{X}_n$, the second in $\mathbb{X}_n^{\jmath}$).



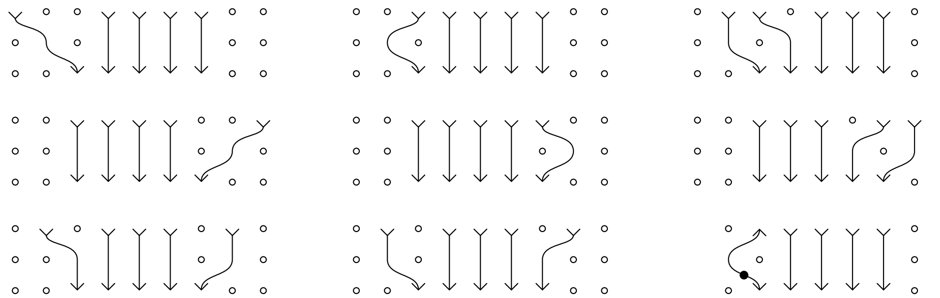
Suppose we are given a d-admissible weight sequence as in (17)

$$\underline{\delta} = \lambda^1 \to \lambda^2 \to \ldots \to \lambda^d.$$

We are going to represent this diagrammatically by what we call a Verma path of length d (depending on the parameter $\delta \in \mathbb{Z}_{\geq 0}$, similar to the diagrams in [BS11b], [BS12]. Note that, since the weights λ^i and λ^{i+1} differ only by some $\pm\epsilon_j$ and let $r_i = |\lambda^i_j|$ and $s_i = |\lambda^i_j \pm 1|$, the associated weight diagrams differ precisely at the coordinates r_i and s_i. We first draw the weight sequence as a sequence of weight diagrams from bottom to top. Then for each $i = 1, \ldots, d-1$, we insert vertical line segments connecting all coordinates strictly smaller or strictly larger than r_i and s_i that are labelled $\vee$ or $\wedge$ in λ^i and λ^{i+1}. Then connect the remaining coordinates r_i and s_i of λ^i and λ^{i+1} as in the appropriate one of the following pictures:



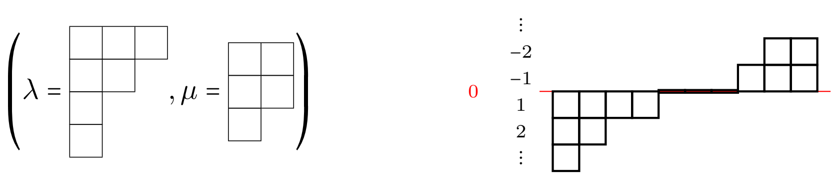
The result is a *Verma path* of length d. Note that in the third row all the given connections might be decorated with a $\bullet$ in a similar way to the fourth row. For this replace the symbol $\diamond$ with either $\wedge$ or $\vee$ such that the weight diagram has an even number of $\vee$'s. If the resulting picture does not look like one of the pictures in the first or second row, i.e. the strand is not oriented then it must be decorated with a $\bullet$.



(31)

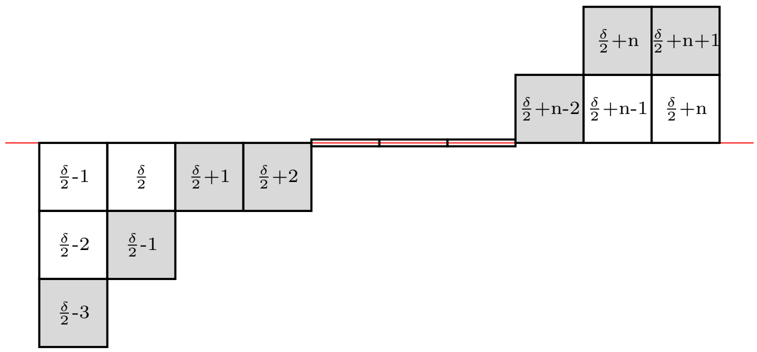
Example 3.3. The first eight pictures in Figure 31 display the Verma paths for $d = 2$ and $\delta > 1$, representing the eight Verma modules appearing in a Verma filtration of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$. In case $\delta = 1$, the first picture has to be replaced by the last one. Note the dot in the last Verma path that indicates that the symbol $\diamond$ at position 0 for λ^1 would need to be substituted with a $\wedge$ to obtain an even number of $\vee$'s.

To any weight λ appearing in a Verma path of length $d \leq n$ we assign a bipartition $\varphi(\lambda) = (\lambda^{(1)}, \lambda^{(2)})$ with $\lambda_i^{(1)} = |m_i|$ and $\lambda_i^{(2)} = m_{s+i}$, where the m_i are defined by $\lambda - \underline{\delta} = \sum_{i=1}^{n} m_i \epsilon_i$ with $m_i \leq 0$ for $1 \leq i \leq s$ and $m_i > 0$ for $i > s$. It is convenient to draw here the bipartition $\varphi(\lambda) \in \mathcal{T}^2(d)$, $d \leq n$, by taking the first diagram transposed (i.e. reflected in the diagonal D through the origin) and the second diagram reflected in the line D' through the origin orthogonal to D and then arrange them in the plane such that reflected origins are positioned at the coordinates $(0,0)$ and $(0,n)$ respectively. The ith column contains then m_i boxes arranged below the x-axis if $m_i < 0$ and above if $m_i > 0$. Let $(\lambda^t, {}^t\mu)$ denote the resulting diagram. For example, the bipartition $(3,2,1,1), (2,2,1)$ is displayed as follows:



(32)

We label its columns from left to right by $0, 1, 2, \dots$ and the rows increasing from top to bottom such that the λ^t has rows $1, 2, \dots$ and ${}^t\mu$ has rows $-1, -2, \cdots$ (with a "ground state line" counted 0 between the two reflected partitions). The *content* $c(b)$ of a box b is its column number minus its row number, $c(b) = \mathrm{col}(b) - \mathrm{row}(b)$ and the content with charge $\frac{\delta}{2}$ is defined as $c_\delta(b) = \frac{\delta}{2} + c(b)$; in our example we have



(33)

Then we can read off the corresponding diagrammatic weight $\lambda \in \Lambda$ by setting $I(\lambda) = \{\mathrm{cont}_\delta(\lambda,\mu)\} = \{c_\delta(b_i) \mid 1 \leq i \leq n\}$, where the b_i are the boundary boxes of the columns. In the above example (with the relevant boxes shaded grey) we obtain $I(\lambda) = \frac{\delta}{2} + \{-3,-1,1,2,\ldots,n-4,n-2,n,n+1\}$. In particular, $\underline{\delta}$ corresponds to the bipartition $(\varnothing,\varnothing)$. In this way, any $\mathbf{t} \in \mathcal{T}_d$ defines a sequence $\mathrm{cont}(\mathbf{t})$ of diagrammatic weights.

Lemma 3.4. *Let $n, \delta \in \mathbb{Z}_{\geq 0}$. Assume $n \geq \delta$. Then the assignment $\mathbf{t} \mapsto \mathrm{cont}(\mathbf{t})$ defines a bijection, with inverse map $\lambda \mapsto \varphi(\lambda)$,*

$$\begin{array}{rc} \mathcal{T}_d = & \{\textit{up-down bitableaux of length } d\} \\ & \updownarrow \\ P_d(\delta) := & \{\textit{Verma paths of length } d \textit{ starting at } \delta \ \ \} \end{array}$$

inducing also a bijection between $\bigcup_{0 \leq k \leq \lfloor \frac{d}{2} \rfloor} \mathcal{P}^2(d-2k)$ and the set of weights appearing at the end of a path in $P_d(\delta)$.

Proof. The empty bipartition $(\varnothing,\varnothing)$ corresponds to the weight $\underline{\delta}+\rho$. Now adding (resp. removing) a box in the jth column of the first partition corresponds to subtracting ϵ_j from (resp. adding ϵ_j to) the weight. On the other hand, adding (resp. removing) a box in the $(n-j)$th column of the second partition corresponds to adding ϵ_j to (resp. subtracting ϵ_{n-j} from) the weight. In any case, the result is an allowed weight if and only if adding (resp. removing) the box produces a new bipartition. Then the lemma follows from the definitions. $\square$

For $M^{\mathfrak{p}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}(n)$ we denote by $L(\lambda)$ its irreducible quotient and by $P^{\mathfrak{p}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}(n)$ its projective cover. For a module M in category $\mathcal{O}^{\mathfrak{p}}(n)$ with Verma flag (i.e with a filtration with subquotients isomorphic to Verma modules) we denote by $\lambda_M = [M{:}M^{\mathfrak{p}}(\lambda)]$ the multiplicity of $M^{\mathfrak{p}}(\lambda)$ in such a flag. By BGG reciprocity we have $[P(\mu){:}M^{\mathfrak{p}}(\nu)] = [M^{\mathfrak{p}}(\nu) : L(\mu)]$ where the latter denotes the Jordan-Hoelder multiplicity of $L(\mu)$ in $M^{\mathfrak{p}}(\nu)$.

Lemma 3.5. *The following holds in $\mathcal{O}^{\mathfrak{p}}(n)$:*

(1) $\dim \mathrm{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}(\lambda), P^{\mathfrak{p}}(\mu)) = \sum_{\nu \in \Lambda} \nu_{P^{\mathfrak{p}}(\lambda)} \nu_{P^{\mathfrak{p}}(\mu)}$, *the number of Verma modules which appear at the same time in $P(\lambda)$ and in $P(\mu)$.*

(2) If P is a projective object then $\dim \operatorname{End}_{\mathfrak{g}}(P) = \sum_{\nu\in\Lambda}(\nu_P)^2$.

Proof. Let $N = \{\nu \in \Lambda \mid \nu_{P(\mu)} \neq 0, \nu_{P(\lambda)} \neq 0\}$. Then $\dim \operatorname{Hom}(P(\lambda), P(\mu)) = [P(\mu) : L(\lambda)] = \sum_{\nu\in\Lambda}[P(\mu) : M(\nu)][M(\nu) : L(\lambda)] = \sum_{\nu\in\Lambda} \nu_{P(\mu)}\nu_{P(\lambda)} = \sum_{\nu\in N} 1\cdot 1$, where for the last equality we used that the multiplicities are at most 1. This follows for regular weights for instance from [LS13, Theorem 2.1] and for singular weights from Lemma 3.10 below. Part (1) follows. Recall that P decomposes into a direct sum of indecomposable projective modules $P = \oplus_\mu a_\mu P(\mu)$, where a_μ is the multiplicity of $P(\mu)$. Then

$$\begin{aligned}
\dim \operatorname{End}(P) &= \sum_{\lambda,\mu\in\Lambda} a_\lambda a_\mu \dim \operatorname{Hom}(P(\lambda), P(\mu)) \\
&= \sum_{\lambda,\mu\in\Lambda} a_\lambda a_\mu \sum_{\nu\in\Lambda} \nu_{P(\mu)}\nu_{P(\lambda)} = \sum_{\nu\in\Lambda}\left(\sum_{\lambda,\mu\in\Lambda} a_\lambda a_\mu \nu_{P(\mu)}\nu_{P(\lambda)}\right) \\
&= \sum_{\nu\in\Lambda}\left(\sum_{\lambda\in\Lambda} a_\lambda \nu_{P(\mu)} \sum_{\mu\in\Lambda} a_\mu \nu_{P(\lambda)}\right) = \sum_{\nu\in\Lambda} \nu_P \nu_P = \sum_{\nu}(\nu_P)^2.
\end{aligned}$$

□

Corollary 3.6. *We have* $\dim \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} = 2^d(2d-1)!!$. *Moreover,* $\operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} \cong \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{-\delta}) \otimes V^{\otimes d}$ *as algebras.*

Proof. Let first $\delta \geq 0$. Since then $M^{\mathfrak{p}}(\underline{\delta})$ is projective in $\mathcal{O}^{\mathfrak{p}}(n)$, so is $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ and therefore it has a Verma filtration. By Lemma 2.16 and the paragraph afterwards $\nu_{M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes d}}$ equals the number of Verma paths of length d ending at $\nu + \rho$. Then the statement follows directly from Lemmas 3.4, Lemma 3.5 and the equality (30). If $\delta < 0$ then $M^{\mathfrak{p}}(\underline{-\delta})$ is a tilting module and the algebra isomorphism follows from Ringel self-duality [MS08, Proposition 4.4] of $\mathcal{O}^{\mathfrak{p}}(n)$ and we are done. □

3.3. **Special projective functors.** In analogy to [BS11b] we introduce special translation functors which will later categorify the actions of the coideal subalgebras $\mathcal{H}$ and $\mathcal{H}^{\jmath}$ in Section 6.

For this let Γ be a diagrammatic block, i.e. given by a block sequence and a parity for the number of ∨'s and ×'s. Furthermore fix $i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ if Γ is supported on the integers and $i \in \mathbb{Z}_{\geq 0}$ if Γ is supported on half-integers.

Case $i \geq 1$: We denote by $\Gamma_{i,+}$ and $\Gamma_{i,-}$ the blocks, whose corresponding block diagram differs from Γ diagrammatically only at the vertices $i - \frac{1}{2}$ and $i + \frac{1}{2}$ as displayed in the following list (note that it has the same type of integrality and the same parity):

$\Gamma_{i,+}$	×○		●●			●○	×●
$\Gamma_{i,-}$	○×	●●		○●	●×		
Γ	●●	×○	○×	●○	×●	○●	●×

(34)

Case $i = \frac{1}{2}$: We denote by $\Gamma_{\frac{1}{2},+}$ and $\Gamma_{\frac{1}{2},-}$ the blocks or in one instance the sum of two blocks, whose corresponding block diagram differs from Γ

diagrammatically only at the vertices 0 and 1 as displayed in the following list (note that it has the same type of integrality):

$$\begin{array}{c||c|c|c|c|} \Gamma_{\frac{1}{2},+} & & & \diamond\bullet & \diamond\circ \\ \Gamma_{\frac{1}{2},-} & \circ\times & \circ\ \bullet\ +\overline{\circ\bullet} & & \\ \hline \Gamma & \diamond\bullet & \diamond\circ & \circ\times & \circ\bullet \end{array} \tag{35}$$

Here the entry $\circ\ \bullet\ +\overline{\circ\bullet}$ means that we take the direct sum of the two blocks with the same underlying block diagram but the two possible choices of parity.

Case $i = 0$**:** In this case the block Γ_0 has the same block diagram as Γ but the opposite parity. Let

$$\mathrm{pr}^1_\Gamma : \mathcal{O}^{\mathfrak{p}}_1(n) \to \mathcal{O}^{\mathfrak{p}}_1(n)_\Gamma \qquad \mathrm{pr}^{\halfnote}_\Gamma : \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n) \to \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n)_\Gamma,$$

denote the projections onto the summand $\mathcal{O}^{\mathfrak{p}}_1(n)_\Gamma$, resp. $\mathcal{O}^{\mathfrak{p}}_{\halfnote}(n)_\Gamma$, depending on whether Γ is supported on the integers or half-integers.

Definition 3.7. We have the *special projective functors* $\mathcal{F}_{i,+}$, $\mathcal{F}_{i,-}$ for $i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ defined by

$$\mathcal{F}_{i,-} := \bigoplus_\Gamma \mathrm{pr}^1_{\Gamma_{i,-}} \circ (? \otimes V) \circ \mathrm{pr}^1_\Gamma \quad : \quad \mathcal{O}^{\mathfrak{p}}_1(n) \to \mathcal{O}^{\mathfrak{p}}_1(n),$$
$$\mathcal{F}_{i,+} := \bigoplus_\Gamma \mathrm{pr}^1_{\Gamma_{i,+}} \circ (? \otimes V) \circ \mathrm{pr}^1_\Gamma \quad : \quad \mathcal{O}^{\mathfrak{p}}_1(n) \to \mathcal{O}^{\mathfrak{p}}_1(n).$$

and $\mathcal{F}_{i,+}$, $\mathcal{F}_{i,-}$ for $i \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F}_0$ defined by

$$\mathcal{F}_{i,-} := \bigoplus_\Gamma \mathrm{pr}^{\halfnote}_{\Gamma_{i,-}} \circ (? \otimes V) \circ \mathrm{pr}^{\halfnote}_\Gamma \quad : \quad \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n) \to \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n),$$
$$\mathcal{F}_{i,+} := \bigoplus_\Gamma \mathrm{pr}^{\halfnote}_{\Gamma_{i,+}} \circ (? \otimes V) \circ \mathrm{pr}^{\halfnote}_\Gamma \quad : \quad \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n) \to \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n),$$
$$\mathcal{F}_0 := \bigoplus_\Gamma \mathrm{pr}^{\halfnote}_{\Gamma_0} \circ (? \otimes V) \circ \mathrm{pr}^{\halfnote}_\Gamma \quad : \quad \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n) \to \mathcal{O}^{\mathfrak{p}}_{\halfnote}(n).$$

In each case the direct sums are over all blocks Γ, where $\Gamma_{i,-}$, resp. $\Gamma_{i,+}$, resp. Γ_0 are defined.

Note that $\mathcal{F}_{i,-}$ and $\mathcal{F}_{i,+}$ are biadjoint and F_0 is selfadjoint, since V is self-dual. Hence the functors are exact and send projectives to projectives. The following Lemma, whose proof can be found in the appendix, describes the effect of taking tensor product with V on projective and simple modules. At the end of this section we give a diagrammatic interpretation of the action on indecomposable projective modules. The symbols $\langle i \rangle$ in the lemma refer to a grading shift which should be ignored. It only makes sense in the graded setup of Lemma 5.3.

Lemma 3.8. *(1) There are isomorphisms of functors*

$$\mathcal{F} := (? \otimes V) \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (\mathcal{F}_{i,-} \oplus \mathcal{F}_{i,+}) : \mathcal{O}^{\mathfrak{p}}_1(n) \to \mathcal{O}^{\mathfrak{p}}_1(n)$$

and

$$\mathcal{F}^{\lrcorner} := (? \otimes V) \cong \bigoplus_{i \in \mathbb{Z}_{>0}} (\mathcal{F}_{i,-} \oplus \mathcal{F}_{i,+}) \oplus \mathcal{F}_0 : \mathcal{O}^{\mathfrak{p}}_{\lrcorner}(n) \to \mathcal{O}^{\mathfrak{p}}_{\lrcorner}(n).$$

(2) *Let λ be a diagrammatic weight and $i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ if $\lambda \in \mathbb{X}_n$ and $i \in \mathbb{Z}_{\geq 0}$ if $\lambda \in \mathbb{X}_n^{\lrcorner}$. Furthermore assume that $i \geq 1$ in both cases. For symbols $x, y \in \{\circ, \wedge, \vee, \times\}$ we write λ_{xy} for the diagrammatic weight obtained from λ by relabelling the $(i - \frac{1}{2})$th entry to x and the $(i + \frac{1}{2})$th entry to y.*

 (i) *If $\lambda = \lambda_{\vee\circ}$ then $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\circ\vee})$, $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\vee})$, $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_{\circ\vee})$.*

 (ii) *If $\lambda = \lambda_{\wedge\circ}$ then $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\circ\wedge})$, $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\wedge})$, $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_{\wedge\circ})$.*

 (iii) *If $\lambda = \lambda_{\times\vee}$ then $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\vee\times})$, $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\vee\times})$, $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_{\vee\times})$.*

 (iv) *If $\lambda = \lambda_{\times\wedge}$ then $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\wedge\times})$, $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\wedge\times})$, $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_{\wedge\times})$.*

 (v) *If $\lambda = \lambda_{\vee\wedge}$ then $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\circ\times}) \oplus P(\lambda_{\circ\times})$, $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\times})$ and $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_{\circ\times})$.*

 (vi) *If $\lambda = \lambda_{\wedge\vee}$ then $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\times})\langle 1\rangle$ and $\mathcal{F}_{i,-}L(\lambda) = \{0\}$.*

 (vii) *If $\lambda = \lambda_{\times\circ}$ then:*

 (a) *$\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\vee\wedge})$;*

 (b) *there is a short exact sequence*

$$0 \to M^{\mathfrak{p}}(\lambda_{\wedge\vee}) \to \mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) \to M^{\mathfrak{p}}(\lambda_{\vee\wedge})\langle -1\rangle \to 0;$$

 (c) *$[\mathcal{F}_{i,-}L(\lambda) : L(\lambda_{\vee\wedge})] = 2$ and all other composition factors are of the form $L(\mu)$ for μ with $\mu = \mu_{\vee\vee}$, $\mu = \mu_{\wedge\wedge}$ or $\mu = \mu_{\wedge\vee}$;*

 (d) *$\mathcal{F}_{i,-}L(\lambda)$ has irreducible socle and head isomorphic to $L(\lambda_{\vee\wedge})$.*

 (viii) *If $\lambda = \lambda_{\vee\vee}$ then $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) = \mathcal{F}_{i,-}L(\lambda) = \{0\}$.*

 (ix) *If $\lambda = \lambda_{\wedge\wedge}$ then $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda) = \mathcal{F}_{i,-}L(\lambda) = \{0\}$.*

 (x) *For the dual statement about $\mathcal{F}_{i,+}$, interchange all occurrences of $\circ$ and $\times$.*

(3) *Let $\lambda \in \mathbb{X}_n$. For symbols $x, y \in \{\circ, \wedge, \vee, \times, \diamond\}$ we write λ_{xy} for the diagrammatic weight obtained from λ by relabelling the 0th entry to x and the 1st entry to y*

 (xi) *If $\lambda = \lambda_{\diamond\vee}$ then $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\circ\times})$, $\mathcal{F}_{\frac{1}{2},-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\times})\langle 1\rangle$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) \cong L(\lambda_{\circ\times})$.*

 (xii) *If $\lambda = \lambda_{\diamond\wedge}$ then $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\circ\times})$, $\mathcal{F}_{\frac{1}{2},-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\times})$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) \cong L(\lambda_{\circ\times})$.*

 (xiii) *If $\lambda = \lambda_{\diamond\circ}$ then $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\circ\wedge}) \oplus P(\lambda_{\circ\vee})$, $\mathcal{F}_{\frac{1}{2},-}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\wedge}) \oplus M^{\mathfrak{p}}(\lambda_{\circ\vee})$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) \cong L(\lambda_{\circ\wedge}) \oplus L(\lambda_{\circ\vee})$.*

 (xiv) *If $\lambda = \lambda_{\circ\vee}$ then we have $\mathcal{F}_{\frac{1}{2},+}P(\lambda) \cong P(\lambda_{\diamond\circ})$, $\mathcal{F}_{\frac{1}{2},+}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\diamond\circ})$, $\mathcal{F}_{\frac{1}{2},+}L(\lambda) = \{0\}$.*

(xv) *If* $\lambda = \lambda_{\circ\wedge}$ *then* $\mathcal{F}_{\frac{1}{2},+}P(\lambda) \cong P(\lambda_{\diamond\circ})$, $\mathcal{F}_{\frac{1}{2},+}M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\diamond\circ})$, $\mathcal{F}_{\frac{1}{2},+}L(\lambda) = \{0\}$.

(xvi) *If* $\lambda = \lambda_{\circ\times}$ *then:*

(a) $\mathcal{F}_{\frac{1}{2},+}P(\lambda) \cong P(\lambda_{\diamond\wedge})$;

(b) there is a short exact sequence

$$0 \to M^{\mathfrak{p}}(\lambda_{\diamond\vee}) \to \mathcal{F}_{\frac{1}{2},+}M^{\mathfrak{p}}(\lambda) \to M^{\mathfrak{p}}(\lambda_{\diamond\wedge})\langle -1\rangle \to 0;$$

(c) $[\mathcal{F}_{\frac{1}{2},+}L(\lambda){:}L(\diamond\wedge)] = 2$ *and all other composition factors are of the form* $L(\mu)$ *for* μ *with* $\mu = \mu_{\diamond\vee}$,

(xvii) *If* $\lambda = \lambda_{\diamond\times}$ *then we have* $\mathcal{F}_{\frac{1}{2},-}P(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}M^{\mathfrak{p}}(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) = \{0\}$.

(xviii) *If* $\lambda = \lambda_{\circ\circ}$ *then we have* $\mathcal{F}_{\frac{1}{2},-}P(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}M^{\mathfrak{p}}(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) = \{0\}$.

(4) Finally assume $\lambda \in \mathbb{X}_n^{\downarrow}$. *For symbols* $x \in \{\circ, \wedge, \vee, \times\}$ *we write* λ_x *for the diagrammatic weight obtained from* λ *by relabelling the* $\frac{1}{2}$*th entry to* x. *Then we have*

(xiv) *If* $\lambda = \lambda_\wedge$ *then* $\mathcal{F}_0P(\lambda) \cong P(\lambda_\vee)$, $\mathcal{F}_0M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_\vee)$, $\mathcal{F}_0L(\lambda) \cong L(\lambda_\vee)$.

(xiv) *If* $\lambda = \lambda_\vee$ *then* $\mathcal{F}_0P(\lambda) \cong P(\lambda_\wedge)$, $\mathcal{F}_0M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_\wedge)$, $\mathcal{F}_0L(\lambda) \cong L(\lambda_\wedge)$.

Remark 3.9. Note that each of the $\mathcal{F}_{i,+}$, $\mathcal{F}_{i,-}$, F_0 is a translation functor in the sense of [Hum08]. It is either an equivalence of categories, a translation functor to the wall, a translation functor out of the wall or, in the special case of $\mathcal{F}_{\frac{1}{2},-}$ and the choice of a specific block diagram a direct sum of two translation functors out of the wall. In contrast to the type A situation from [BS11b] not all translations to walls appear however as summands.

3.4. **Projective modules and cup diagrams.** In this section we indicate how to make explicit calculations related to projective modules using decorated cup diagrams following [LS13]. We briefly recall from [ES13b, Definition 3.5] how one associates a *(decorated) cup diagram*, denoted $\underline{\lambda}$, to a diagrammatic weight λ. For a more detailed construction we refer to [ES13b].

First draw the diagrammatic weight as weight diagram by putting the symbol λ_i at coordinate i on the real positive line. Start by connecting neighboured coordinates in the weight diagram labelled $\vee\wedge$ successively by an arc (ignoring already joint coordinates and coordinates not having the symbols $\wedge$ or $\vee$) as long as possible and attach to each remaining $\vee$ a vertical ray. Then connect from left to right pairs of two neighboured $\wedge$'s by a dotted arc, i.e. an arc decorated with a $\bullet$. If a single $\wedge$ remains, attach a vertical ray decorated with a $\bullet$ to it. Finally forget the labelling at the vertices other then $\circ$ and $\times$. Putting a diagrammatic weight μ on top of a cup diagram $\underline{\lambda}$ we call the resulting diagram oriented if the labels $\circ$ and $\times$ in μ and $\underline{\lambda}$ match precisely, undecorated rays are labelled $\vee$ and decorated rays are labelled $\wedge$ and every undecorated cup has exactly one $\wedge$ and one $\vee$ whereas any decorated cup has either two $\vee$'s or two $\wedge$'s at its endpoints.

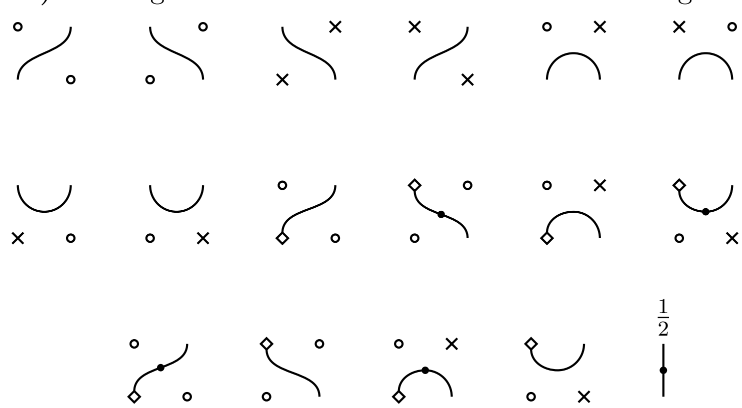
The multiplicities $\mu_{P(\lambda)}$ are then easily calculated:

Lemma 3.10.

$$\mu_{P(\lambda)} = \begin{cases} 1 & \textit{if } \mu\underline{\lambda} \textit{ is oriented} \\ 0 & \textit{if } \mu\underline{\lambda} \textit{ is not oriented.} \end{cases}$$

Proof. For regular weights, i.e. weights which do not contain $\times$'s or $\circ$'s this was proved is [LS13, Theorem 2.1]. (Lemma 4.3 therein and the paragraph afterwards makes the translation into our setup.) For singular weights observe that diagrammatically the numbers stay the same if we remove the $\times$'s and $\circ$'s. This corresponds Lie theoretically to the Enright-Shelton equivalence [ES87] between singular blocks and regular blocks for smaller rank Lie algebras, and hence the multiplicities agree. □

The action of the special projective functors on projectives can be calculated easily in terms of yet another diagram calculus inside the Temperley-Lieb category of type D, [Gre98]. Let $\mathcal{C}$ be the free abelian group generated by $\underline{\lambda}$, $\lambda \in \Lambda$. To any indecomposable projective object $P(\lambda)$ we assign the cup diagram $\underline{\lambda} \in \mathcal{C}$ and extend linearly to direct sums. To the functor $\mathcal{F} := (? \otimes V)$ we assign the formal sum $\mathcal{F}^{\mathrm{diag}}$ of decorated tangles:

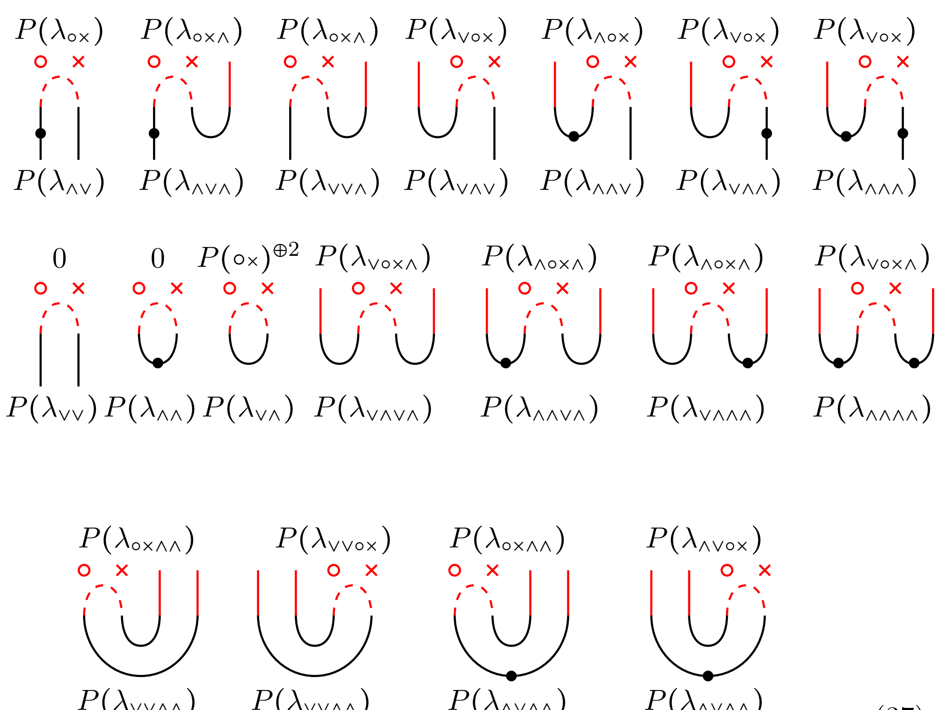


(36)

Each of these tangle diagrams induces a group homomorphism of $\mathcal{C}$. On a cup diagram $\underline{\lambda}$ the action produces a multiple of a cup diagram which is computed as follows: first concatenate and remove any pair of dots on the same line or internal circle. If the labels $\circ$, $\times$ and $\diamond$ do not match or the result contains a circle with one dot or a cap without dots then it acts by zero. Otherwise remove each occurring (undotted) circle and (dotted) cap. The resulting diagram is again a cup diagram $\underline{\lambda}$ (where we always consider two diagrams as the same if they differ only by an isotopy in the plane and moving decorations, i.e. moving $\bullet$'s, along lines) and the result of the action is $2^c\underline{\lambda}$, where c is the number of circles removed. We extend this linearly to an action of $\mathcal{F}^{\mathrm{diag}}$ on $\mathcal{C}$.

Lemma 3.11. *Fix the isomorphism of free abelian groups* $\Phi : K_0(\mathcal{O}^{\mathfrak{p}}(n)) \cong \mathcal{C}$*, which sends the class of* $P(\lambda)$ *to* $\underline{\lambda}$*. Then* $\mathcal{F}^{\mathrm{diag}}(\Phi[P]) = [P \otimes V]$ *for any projective module* $P \in \mathcal{O}^{\mathfrak{p}}(n)$*.*

Proof. In the case of columns 4-7 in (34), the group homomorphism $K_0(\mathcal{F}_{i,-}) : K_0(\mathcal{O}^{\mathfrak{p}}(n)_\Gamma) \to K_0(\mathcal{O}^{\mathfrak{p}}(n)_{\Gamma_{i,-}})$ induced by $\mathcal{F}_{i,-}$ is an isomorphism when expressed in the basis of the isomorphism classes of Verma modules; it has an inverse, the morphism induced by $\mathcal{F}_{i,+}$. They both agree with the morphism induced from the restriction of $\mathcal{F}$ to these blocks. Hence indecomposable projectives are sent to the corresponding indecomposable projectives. The corresponding statement on the diagram side is obvious, involving the first four diagrams from (36) only. In the case of column 2 or 3 in (34) we can apply to a cup diagram $\underline{\lambda}$ exactly one of the first two diagrams from the second line of (36). In each case the result is $\underline{\lambda}$ with an extra cup and we are done by Proposition 3.8 (vii). Column 1 is the most involved situation. In this case $\underline{\lambda}$ looks locally as one of the following 18 diagrams D_i, $1 \le i \le 18$ drawn in thick black lines or the same picture with $\circ$ and $\times$ swapped. Together with the dashed lines they indicate the result after applying the diagrammatic functor (the first two in the second line give zero).



(37)

In the first seven non-zero pictures, we have only one possible orientation for the displayed resulting diagram. On the other hand, there are at most two orientations for each D_i of which exactly one corresponds to a Verma module which is not annihilated by $\mathcal{F}$ with the image in our chosen blocks except of the last and penultimate diagram where none of the two is annihilated. Similar arguments can be used for the functors $\mathcal{F}_{\frac{1}{2},+}$, $\mathcal{F}_{\frac{1}{2},-}$, and $\mathcal{F}_0$. Comparing with Proposition 3.8 using Lemma 3.10, we see that $\mathcal{F}^{\mathrm{diag}}(\Phi[P]) = [P \otimes V]$ in the basis of Verma modules. □

As a special case we obtain the following:

Corollary 3.12. *Assume we are in case (vi) of Proposition 3.8 and assume there is no* $\vee$ *to the left of our fixed* $\wedge\vee$ *pair then* $\mathcal{F}_{i,-}P^{\mathfrak{p}}(\lambda) \cong P^{\mathfrak{p}}(\lambda_{\circ\times})$.

Proof. By assumption and construction, the cup diagram λ looks locally at our two fixed vertices as displayed in one of the pictures in the top line of the following diagram. Applying $\mathcal{F}^{\mathrm{diag}}$ gives for each diagram precisely one new diagram in the required block as displayed. Hence the statement follows by applying φ^{-1} and the definition of $\mathcal{F}_{i,-}$. □

4. Cyclotomic quotients

For this whole section we will always assume $n \geq 2d$**.** This is important for all statements that involve the idempotent z_d introduced in Definition 4.3.

The action of the commuting y's decomposes $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ and $\mathbb{W}_d(\alpha,\beta)$ into generalized eigenspaces. Fix the sets $J = \frac{\delta}{2} + \mathbb{Z}$ and $J_< = \frac{\delta}{2} + \mathbb{Z}_{<(n-1)}$.

Definition 4.1. We identify $\mathbb{W}_d(\alpha,\beta)$ with $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$ via Theorem 3.1. Define the orthogonal *weight idempotents* $\mathrm{e}(\boldsymbol{i})$, $\boldsymbol{i} \in J^d$ characterized by the property that

$$\mathrm{e}(\boldsymbol{i})(y_r - i_r)^m = (y_r - i_r)^m \,\mathrm{e}(\boldsymbol{i}) = 0$$

for each $1 \leq r \leq d$ and $m \gg 0$ (for instance $m \geq d$ is enough).

Then we have the generalized eigenspace decompositions

$$M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d} = \textstyle\sum_{\boldsymbol{i}\in J^d} M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}\,\mathrm{e}(\boldsymbol{i}),$$
$$\mathbb{W}_d(\alpha,\beta) = \textstyle\sum_{\boldsymbol{i}\in J^d} \mathbb{W}_d(\alpha,\beta)\,\mathrm{e}(\boldsymbol{i}) \text{ and } \mathbb{W}_d(\alpha,\beta) = \textstyle\sum_{\boldsymbol{i}\in J^d} \mathrm{e}(\boldsymbol{i})\mathbb{W}_d(\alpha,\beta)$$

as right respectively left modules. The quasi-idempotents e_k only acts between certain generalized eigenspaces:

Lemma 4.2. *Let* $1 \leq k \leq d-1$. *If* $p := \sum_{\boldsymbol{i}\in J^d, i_{k+1}=-i_k} \mathrm{e}(\boldsymbol{i})$ *then we have* $e_k = pe_k = e_k p = pe_k p$. *In particular* $\mathrm{e}(\boldsymbol{i})e_k = 0 = e_k\,\mathrm{e}(\boldsymbol{i})$ *if* $i_{k+1} \neq i_k$.

Proof. Let $m \in \mathbb{W}_d(\alpha,\beta)$ or $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ be contained in the generalized eigenspace for ı, hence $m\,\mathrm{e}(\boldsymbol{i}) \neq 0$. We claim that if $me_k \neq 0$ then $i_{r+1} + i_r = 0$. First note that for $(y_k + y_{k+1} - i_{k+1} - i_k)^r m = \sum_{a=0}^r (y_k - i_k)^a (y_{k+1} - i_{k+1})^{r-a} m = 0$ for $r \gg 0$. Hence m is in the $\mu := i_{k+1} + i_k$-generalized eigenspace for $y_{k+1} + y_k$. Then $0 = m(y_k + y_{k+1} - \mu)^r e_k = me_k(-\mu)^k$ by (8a). Hence $\mu = 0$ and the claim follows. Since the modules have a generalized eigenspace decomposition $pe_k = 0 = e_k p$. The rest follows analogously. □

4.1. **Semisimplicity.**

Definition 4.3. The *Brauer algebra idempotent* is defined as

$$z_d \quad = \sum_{\boldsymbol{i}\in(J_<)^d} \mathrm{e}(\boldsymbol{i}). \tag{38}$$

Multiplication by $\mathrm{e}(\boldsymbol{i})$ projects any $\mathbb{W}_d(\alpha,\beta)$-module onto its $\boldsymbol{i}$*-weight space*, that is, the simultaneous generalised eigenspace for the commuting operators $y_1,\dots,y_r$ with respective eigenvalues $i_1,\dots,i_r$. By (33), Lemma 3.4 and our assumption on n being large, the element z_d is just the projection onto the blocks containing Verma modules indexed by bipartitions of the form $(\lambda,\varnothing)$ where λ is an up-down-tableau. (The corresponding weights are precisely those λ which satisfy $I(\lambda)\subset[-(\frac{\delta}{2}+n-1),\frac{\delta}{2}+n-1].$)

Proposition 4.4. *The algebra* $z_d\mathbb{W}_d(\alpha,\beta)z_d$ *has dimension* $(2d-1)!!$.

Proof. Using Lemma 3.4 and the fact that the algebra has a basis given by Verma paths that correspond to bipartitions of the form $(\lambda,\varnothing)$ gives the result. □

Theorem 4.5.

(1) *The algebra* $\mathbb{W}_d(\alpha,\beta)$ *is semisimple if and only if* $\delta\geq d-1$.
(2) *The algebra* $z_d\mathbb{W}_d(\alpha,\beta)z_d$ *is semisimple if and only if* $\delta\neq 0$ *and* $\delta\geq d-1$ *or* $\delta=0$ *and* $d=1,3,5$.

Proof. Note that $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes d})$ is semisimple if and only if $M^{\mathfrak{p}}(\underline{\delta})\otimes V^{\otimes d}$ is a direct sum of projective Verma modules. Or equivalently if all occurring Verma modules have highest weight λ such that $\lambda+\rho$ is dominant. This is equivalent to the statement that the corresponding diagrammatic weights do not contain two $\wedge$'s any pair $\vee$, $\wedge$ (in this order) or a pair $\diamond\wedge$. Starting from the diagrammatic weight $\underline{\delta}$ we need precisely δ steps to create an $\wedge$ left to a $\vee$, hence $\delta+2$ steps to create a $\vee$, $\wedge$; we also need $2(\frac{\delta}{2}+1)$ steps to create a pair of two $\wedge$'s and $\delta+2$ steps to create $\diamond$, $\wedge$. Hence both algebras are semisimple at least if $d<\delta+2$. Now in case (1), we always can add ϵ_n without changing those pairs, an hence the algebra is not semisimple for all $d\geq\delta+2$. In case (2) with $\delta\neq 0,1$ we can always find some $\circ$ at some position $<n$ and hence add or subtract some appropriate ϵ_j without changing the pair. That means the truncated algebra is not semisimple for $d\geq\delta+2$. For $\delta=1$ we need 3 steps to create a weight starting with $\diamond\wedge\circ\vee$. Hence the algebra is not semisimple and stays not semisimple, since we can always repeatedly swap the $\circ$ with the $\vee$ without changing the $\diamond\wedge$ pair. Hence (2) holds for any $\delta>0$. In case $\delta=0$ one can calculate directly that it is semisimple in the cases $d=1,d=3,d=5$, but not in the cases $d=2,4,6$ and for $=7$ we obtain a weight starting with $\diamond\wedge\wedge\vee$. Since we can change the last two symbols $\wedge\vee$ into $\circ\times$ and back again without loosing the $\diamond\wedge$-pair, the algebra stays not semisimple for $d\geq 7$ and $\delta=0$. The theorem follows. □

4.2. **The basic algebra underlying** $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$**.** Our next goal is to determine for $\delta \geq$ the projective modules appearing in $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ and hence the basic algebra underlying $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$ for $\delta \in \mathbb{Z}$. Our main tool here is the notion of $\mathrm{ht}_{\underline{\delta}}$ which measures the minimal length of a Verma path needed to create a Verma module of a given weight.

Definition 4.6. Let $\delta, d \in \mathbb{Z}$, $\delta \geq 0$, $d \geq 1$ and $\mu \in \Lambda$. The δ-height of μ, $\mathrm{ht}_{\underline{\delta}}$ is defined as $\mathrm{ht}_{\underline{\delta}}(\mu) = \sum_{i=1}^{n} |(\underline{\delta} + \rho)_i - (\mu + \rho)_i| = \sum_{i=1}^{n} |\underline{\delta}_i - \mu_i|$.

Note that, by Lemma 2.16, the highest weights λ occurring in a Verma filtration of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$ satisfy $\mathrm{ht}_{\underline{\delta}}(\lambda) \leq d$ and $\mathrm{ht}_{\underline{\delta}}(\lambda)$ is precisely the number of boxes in $\varphi(\lambda)$, see (32), (33).

Proposition 4.7. *Let $\delta, d \in \mathbb{Z}$, $\delta \geq 0$, $d \geq 1$. Then $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d} \cong \oplus_{\mu \in \Lambda} a_\mu P(\mu)$ with multiplicities a_μ and $a_\mu \neq 0$ if and only if $\mathrm{ht}_{\underline{\delta}} \leq d$ and $d - \mathrm{ht}_{\underline{\delta}}$ is even.*

Proof. Starting with $d = 0$ there is only one weight with δ-height zero, namely $\underline{\delta}$ and the statement is clear. For $d = 1$ there are two weights, $\underline{\delta} - \epsilon_1$ and $\underline{\delta} + \epsilon_n$. Again the statement is clear, since the corresponding two Verma modules are projective. Now we want to prove the lemma for arbitrary d assuming it for all smaller ones. Note that $a_\mu \neq 0$ implies that $M^{\mathfrak{p}}(\mu)$ appears in a Verma flag of $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$, and hence $\mathrm{ht}_{\underline{\delta}}(\mu) \leq d$. Moreover, the highest weights of the occurring Verma modules change in each step by $\pm\epsilon_j$ which changes the δ-height either by 1 or -1. By induction, $d - \mathrm{ht}_{\underline{\delta}}(\mu)$ is even.

Conversely, assume μ is a weight with $\mathrm{ht}_{\underline{\delta}}(\mu) = d - 2k$ for some $k \geq 0$. If $k > 0$ then by induction hypothesis $P(\mu)$ appears in $M(\underline{\delta}) \otimes V^{\otimes d-2}$. The weight μ contains at least one neighboured $\wedge\circ$, $\vee\circ$, or $\times\circ$ pair (since only finitely many vertices are labelled $\wedge$, $\vee$ or $\times$). Hence we are in one of the cases (i), (ii) or (vii) of Proposition 3.8 so $P(\mu)$ is a summand of $\mathcal{F}_{i,+}\mathcal{F}_{i,-}P(\mu)$ and therefore also a summand in $M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}$. Now we have to proceed via a case-by case argument for $k = 0$:

(1) Assume first that there exist $i \in I(\mu)$, $-i \notin I(\mu)$ with $i := (\mu + \rho)_a > 0$ and $\underline{\delta}_a - \mu_a > 0$ for some a. (In the diagram picture this means that the ath $\vee$ in $\underline{\delta}$ was moved to the left and gives a $\vee$ in μ at position i). We assume that a was chosen to be maximal amongst these (that is the rightmost $\vee$ which got moved to the left to create some $\vee$). Locally at the positions i and $i + 1$ the diagram for μ could look as follows
   (a) $\vee\circ$. In this case set $\nu = \circ\vee$
   (b) $\vee\wedge$. In this case set $\nu = \circ\times$
   (c) $\vee\times$. In this case set $\nu = \times\vee$
   (d) $\vee\vee$. Since a was maximal, this means that the $\vee$ at position $i+1$ forces $i + 1 = (\mu + \rho)_{a+1}$ and at the same time $\underline{\delta}_{a+1} - \mu_{a+1} \leq 0$ which is a contradiction.

   In each of the first three cases $P(\mu)$ appears as a summand in $P(\nu) \otimes V$ (by Lemma 3.8 (iii), (vii), (x)) and $\mathrm{ht}_{\underline{\delta}}(\nu) = \mathrm{ht}_{\underline{\delta}}(\mu) - 1$. By induction hypothesis $P(\nu)$ appears as summand in $M(\underline{\delta}) \otimes V^{\otimes d-1}$ and we are done.

(2) Assume, (1) does not hold and moreover that if $i := (\mu+\rho)_a > 0$ for some a and $-i \notin I$ then $\underline{\delta}_a - \mu_a = 0$. (Diagrammatically this means that $\vee's$ did either not move or turned into $\wedge's$, creating possibly some $\times$'s). Choose, if it exists, $-j \in I$ maximal such that $-j < 0$. (Diagrammatically we look at the $\wedge$'s and $\times$'s and choose the leftmost. It sits at position j). Locally at the positions $j-1$ and j, for $j > \frac{1}{2}$, the diagram for μ could look as follows
   (a) $\diamond\wedge$. In this case set $\nu = \circ\times$.
   (b) $\vee\wedge$. In this case set $\nu = \times\circ$.
   (c) $\circ\wedge$. In this case set $\nu = \wedge\circ$.
   (d) $\vee\times$. In this case set $\nu = \times\vee$.
   (e) $\circ\times$. In this case set $\nu = \wedge\vee$. The $\vee$ in $\underline{\delta}$ at place $j-1$ got moved, hence turned into an $\wedge$ or created a $\times$. But then the same happened to all the $\vee$'s further to the left. In particular, ν has no $\vee$ left of the $\wedge$ at position j and hence we can apply Lemma 3.12 to $P(\nu)$.

   Again, in each of the last four cases $P(\mu)$ appears as a summand in $P(\nu) \otimes V$ by Proposition 3.8 and by induction hypothesis $P(\nu)$ appears as summand in $M(\underline{\delta}) \otimes V^{\otimes d-1}$. In case no $\wedge$ or $\times$ exists we either have $\mu = \underline{\delta}$ and there is nothing to do or μ has a $\circ\vee$ or $\vee\circ$ pair which we swap to create a new weight ν and argue as above using Proposition 3.8 (i) or (iii) with (x).

(3) Assume finally that neither (1) nor (2) holds. The arguments are completely analogous to (2) except for the case
   (e') $\circ\times$. In case there is no $\vee$ to the left of the cross we set $\nu = \wedge\vee$ and argue as before using Lemma 3.12. Otherwise such a $\vee$ did not get moved or got moved to the right, since we excluded case (1). Since it is separated from our $\times$ at position j by a $\circ$, our $\times$ is created from a $\vee$ which got moved to the right. Hence changing $\circ\times$ to $\vee\wedge$ decreases the height by 1. Then we can argue again using Proposition 3.8, this time part (v).

□

Lemma 4.8. *Let $\delta, d \in \mathbb{Z}$, $\delta \geq 0$, $d \geq 1$. The set*

$$\mathcal{S} = \mathcal{S}(\delta, d) = \{\mu \in \Lambda \mid \mathrm{ht}_{\underline{\delta}}(\mu) \leq d, \mathrm{ht}_{\underline{\delta}}(\mu) \equiv d \mod 2\} \tag{39}$$

is saturated in the following sense: if $\mu \in \mathcal{S}$ and $\nu \in \Lambda$ such that μ and ν are in the same block and $\nu \geq \lambda$ then $\nu \in \mathcal{S}$.

Proof. Assume μ is in the same block as ν, $\mu < \nu$. Then the diagrammatic weight for ν is obtained from μ by applying a finite sequence of changes from $\wedge\wedge$ to $\vee\vee$ or from $\vee\wedge$ to $\wedge\vee$ at positions only separated by $\circ'$ and $\times$'s. Hence it is enough to consider these basic changes.

Change from $\vee\wedge$ to $\wedge\vee$: Let $0 \leq i \leq i+j$ be the positions of the two symbols. In the ϵ-basis $\mu+\rho$ and $\nu+\rho$ are then of the form

$$(\overbrace{-x_1,\dots-x_a}, -(i+j), \overbrace{-y_1,\dots-y_b}, \overbrace{z_1,\dots,z_c}, \overbrace{u_1,\dots u_r}, i, \overbrace{v_1,\dots,v_s}, \overbrace{w_1,\dots w_t})$$
$$(\overbrace{-x_1,\dots-x_a}, \overbrace{-y_1,\dots-y_b}, -i\,\overbrace{z_1,\dots,z_c}, \overbrace{u_1,\dots u_r}, \overbrace{v_1,\dots,v_b}, i+j, \overbrace{w_1,\dots w_t})$$

where $a,b,c,r,t \geq 0$ with the $x_k > 0$ indicating the $\wedge$'s to the right of position $i+j$, the $y_k > 0$ indicating the $\wedge$'s as part of the crosses between positions i and $i+j$, the $z_k > 0$ indicating the $\wedge$'s to the left of position i, the $u_k > 0$ indicating the $\vee$'s to the left of position i, the $v_k > 0$ indicating the $\vee$'s as part of the crosses between positions i and $i+j$ and the $w_k > 0$ indicating the $\vee$'s to the right of position $i+j$. Hence to determine $\mathrm{ht}_{\underline{\delta}}(\mu) - \mathrm{ht}_{\underline{\delta}}(\nu)$ we could without loss of generality assume that the x's, z's, u's and w's are zero. Recalling that the vector $\delta+\rho$ is an increasing sequence of consecutive numbers, we are left with

$$\begin{aligned}
&\mathrm{ht}_{\underline{\delta}}(\mu) - \mathrm{ht}_{\underline{\delta}}(\nu)\\
= \;& (|m+i+j| + \sum_{k=1}^{b} |m+k+y_k| - \sum_{k=1}^{b} |m+k-1+y_k| - |m+b+i|)\\
&+(|p-i| + \sum_{k=1}^{b} |p+k-v_k| - \sum_{k=1}^{b} |p+k-1-v_k| - |p+b-i-j|)
\end{aligned}$$

for some non-negative m, p which are integers in case i, j are integers and half-integers in case i,j are half-integers. The absolute values in the first summand can be removed and hence gives j in total. Set $v_0 = i$ and $v_{b+1} = i+j$, and choose $-1 \leq k_0 \leq b+1$ such that $p+k-v_k > 0$ for $0 \leq k \leq k_0$ and $p+k-v_k \leq 0$ for $b+1 \geq k > k_0$.

$$\begin{aligned}
&\mathrm{ht}_{\underline{\delta}}(\mu) - \mathrm{ht}_{\underline{\delta}}(\nu)\\
= \;& j + |p-v_0| - |p+b-v_{b+1}| + \sum_{1\leq k\leq k_0, b} ((p+k-v_k) - (p+k-1-v_k))\\
&+ \sum_{k_0+1\leq k\leq b} ((v_k-p-k) - (v_k-p-k+1))\\
= \;& j + |p-v_0| - |p+b-v_{b+1}| + k_0 - (b-k_0) = \begin{cases} 2j & \text{if } k_0 = b+1,\\ 2(p-i+k_0) & \text{if } 0 \leq k_0 \leq b,\\ 0 & \text{if } k_0 = -1. \end{cases}
\end{aligned}$$

In any case $\mathrm{ht}_{\underline{\delta}}(\mu) \geq \mathrm{ht}_{\underline{\delta}}(\nu)$ and their difference is even, since j, k_0 and $p-i$ are integral. Change from $\wedge\wedge$ to $\vee\vee$: It is easy to check that the difference of the heights is even. We claim that if μ' and ν' are weights such that ν' is obtained from μ' just by changing an $\wedge$ at some position i which has no other $\wedge$ or $\vee$ to its left into a $\vee$ then $\mathrm{ht}_{\underline{\delta}}(nu') \leq \mathrm{ht}_{\underline{\delta}}(\mu)$. Together with the previous paragraph this claim proves the lemma. In the ϵ-basis, the weights

$\mu' + \rho$ and $\nu' + \rho$ have the following form

$$(\overbrace{-x_1, \dots - x_a}, \overbrace{-y_1, \dots - y_b}, \overbrace{z_1, \dots, z_b}, i, \overbrace{w_1, \dots w_c})$$
$$(\overbrace{-x_1, \dots - x_a}, -i, \overbrace{-y_1, \dots - y_b}, \overbrace{z_1, \dots, z_b}, \overbrace{w_1, \dots w_c})$$

where $a, b, c \geq 0$ with the $x_k > 0$ indicating the $\wedge$'s to the right of position i, the $y_k > 0$ indicating the $\wedge$'s as part of the crosses to the left of i, the $z_k > 0$ indicating the $\vee$'s as part of the crosses to the left of i and the $w_k > 0$ indicating the $\vee$'s to the right of position i. Again, the x's and w's are irrelevant for the difference of the heights. Note that the number of y's equals the number of z's. With the arguments from above we get

$$\begin{aligned} \mathrm{ht}_{\underline{\delta}}(\mu') - \mathrm{ht}_{\underline{\delta}}(\nu') &= (|m+i| + \sum_{k=1}^{b}(m+k+y_k) - \sum_{k=1}^{b}(m+k-1+y_k)) \\ &+ (\sum_{k=1}^{b}(|p+k-z_k| - \sum_{k=1}^{b}|p+k-1-z_k|) - |p+b-i|) \end{aligned}$$

with $p = m + b + 1$. The first sum equals b, the second $\geq -b$ and hence $\mathrm{ht}_{\underline{\delta}}(\mu') - \mathrm{ht}_{\underline{\delta}}(\nu') \geq m+i-|m+b+1-i|$. This is obviously positive if $m+b+1-i < 0$ and equals $-2b + 2i - 1$ otherwise. On the other hand, $b < i$ and our claim follows. □

4.3. **Quasi-hereditaryness.** The category $\mathcal{O}^{\mathfrak{p}}(n)$ is a highest weight category in the sense of [CPS88], i.e. it is a category $\mathcal{C}$ with a poset $(\mathcal{S}, \leq)$ satisfying

- $\mathcal{C}$ is an $\mathbb{C}$-linear Artinian category equipped with a duality, that is, a contravariant involutive equivalence of categories $\circledast : \mathcal{C} \to \mathcal{C}$;
- for each $\lambda \in \mathcal{S}$ there is a given object $L(\lambda) \cong L(\lambda)^{\circledast} \in \mathcal{C}$ such that $\{L(\lambda) | \lambda \in \mathcal{S}\}$ is a complete set of representatives for the isomorphism classes of irreducible objects in $\mathcal{C}$;
- each $L(\lambda)$ has a projective cover $P(\lambda) \in \mathcal{C}$ such that all composition factors of $P(\lambda)$ are isomorphic to $L(\mu)$'s for $\mu \in \mathcal{S}$;
- writing $V(\lambda)$ for the largest quotient of $P(\lambda)$ with the property that all composition factors of its radical are isomorphic to $L(\mu)$'s for $\mu > \lambda$, the object $P(\lambda)$ has a filtration with $V(\lambda)$ at the top and all other factors isomorphic to $V(\nu)$'s for $\nu < \lambda$.

The poset $\mathcal{S}$ is in our example the set of Λ highest weight vectors for irreducible modules with the reversed Bruhat ordering $\leq_{\mathrm{rBruhat}}$; $V(\lambda)$ is the Verma module of highest weight λ and the duality is the standard duality from $\mathcal{O}$, [Hum08].

The category $\mathbb{W}_d(\alpha, \beta)^{\mathrm{opp}}$ – mod of finite dimensional (right) $\mathbb{W}_d(\alpha, \beta)$-modules is by [AMR06, Corollary 8.6] also a highest weight category, with poset the set of bipartitions of $d - 2t$ for $1 \leq t \leq \lfloor \frac{d}{2} \rfloor$ equipped with the dominance ordering defined below, $V(\lambda)$ the cell module from [AMR06, Section 6], and $\circledast$ the ordinary vector space duality using Remark 2.2.

The category of finite dimensional $\mathrm{Br}_d(\delta)$-modules with $\delta \neq 0$ is by [KX98, Theorem 1.3], [CDVM09a, Corollary 2.3] a highest weight category, with poset the set of partitions of $d-2t$ for $1 \leq t \leq \lfloor \frac{d}{2} \rfloor$ with the dominance ordering, $V(\lambda)$ the cell module from [CDVM09a, Lemma 2.4] and $\circledast$, the ordinary vector space duality using the analogue of Remark 2.2 for the Brauer algebra. In case $\delta = 0$ it is a highest weight category if and only if $\delta \neq 0$ or $\delta = 0$ and d odd.

Theorem 4.9. *For fixed δ, d and $\mathcal{S} = \mathcal{S}(\delta, d)$ as in (39) let $\mathcal{O}^{\mathfrak{p}}(n,\delta,d)$ be the full subategory of $\mathcal{O}^{\mathfrak{p}}(n)$ given by all modules with composition factors $L(\mu)$ with $\mu \in \mathcal{S}$. Then*

(1) $\mathcal{O}^{\mathfrak{p}}(n,\delta,d)$ inherits the structure of a highest weight category from $\mathcal{O}^{\mathfrak{p}}(n)$ with poset $(\mathcal{S}, \leq_{\mathrm{rBruhat}})$ and the $V(\lambda)$ for $\lambda \in \mathcal{S}$. The duality on $\mathcal{O}^{\mathfrak{p}}(n)$ restricts to the duality on the subcategory.

(2) The functor $\mathcal{E} = \mathrm{Hom}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}, ?)$ defines an equivalence

$$\mathcal{E}: \quad \mathcal{O}^{\mathfrak{p}}(n,\delta,d) \quad \cong \quad \mathbb{W}_d(\alpha,\beta)^{\mathrm{opp}} - \mathrm{mod}$$

of highest weight categories, i.e. $\mathcal{E}P(\lambda) \cong P(\varphi(\lambda))$, $\mathcal{E}V(\lambda) \cong V(\varphi(\lambda))$, $\mathcal{E}L(\lambda) \cong L(\varphi(\lambda))$ for any $\lambda \in \mathcal{S}(\delta,\lambda)$ and $\mathcal{E}$ interchanges the dualities.

Proof. The first statement follows directly from the fact, Lemma 4.8, that $\mathcal{S}$ is saturated, see [Don98, Appendix]. The functor $\mathcal{E}$ is an equivalence of categories, hence induces a highest weight structure on $\mathbb{W}_d(\alpha,\beta)^{\mathrm{opp}} - \mathrm{mod}$ and it is enough to show that it agrees with the one defined in [AMR06]. Using Remark 2.2 we can find an isomorphism $\mathbb{W}_d(\alpha,\beta)^{\mathrm{opp}} \cong \mathbb{W}_d(\alpha,\beta)$ which we fix. Theorem 3.1 induces an isomorphism $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} \cong \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$. Under the identifications $\mathcal{O}^{\mathfrak{p}}(n,\delta,d) = \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} - \mathrm{mod} = \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}) - \mathrm{mod}$, the duality on $\mathcal{O}^{\mathfrak{p}}(n,\delta,d)$ corresponds to the ordinary duality $\mathrm{Hom}_{\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})}(?, \mathbb{C})$ and hence our equivalence interchanges the dualities. Since ϕ induces a bijection between the labelling sets of the two highest weight structures, it is enough to show that one ordering refines the other which is done in Lemma 4.12 below. □

Remark 4.10. Due to the fact that Verma path always only include diagrammatic weights with the same integrality as $\underline{\delta}$, $\mathcal{O}^{\mathfrak{p}}(n,\delta,d)$ is either a subcategory of $\mathcal{O}^{\mathfrak{p}}_{1}(n)$ or $\mathcal{O}^{\mathfrak{p}}_{\frac{1}{2}}(n)$ depending on whether δ is even or odd.

Definition 4.11. The *dominance ordering* on $\mathcal{P}^1$ is defined by

$$\lambda \trianglerighteq \mu \text{ if } |\lambda| > |\mu| \text{ or } |\lambda| = |\mu| \quad \text{and} \quad \sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j \text{ for any } k \geq 1;$$

and on $\mathcal{P}^2$ by $\lambda = (\lambda^1, \lambda^2) \trianglerighteq \mu = (\mu^1, \mu^2)$ if $|\lambda^{(1)}| + |\lambda^{(2)}| > |\mu^{(1)}| + |\mu^{(2)}|$ or $|\lambda^{(1)}| + |\lambda^{(2)}| = |\mu^{(1)}| + |\mu^{(2)}|$, $\lambda^{(1)} \trianglerighteq \mu^{(1)}$ and $|\lambda^{(1)}| + \sum_{j=1}^{k} \lambda^{(2)}_j \geq |\mu^{(1)}| + \sum_{j=1}^{k} \mu^{(2)}_j$ for all $k \geq 0$.

The reversed partial ordering on $\mathcal{S}$ is weaker than the dominance ordering on bipartitions:

Lemma 4.12. *For any $\lambda, \mu \in \mathcal{S}(\delta, d)$ we have*

$$\lambda \leq_{\mathrm{rBruhat}} \mu \quad \Rightarrow \quad \varphi(\lambda) \trianglerighteq \varphi(\mu).$$

Proof. Recall that the Bruhat ordering is generated by basic swaps, turning two symbols $\wedge\vee$ (in this order with only $\circ$'s and $\times$'s between them) into a $\vee\wedge$ or two $\wedge$'s into two $\vee$'s making the weight smaller. In the first case, the $\wedge$ moves to the right and a $\vee$ moves to the left. The $\wedge$ increases the number of boxes in the first partition, whereas the $\vee$ increases the number of boxes in the first partition or decreases the number of boxes in the second partition. In any case however, the total number of boxes removed is at most the total number of boxes added. Similarly, in case two if two $\wedge$'s turn into two $\vee$'s, the total number of boxes increases or stays the same, but the number of boxes in the first partition increases. In both cases, the newly created bipartition is larger in the dominance order. $\square$

Due to our assumptions on $n \geq 2d$ and thus the simple structure of the idempotent z_d, the category $z_d \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} z_d - \mathrm{mod}$ is equivalent to a quotient category of $\mathcal{O}^{\mathfrak{p}}(n, \delta, d)$.

For $\mathcal{O}^{\mathfrak{p}}_1(n)$, resp. $\mathcal{O}^{\mathfrak{p}}_{\rotatebox{180}{1}}(n)$, we denote by p_δ the projection onto those blocks whose block diagram Θ satisfies $P_\bullet(\Theta) \cup P_\times(\Theta) \subset [0, \frac{\delta}{2} + n - 1]$

For even δ we put $\widetilde{\mathcal{F}} = p_\delta \mathcal{F} p_\delta$, otherwise we put $\widetilde{\mathcal{F}} = p_\delta \mathcal{F}^{\rotatebox{180}{1}} p_\delta$. Furthermore let $P_d = \widetilde{\mathcal{F}}^d(M^{\mathfrak{p}}(\underline{\delta}))$. Then $\operatorname{End}_{\mathfrak{g}}(P_d) = z_d \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d}) z_d$ and we have the quotient functor

$$\operatorname{Hom}_{\mathfrak{g}}(P_d, ?): \quad \mathcal{O}(n, \delta, d) \quad \to \quad z_d \operatorname{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} z_d - \mathrm{mod}.$$

We will denote the corresponding quotient category by $\mathcal{O}^{\mathfrak{p}}(n, \delta, d)'$.

Theorem 4.13. *The highest weight structure on $\mathcal{O}^{\mathfrak{p}}(n, \delta, d)$ induces a highest weight structure on $\mathcal{O}^{\mathfrak{p}}(n, \delta, d)'$ iff $\delta \neq 0$ or $\delta = 0$ and d odd or $d = 0$.*

Proof. A Verma path is compatible with the truncation if any two steps differ by some $\pm\epsilon_j$ for $j < n$. Let $\mathcal{S}'$ be the set of weights $\lambda \in \mathcal{S}(\delta, d)$ which appear as endpoints of a compatible Verma path of length d. The paths correspond to up-down-d-bitableaux where the second component is always the empty partition. Ignoring the second component we identify these paths with up-down-d-tableaux. For the algebra to be quasi-hereditary is is enough that all indecomposable projective modules indexed by these up-down-d-tableaux appear as summands in P_d. For $d = 0$ there is nothing to check and for $d = 1$ the partition with one box corresponds to $P(\lambda_{\wedge\circ\times\wedge}) = M^{\mathfrak{p}}(\underline{\delta} - \epsilon_1)$. Now assume the claim is true for $d - 2$. Hence all partitions with $d - 2k$ boxes arise as labels of indecomposable projectives in P_{d-2}. Let now λ be such a partition. If $\delta > 0$ then the associated weight diagram has at least one $\circ$ and hence at least one of the configurations displayed in the bottom line of the following table at positions i, $i+1$ smaller than $\frac{\delta}{2}+n$:

μ	$\wedge\circ$	$\vee\circ$	$\vee\wedge$	$\circ\wedge$	$\circ\vee$	$\vee\wedge$
λ	$\circ\wedge$	$\circ\vee$	$\circ\times$	$\wedge\circ$	$\vee\circ$	$\times\circ$

In each case we choose the weight μ to be equal to λ except at the positions i, $i+1$ where it is as shown in the table. Then $\widetilde{\mathcal{F}}P(\lambda)$ contains $P(\mu)$ as direct summand and $\widetilde{\mathcal{F}}P(\mu)$ contains $P(\lambda)$ as summand. Hence $P(\lambda)$ appears as a summand in P_d. Now assume λ is a weight corresponding to a partition with d boxes and $\delta > 0$. If $d > 0$ then there is at least one box which can be removed from the partition, hence λ contains at least one of the configurations displayed in the top line of the following table at positions i, $i+1$ smaller than $\frac{\delta}{2}+n$:

										at $\frac{1}{2}$
λ	$\circ\wedge$	$\vee\wedge$	$\circ\times$	$\vee\times$	$\vee\circ$	$\times\circ$	$\vee\wedge$	$\times\wedge$	$\diamond\circ$	$\wedge$
μ	$\wedge\circ$	$\times\circ$	$\wedge\vee$	$\times\vee$	$\circ\vee$	$\wedge\vee$	$\circ\times$	$\wedge\times$	$\circ\vee$	$\vee$

In this case we create by Lemma 3.11, (37) and Corollary 3.12 a weight μ whose partition has $d-1$ boxes such that $\widetilde{\mathcal{F}}P(\mu)$ contains $P(\lambda)$ as direct summand and we are done by induction.

It remains to consider the case $\delta = 0$. For $d = 0$, $d = 1$ the assertion is clear. For $d = 2$ we have $P_d = P(a) \oplus P(b)$, where a and b correspond to the partitions (2) and $(1,1)$ respectively and $\mathrm{End}_{\mathfrak{g}}(P_2) \cong \mathbb{C}[x]/(x^2) \oplus \mathbb{C}$ which has infinite global dimension and hence is not quasi-hereditary. In case $\delta = 0$ and $d = 0$ then every weight contains at east one $\circ$ and we can argue as in the case $\delta > 0$. For d odd observe that the occurring weights λ always contain at least one $\wedge$ or $\times$, and P_d contains a summand isomorphic to P_2. Hence the algebra is not quasi-hereditary. $\square$

Remark 4.14. The truncated algebra $z_d \,\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} z_d$ inherits always a cellular algebra structure in the sense of [GL96] from $\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}}$, see [KX98]. By Theorem 4.13, $z_d \,\mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})^{\mathrm{opp}} z_d$ is quasi-hereditary if and only if $\delta \neq 0$ or $\delta = 0$ and d odd.

5. Koszulness and Gradings

The following theorem shows the existence of a hidden grading on the cyclotomic VW-algebras. Recall that any block $\mathcal{B}$ of ordinary or parabolic category $\mathcal{O}$ for $\mathfrak{g}$ is equivalent to the category of modules over the endomorphism ring $A = \mathrm{End}_{\mathfrak{g}} P_{\mathcal{B}}$ of a minimal projective generator P. By [BGS96], A has a natural Koszul grading which we fix. The category of finite dimensional graded A-modules is called the *graded version* of $\mathcal{B}$ and denoted $\hat{\mathcal{B}}$. For a block of $\mathcal{O}(n)$, the algebra A is precisely the Khovanov algebra attached to the block in [ES13b].

Theorem 5.1. *Let $d, \delta \in \mathbb{Z}_{\geq 0}$.*

(1) The algebra $\mathbb{W}_d(\alpha,\beta)$ is Morita equivalent to a Koszul algebra.

(2) The algebra $z_d\mathbb{W}_d(\alpha,\beta)z_d$ is Morita equivalent to a Koszul algebra if and only if $\delta \neq 0$ or $\delta = 0$ and d odd or $d = 0$.

Proof. The category $\mathcal{O}^{\mathfrak{p}}(n)$ is a highest weight category and each block is equivalent to a category of finite dimensional modules over a Koszul algebra $A = \mathrm{End}_{\mathfrak{g}}(P)$, [BGS96]. Then A is standard Koszul by [ÁDL03, Corollary 3.8]. Since the set $\mathcal{S}$ is saturated, the algebra $\mathbb{W}_d(\alpha,\beta)$ is again standard Koszul by [ÁDL03, Proposition 1.11]. Since it is standard Koszul and quasi-hereditary it is Koszul by [ÁDL03, Theorem 1]. The second statement follows by the same arguments using Theorem 4.13. □

Any indecomposable projective, Verma or simple module in $\mathcal{B}$ as above has a graded lift in $\hat{\mathcal{B}}$ and this lift is unique up to isomorphism and overall shift in the grading, [Str03]. We choose a lift such that the head is concentrated in degree zero and denote it by the same symbol as the original module. For an explicit description of these modules for $\mathcal{O}(n)$ see [ES13b].

Remark 5.2. The Koszul grading on the basic algebra underlying $\mathbb{W}_d(\alpha,\beta)$ or $z_d\mathbb{W}_d(\alpha,\beta)z_d$ is unique up to isomorphism. Lemma 3.10 refines then to graded decomposition numbers given by parabolic Kazhdan-Lusztig polynomials of type (D_n, A_{n-1}), see [LS13] for explicit formulas and [ES13b] for an explicit description of the underlying basic Koszul algebra.

Lemma 5.3. *The special projective functors from Definition 3.7 lift to the graded functors $\hat{\mathcal{F}} := (?\otimes V) \cong \bigoplus_{i\in\mathbb{Z}_{\geq 0}+\frac{1}{2}}(\hat{\mathcal{F}}_{i,-}\oplus\hat{\mathcal{F}}_{i,+}) : \mathcal{O}^{\mathfrak{p}}_1(n) \to \hat{\mathcal{O}}^{\mathfrak{p}}{}_1(n)$ and $\hat{\mathcal{F}}^{\lrcorner} := (?\otimes V) \cong \bigoplus_{i\in\mathbb{Z}_{>0}}(\hat{\mathcal{F}}_{i,-}\oplus\hat{\mathcal{F}}_{i,+})\oplus\hat{\mathcal{F}}_0 : \hat{\mathcal{O}}^{\mathfrak{p}}{}_{\lrcorner}(n) \to \hat{\mathcal{O}}^{\mathfrak{p}}{}_{\lrcorner}(n)$ satisfying now the properties of Lemma 3.8 by replacing the $\mathcal{F}$ by $\hat{\mathcal{F}}$. Then $\langle i\rangle$, $i\in\mathbb{Z}$ denotes the grading shift up by i.*

Proof. By Remark 3.9 each of these functors is a translation functor to or out of a wall or an equivalence of categories, hence the existence of graded lifts is given by [Str03]. To deduce the formulas note that our normalization is slightly different here, since we shift the translation functor to the wall in [Str03] by $\langle 1\rangle$ and the translation functor out of the wall by $\langle -1\rangle$. Alternatively one could also mimic the arguments in [BS11b] for the Khovanov algebra of type D from [ES13b]. □

Note that given a graded abelian category $\mathcal{A}$, the Grothendieck group is a $\mathbb{Z}[q,q^{-1}]$-module, hence a $\mathbb{Q}(q)$-vector space by extension of scalars, by letting act q^r by shifting the grading by $+r$. We denote this $\mathbb{Q}(q)$-vector space by $K_0(\mathcal{A})$. In particular, we have $K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}}(n)\right)$ with the induced action of the graded special projective functors. We describe this now in detail.

6. Coideal subalgebras

6.1. **Quantum groups.** The following constructions fall again into two families, the even/half-integer family and the odd/integer family. To be able to treat these cases simultaneously as much as possible we introduce the following indexing sets:

$$\begin{array}{llll} I = \mathbb{Z} + \frac{1}{2}, & I^{++} = \{i \in I \mid i \geq \frac{3}{2}\}, & J = \mathbb{Z}, & J^{+} = \{j \in J \mid j \geq 1\}, \\ I^{\daleth} = \mathbb{Z}, & I^{\daleth,++} = \{i \in I \mid i \geq 1\}, & J^{\daleth} = \mathbb{Z} + \frac{1}{2}, & J^{\daleth,+} = \{j \in J \mid j \geq \frac{1}{2}\}. \end{array}$$

Definition 6.1. Let $\mathcal{U} = \mathcal{U}_q(\mathfrak{gl}_{\mathbb{Z}})$ be the *(generic) quantized universal enveloping algebra* for $\mathfrak{gl}_{\mathbb{Z}}$. That means $\mathcal{U}$ is the associative unital algebra over the field of rational functions $\mathbb{Q}(q)$ in an indeterminate q with generators $D_j^{\pm 1}$ for $j \in J$ and E_i, F_i for $i \in I$ subject to the following relations:

(i) all $D_j^{\pm 1}$ commute and $D_j D_j^{-1} = 1$ for $j \in J$,

(ii) For $i \in I$ and $j \in J$:

$$D_j E_i D_j^{-1} = \begin{cases} qE_i & \text{for } j = i - \frac{1}{2} \\ q^{-1}E_i & \text{for } j = i + \frac{1}{2} \\ E_i & \text{otherwise} \end{cases} \qquad D_j F_i D_j^{-1} = \begin{cases} qF_i & \text{for } j = i + \frac{1}{2} \\ q^{-1}F_i & \text{for } j = i - \frac{1}{2} \\ F_i & \text{otherwise} \end{cases}$$

(iii) E_i and $F_{i'}$ commute unless $i = i'$ and in this case

$$E_iF_i - F_iE_i = \frac{D_{i-\frac{1}{2}}D^{-1}_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}D^{-1}_{i-\frac{1}{2}}}{q - q^{-1}}.$$

(iv) If $|i - i'| = 1$ then

$$E_i^2 E_{i'} - (q + q^{-1})E_iE_{i'}E_i + E_{i'}E_i^2 = 0$$

in all other cases $E_iE_{i'} = E_{i'}E_i$.

(v) Analogously, if $|i - i'| = 1$ then

$$F_i^2 F_{i'} - (q + q^{-1})F_iF_{i'}F_i + F_{i'}F_i^2 = 0$$

in all other cases $F_iF_{i'} = F_{i'}F_i$.

The *quantized enveloping algebra* $\mathcal{U}^{\daleth} = \mathcal{U}_q(\mathfrak{gl}_{\mathbb{Z}^{\daleth}})$ is defined in exactly the same way by replacing I with $I^{\daleth}$ and J with $J^{\daleth}$ in the definitions.

In $\mathcal{U}$, resp. $\mathcal{U}^{\daleth}$, let $K_i := D_{i-\frac{1}{2}}D^{-1}_{i+\frac{1}{2}}$ for $i \in I$, resp. $i \in I^{\daleth}$. From this definition it follows immediately that

$$K_iE_iK_i^{-1} = q^2E_i \qquad \text{and} \qquad K_iF_iK_i^{-1} = q^{-2}F_i.$$

Both $\mathcal{U}$ and $\mathcal{U}^{\daleth}$ are Hopf algebras with comultiplication Δ defined on generators by

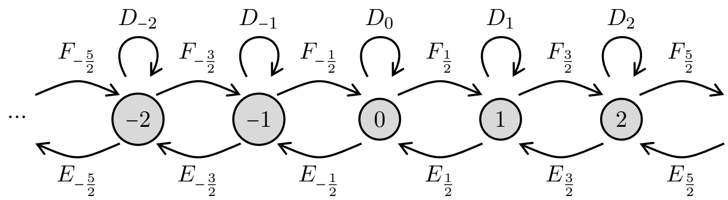
$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(D_j^{\pm 1}) = D_j^{\pm 1} \otimes D_j^{\pm 1}.$$

Definition 6.2. Let $\mathbb{V}$ be the vector space on basis $\{v_l \mid l \in J\}$. The *natural representation* of $\mathcal{U}$ is the representation on $\mathbb{V}$ given by the rules

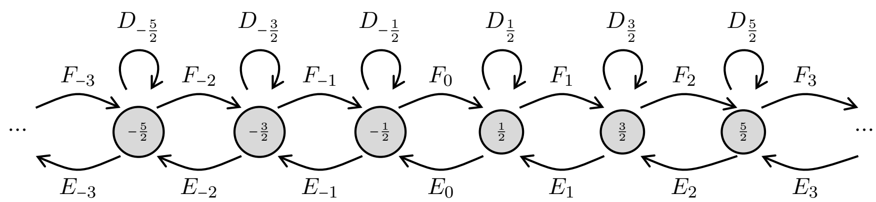
$$E_iv_l = \delta_{i+\frac{1}{2},l}v_{l-1} \qquad F_iv_l = \delta_{i-\frac{1}{2},l}v_{l+1} \qquad D_j^{\pm 1}v_l = q^{\pm\delta_{j,l}}v_l$$

for $i \in I$ and $j \in J$. The *natural representation* $\mathbb{V}^{\daleth}$ of $\mathcal{U}^{\daleth}$ is defined analogously by using $J^{\daleth}$ as the indexing set for the basis; all formulas stay the same.

Remark 6.3. The action on $\mathbb{V}$ can be illustrated as follows, showing how the F_i's and E_i's move between weight spaces and on which weight space D_j acts by multiplication with q, for $\mathbb{V}$



and similarly for $\mathbb{V}^{\lrcorner}$



The modules we are interested in are the quantum analogues of the fundamental representations of $\mathfrak{gl}_{\mathbb{Z}}$.

Definition 6.4. The k-th *quantum exterior power* $\bigwedge^k \mathbb{V}$ is the $\mathcal{U}$-submodule of $\bigotimes^k \mathbb{V}$ with basis given by the vectors

$$v_{i_1} \wedge \cdots \wedge v_{i_k} := \sum_{w \in S_k} (-q)^{\ell(w)} v_{i_{w(1)}} \otimes \cdots \otimes v_{i_{w(k)}} \tag{40}$$

for all $i_1 < \cdots < i_k$ from the index set J. Here, $\ell(w)$ denotes the usual length of a permutation w in the symmetric group S_k.

The corresponding module $\bigwedge^k \mathbb{V}^{\lrcorner}$ for $\mathcal{U}^{\lrcorner}$ is defined in the same way by again just changing the indexing set for the basis.

Definition 6.5. The *quantized universal enveloping algebra* $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$ is defined as the associative unital algebra over the field of rational functions $\mathbb{Q}(q^{1/2})$ with the sets of generators

$$\{E_i \mid i \in I^{++}\} \cup \{F_i \mid i \in I^{++}\} \cup \{D_j^{\pm 1} \mid j \in J^+\},$$

which satisfy the same relations as the generators of $\mathcal{U}$ by identifying $X_i \in U_q(\mathfrak{gl}_{\mathbb{N}})$ with X_i in $\mathcal{U}$ for $X \in \{E, F, D\}$.

Let $\mathbb{W}$ denote the natural representation of $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$ with basis $\{w_j \mid j \in J^+\}$ and consider the $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$-module $\bigwedge^r \mathbb{W} \otimes \bigwedge^{k-r} \mathbb{W}$. Each factor has the same type of basis as $\mathbb{V}$ above. Hence the tensor product has the obvious monomial basis

$$\left\{ (w_{i_1} \wedge \cdots \wedge w_{i_r}) \otimes (w_{j_1} \wedge \cdots \wedge w_{j_{k-r}}) \;\middle|\; \begin{array}{l} i_1, \ldots, i_r, j_1, \ldots, j_{k-r} \in J^+, \\ i_1 < \cdots < i_r,\ j_1 < \cdots < j_{k-r} \end{array} \right\}. \tag{41}$$

As before we can change the indexing set for the generators of $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$ to $I^{\daleth,++}$ and $J^{\daleth,+}$ and would obtain an isomorphic algebra with corresponding natural representation $\mathbb{W}^{\daleth}$.

Remark 6.6. Although the above definition of $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$ gives a natural embedding of Hopf algebras into both $\mathcal{U}$ and $\mathcal{U}^{\daleth}$ we will be interested in another copy of $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$ inside $\mathcal{U}$ and $\mathcal{U}^{\daleth}$ given by certain involution invariant generators.

6.2. **Coideal subalgebra.** We are interested in the quantization of the fixed point Lie algebra $\mathfrak{g}^{\theta}$ for a certain involution θ. Unfortunately, the obvious construction $\mathcal{U}_q(\mathfrak{g}^{\theta})$ does not naturally embed into $\mathcal{U}_q(\mathfrak{g})$. A way around this is to study coideal subalgebras (instead of Hopf subalgebras) of $\mathcal{U}_q(\mathfrak{g})$ which specialize to $\mathfrak{g}^{\theta}$. This was studied in [Let03] and [Let02].

Lemma 6.7. *The assignment*

$$E_i \mapsto qK_{-i}F_{-i}, \quad F_i \mapsto q^{-1}E_{-i}K_{-i}^{-1} \tag{42}$$

$$D_j \mapsto D_{-j}^{-1}, \quad K_i \mapsto K_{-i}, \quad q \mapsto q^{-1} \tag{43}$$

defines a q-antilinear involution θ on both $\mathcal{U}$ and $\mathcal{U}^{\daleth}$.

Proof. We first note that $\theta^2(X) = X$ for any generator X, e.g.

$$\theta^2(E_i) = \theta(qK_{-i}F_{-i}) = q^{-1}K_i q^{-1}E_i K_i^{-1} = E_i.$$

To see that it is well-defined we have to verify the relations. They fall into three families of which we check explicitly one each and omit the analogous calculations. Commuting relations:

$$\theta(D_{i+\frac{1}{2}}E_iD_{i+\frac{1}{2}}^{-1}) = D_{-i-\frac{1}{2}}^{-1}qK_{-i}F_{-i}D_{-i-\frac{1}{2}} = q^2K_{-i}F_{-i} = \theta(q^{-1}E_i).$$

Commutator relations:

$$\begin{aligned}\theta(E_iF_i - F_iE_i) &= K_{-i}F_{-i}E_{-i}K_{-i}^{-1} - E_{-i}F_{-i}\\ &= F_{-i}E_{-i} - E_{-i}F_{-i} = -\frac{D_{-i-\frac{1}{2}}D_{-i+\frac{1}{2}}^{-1} - D_{-i+\frac{1}{2}}D_{-i-\frac{1}{2}}^{-1}}{q-q^{-1}}\\ &= \theta\left(\frac{D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1} - D_{i-\frac{1}{2}}D_{i+\frac{1}{2}}^{-1}}{q^{-1}-q}\right) = \theta\left(\frac{D_{i-\frac{1}{2}}D_{i+\frac{1}{2}}^{-1} - D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1}}{q-q^{-1}}\right),\end{aligned}$$

Quantum Serre relations:

$$\begin{aligned}&\theta\left(E_i^2E_{i+1} - (q+q^{-1})E_iE_{i+1}E_i + E_{i+1}E_i^2\right)\\ =\;& q^3K_{-i}F_{-i}K_{-i}F_{-i}K_{-i-1}F_{-i-1} - (q+q^{-1})q^3K_{-i}F_{-i}K_{-i-1}F_{-i-1}K_{-i}F_{-i}\\ &\quad + q^3K_{-i-1}F_{-i-1}K_{-i}F_{-i}K_{-i}F_{-i}\\ =\;& q^3K_{-i}^2K_{-i-1}\left(F_{-i}^2F_{-i-1} - (q+q^{-1})F_{-i}F_{-i-1}F_{-i} + F_{-i-1}F_{-i}^2\right) = 0\end{aligned}$$

The claim follows. ∎

To study the fixed points of our involution θ, we enlarge the quantum groups $\mathcal{U}$ and $\mathcal{U}^{\lrcorner}$ by also including the elements $d_j^{\pm 1}$ for $j \in J$, resp. $j \in J^{\lrcorner}$, satisfying $d_j^2 = D_j$ and corresponding elements $k_i = d_{i-\frac{1}{2}} d_{i+\frac{1}{2}}^{-1}$. The relations for these elements are the same as for their squares just dividing the powers of q by two, hence we enlarge the base field to $\mathbb{Q}(q^{\frac{1}{2}})$. Denote these algebras by $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{U}}^{\lrcorner}$ respectively.

Lemma 6.8. *The involution $\theta : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$ fixes the following elements $\widetilde{B}_i \in \widetilde{\mathcal{U}}$*

$$\begin{aligned} \widetilde{B}_i &= q^{-\frac{1}{2}} E_i k_i^{-1} k_{-i}^{-1} + q^{-\frac{1}{2}} F_{-i} k_{-i} k_i^{-1} \quad \text{for } i \neq -\tfrac{1}{2}, \\ \widetilde{B}_{-\frac{1}{2}} &= E_{-\frac{1}{2}} k_{-\frac{1}{2}}^{-1} k_{\frac{1}{2}}^{-1} + q^{-1} F_{\frac{1}{2}} k_{\frac{1}{2}} k_{-\frac{1}{2}}^{-1}. \end{aligned}$$

The involution θ fixes the following elements $B_i \in \widetilde{\mathcal{U}}^{\lrcorner}$

$$\begin{aligned} \widetilde{B}_i &= q^{-\frac{1}{2}} E_i k_i^{-1} k_{-i}^{-1} + q^{-\frac{1}{2}} F_{-i} k_{-i} k_i^{-1} \quad \text{for } i \neq 0, \\ \widetilde{B}_0 &= q^{-1} E_0 K_0^{-1} + F_0. \end{aligned}$$

Proof. This is just a straight-forward calculation. For example one checks $\theta\left(q^{-\frac{1}{2}} E_i k_i^{-1} k_{-i}^{-1}\right) = q^{\frac{3}{2}} K_{-i} F_{-i} k_{-i}^{-1} k_i^{-1} = q^{-\frac{1}{2}} F_{-i} k_{-i} k_i^{-1}$. $\square$

Note that the definition of the generators of the coideal subalgebras vary slightly in the literature, see [Let03] and [Let02], although the coideals are classified. For our purposes a slight renormalisation of the generators is more suited.

Definition 6.9. We fix the following elements in $\mathcal{U}$

$$\begin{aligned} B_i &:= q^{\frac{1}{2}} \widetilde{B}_i k_i k_{-i}^{-1} &&= E_i K_{-i}^{-1} + F_{-i} \qquad \text{for } i \neq \pm\tfrac{1}{2}, \\ B_{\frac{1}{2}} &:= q^{\frac{1}{2}} \widetilde{B}_{\frac{1}{2}} k_{\frac{1}{2}} k_{-\frac{1}{2}}^{-1} &&= E_{\frac{1}{2}} K_{-\frac{1}{2}}^{-1} + F_{-\frac{1}{2}}, \\ B_{-\frac{1}{2}} &:= q \widetilde{B}_{-\frac{1}{2}} k_{-\frac{1}{2}} k_{\frac{1}{2}}^{-1} &&= q E_{-\frac{1}{2}} K_{\frac{1}{2}}^{-1} + F_{\frac{1}{2}}. \end{aligned}$$

and also the following elements in $\mathcal{U}^{\lrcorner}$

$$\begin{aligned} B_i &:= q^{\frac{1}{2}} \widetilde{B}_i k_i k_{-i}^{-1} &&= E_i K_{-i}^{-1} + F_{-i} \qquad \text{for } i \neq 0, \\ B_0 &:= \widetilde{B}_0 &&= q^{-1} E_0 K_0^{-1} + F_0. \end{aligned}$$

Definition 6.10. Let $\mathcal{R}$ denote the commutative subalgebra of $\mathcal{U}$ generated by the elements of the form $(D_j D_{-j})^{\pm 1}$ for all $j \in J$. Analogously we have $\mathcal{R}^{\lrcorner}$ the commutative subalgebra of $\mathcal{U}^{\lrcorner}$ generated by all $(D_j D_{-j})^{\pm 1}$ for all $j \in J^{\lrcorner}$.

Even though the generators are not invariant under θ it follows from the definition that they are mapped to their inverse via θ and thus $\mathcal{R}$ and $\mathcal{R}^{\lrcorner}$ themselves are stable under θ.

Remark 6.11. The subalgebras $\mathcal{R}$ and $\mathcal{R}^{\lrcorner}$ contain the elements of the form $K_i K_{-i}^{-1}$. They are slightly larger than the commutative subalgebra of the coideal subalgebra appearing in [Let03], since we are considering $\mathfrak{gl}$ instead of $\mathfrak{sl}$.

Definition 6.12. Let $\mathcal{H}$ be the subalgebra of $\mathcal{U}$ generated by the B_i for $i \in I$ and the subalgebra $\mathcal{R}$. In analogy, let $\mathcal{H}^\jmath$ be the subalgebra of $\mathcal{U}^\jmath$ generated by the B_i for $i \in I^\jmath$ and the subalgebra $\mathcal{R}^\jmath$.

The following lemma is straight-forward and checked on generators.

Lemma 6.13. *We have $\Delta(\mathcal{H}) \subset \mathcal{H} \otimes \mathcal{U}$, resp. $\Delta(\mathcal{H}^\jmath) \subset \mathcal{H}^\jmath \otimes \mathcal{U}^\jmath$, hence $\mathcal{H}$, resp. $\mathcal{H}^\jmath$, is a right coideal subalgebra of $\mathcal{U}$, resp. $\mathcal{U}^\jmath$.*

Remark 6.14. If we truncate our indexing set at m, we obtain the coideal subalgebra of type $AIII$ discussed in [Let03, Section 7] specializing to the Lie subalgebra $\mathfrak{gl}_m \times \mathfrak{gl}_m \subset \mathfrak{gl}_{2m}$ (resp. $\mathfrak{gl}_m \times \mathfrak{gl}_{m+1}$). Our renormalisation does not change this fact as we only multiplied with elements of the form $k_i k_{-i}^{-1}$. See also Section 7 for fixed point Lie algebras.

Although $\mathcal{H}$ and $\mathcal{H}^\jmath$ are not quantum groups, they each contain a copy of the quantum group $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$:

Lemma 6.15. *The assignment*

$$E_i \mapsto B_i, \qquad F_i \mapsto B_{-i}, \qquad D_j^{\pm 1} \mapsto (D_j D_{-j})^{\pm 1},$$

for $i \in I^{++}$ and $j \in J^+$ extends to an injective map of algebras from $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$ to $\mathcal{H}$. We denote the image of this embedding by $\check{\mathcal{U}}$.

Proof. Obviously the images of the $\check{D}_j^{\pm 1}$'s are pairwise inverse elements that form a commutative subalgebra. Furthermore one checks that

$$D_j D_{-j} B_i (D_j D_{-j})^{-1} = \begin{cases} qB_i & \text{if } j = i - \frac{1}{2} \\ q^{-1} B_i & \text{if } j = i + \frac{1}{2} \\ B_i & \text{otherwise} \end{cases}$$

and that the analogous statement for B_{-i} with inverse q's holds as well. Computing the commutator we obtain

$$\begin{aligned} [B_i, B_{-i}] &= [E_i K_{-i}^{-1}, F_i] + [F_{-i}, E_{-i} K_i^{-1}] = [E_i, F_i] K_{-i}^{-1} + [F_{-i}, E_{-i}] K_i^{-1} \\ &= \frac{K_i K_{-i}^{-1} - K_{-i} K_i^{-1}}{q - q^{-1}}. \end{aligned}$$

and $[B_i, B_{-i'}] = 0$ if $i \neq i'$. It remains to verify the quantum Serre relations, which is a long, but straight-forward calculation and therefore omitted. □

Remark 6.16. By using the indexing set $I^{\jmath,++}$ and $J^{\jmath,+}$ for $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$ we obtain the same embedding for $\mathcal{H}^\jmath$ and denote its image by $\check{\mathcal{U}}^\jmath$.

The coideal subalgebras have a Serre type presentation very similar to the quantum groups except that the Serre relations involving the generators $B_{\pm\frac{1}{2}}$ and B_0 are slightly modified, see [Let03, Theorem 7.1] for a general statement. In our case we have:

Proposition 6.17. *The coideal subalgebra $\mathcal{H}$ is isomorphic to the $\mathbb{Q}(q)$-algebra $\check{\mathcal{H}}$ with generators*

$$\{\check{E}_i, \check{F}_i \mid i \in I^{++}\} \cup \{\check{B}_+, \check{B}_-\} \cup \{\check{D}_j^{\pm 1} \mid j \in J^+\} \cup \{\check{D}_0^{\pm 1}\},$$

subject to the following relations

(1) *The* $\check{D}_i^{\pm 1}$ *generate a subalgebra isomorphic to* $\mathcal{R}$.
(2) *The* $\check{E}_i$, $\check{F}_i$, *and* $\check{D}_j^{\pm 1}$ *for* $i \in I^{++}$ *and* $j \in J^{+}$ *generate a subalgebra isomorphic to* $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$.
(3) $\check{D}_0$ *commutes with* $\check{E}_i$ *and* $\check{F}_i$ *for all* $i \in I^{++}$, *while* $\check{B}_+$ *and* $\check{B}^-$ *commute with all other generators except for the following relations*
   (a) *Commutation relations:*

$$\check{D}_1 \check{B}_+ \check{D}_1^{-1} = q^{-1} \check{B}_+, \qquad \check{D}_1 \check{B}_- \check{D}_1^{-1} = q \check{B}_-$$
$$\check{D}_0 \check{B}_+ \check{D}_0^{-1} = q^{2} \check{B}_+, \qquad \check{D}_0 \check{B}_- \check{D}_0^{-1} = q^{-2} \check{B}_-$$

   (b) *Quantum Serre relations:*

$$\begin{aligned}
\check{B}_+^2 \check{E}_{\frac{3}{2}} - (q+q^{-1}) \check{B}_+ \check{E}_{\frac{3}{2}} \check{B}_+ + \check{E}_{\frac{3}{2}} \check{B}_+^2 &= 0,\\
\check{B}_-^2 \check{F}_{\frac{3}{2}} - (q+q^{-1}) \check{B}_- \check{F}_{\frac{3}{2}} \check{B}_- + \check{F}_{\frac{3}{2}} \check{B}_-^2 &= 0,\\
\check{E}_{\frac{3}{2}}^2 \check{B}_+ - (q+q^{-1}) \check{E}_{\frac{3}{2}} \check{B}_+ \check{E}_{\frac{3}{2}} + \check{B}_+ \check{E}_{\frac{3}{2}}^2 &= 0,\\
\check{F}_{\frac{3}{2}}^2 \check{B}_- - (q+q^{-1}) \check{F}_{\frac{3}{2}} \check{B}_- \check{F}_{\frac{3}{2}} + \check{B}_- \check{F}_{\frac{3}{2}}^2 &= 0,
\end{aligned}$$

   (c) *Modified quantum Serre relations:*

$$\begin{aligned}
\check{B}_+^2 \check{B}_- &- (q+q^{-1}) \check{B}_+ \check{B}_- \check{B}_+ + \check{B}_- \check{B}_+^2\\
&= -(q+q^{-1}) \check{B}_+ (q \check{D}_1^{-1} \check{D}_0 + q^{-1} \check{D}_1 \check{D}_0^{-1}),\\
\check{B}_-^2 \check{B}_+ &- (q+q^{-1}) \check{B}_- \check{B}_+ \check{B}_- + \check{B}_+ \check{B}_-^2\\
&= -(q+q^{-1}) \check{B}_- (q^2 \check{D}_1 \check{D}_0^{-1} + q^{-2} \check{D}_1^{-1} \check{D}_0).
\end{aligned}$$

Proof. Up to renormalization these are the relations from [Let03] for the coideal subalgebra of type $AIII$. The isomorphism is then given by $\check{E}_i \mapsto B_i$, $\check{F}_i \mapsto B_{-i}$, $\check{D}_j^{\pm 1} \mapsto (D_j D_{-j})^{\pm 1}$, and $\check{B}_+ \mapsto B_{\frac{1}{2}}$, $\check{B}_- \mapsto B_{-\frac{1}{2}}$, $\check{D}_0 \mapsto D_0^2$. ☐

An analogous statement also holds for $\mathcal{H}^{𝅗𝅥}$.

Proposition 6.18. *The coideal subalgebra* $\mathcal{H}^{𝅗𝅥}$ *is isomorphic to the* $\mathbb{Q}(q)$-*algebra* $\check{\mathcal{H}}^{𝅗𝅥}$ *with generators*

$$\{\check{E}_i, \check{F}_i \mid i \in I^{𝅗𝅥,++}\} \cup \{\check{B}\} \cup \{\check{D}_j^{\pm 1} \mid j \in J^{𝅗𝅥,+}\},$$

subject to the following relations

(1) *The* $\check{D}_i^{\pm 1}$ *generate a subalgebra isomorphic to* $\mathcal{R}^{𝅗𝅥}$.
(2) *The* $\check{E}_i$, $\check{F}_i$, *and* $\check{D}_j^{\pm 1}$ *for* $i \in I^{𝅗𝅥,++}$ *and* $j \in J^{𝅗𝅥,+}$ *generate a subalgebra isomorphic to* $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$.
(3) *The generator* $\check{B}$ *commutes with all other generators except for the following relations:*

$$\begin{aligned}
\check{E}_1^2 \check{B} - (q+q^{-1}) \check{E}_1 \check{B} \check{E}_1 + \check{B} \check{E}_1^2 &= 0,\\
\check{F}_1^2 \check{B} - (q+q^{-1}) \check{F}_1 \check{B} \check{F}_1 + \check{B} \check{F}_1^2 &= 0,
\end{aligned}$$

$$\begin{aligned}\check{B}^2\check{E}_1 - (q+q^{-1})\check{B}\check{E}_1\check{B} + \check{E}_1\check{B}^2 &= \check{E}_1,\\ \check{B}^2\check{F}_1 - (q+q^{-1})\check{B}\check{F}_1\check{B} + \check{F}_1\check{B}^2 &= \check{F}_1.\end{aligned}$$

6.3. **Action on $\bigwedge^n \mathbb{V}$.** In [BS11a], a calculus of diagrammatic weights and diagrammatic algebras was introduced to describe maximal parabolic versions of the BGG-category $\mathcal{O}$ in type A. In particular, the Grothendieck group of the maximal parabolic category $\mathcal{O}$ was identified with the $\mathbb{Q}(q)$-vector space over the basis of diagrammatic weights which in turn was identified with a module for the quantum group $\mathcal{U}_q(\mathfrak{gl}_\mathbb{Z})$. Here we like to develop an analogous picture (based on [ES13b]). The first step is to decompose specific $\mathcal{U}$-modules with respect to $\check{\mathcal{U}}$.

For a sequence $\mathbf{i} = (i_1 < \ldots < i_r)$ of strictly decreasing integers we denote by $\mathbf{i}_+$ the subsequence of strictly positive numbers, by $\mathbf{i}_-$ the subsequence of strictly negative numbers and by $\mathbf{i}_0$ the subsequence consisting of the entry equal to 0 if they exists, otherwise they are empty. Furthermore we denote by $-\mathbf{i}$ the sequence $(-i_r < \ldots < -i_1)$.

Lemma 6.19. *There is an isomorphism of $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$-modules*

$$\Phi : \bigwedge^n \mathbb{V} \longrightarrow \bigoplus_{r=0}^{n}\left(\bigwedge^r \mathbb{W} \otimes \bigwedge^{n-r} \mathbb{W}\right) \oplus \bigoplus_{r=0}^{n-1}\left(\bigwedge^r \mathbb{W} \otimes \mathbb{Q}(q) \otimes \bigwedge^{n-1-r} \mathbb{W}\right)$$

where $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$ acts on the left via the isomorphism with $\check{\mathcal{U}}$ and $\mathbb{Q}(q)$ is the trivial representation with basis $\{w_0\}$. The map is given by

$$\Phi(v_{\boldsymbol{i}}) = \begin{cases} w_{-\boldsymbol{i}_-} \otimes w_{\boldsymbol{i}_0} \otimes w_{\boldsymbol{i}_+} & \text{if } \boldsymbol{i}_0 \neq \varnothing,\\ w_{-\boldsymbol{i}_-} \otimes w_{\boldsymbol{i}_+} & \text{otherwise.}\end{cases}$$

Similarly, there is an isomorphism of $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$-modules

$$\Phi^\jmath : \bigwedge^n \mathbb{V}^\jmath \longrightarrow \bigoplus_{r=0}^{n}\left(\bigwedge^r \mathbb{W}^\jmath \otimes \bigwedge^{n-r} \mathbb{W}^\jmath\right),$$

where $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$ acts on the left via the isomorphism with $\check{\mathcal{U}}^\jmath$ and the map is given by $\Phi^\jmath(v_{\boldsymbol{i}}) = w_{-\boldsymbol{i}_-} \otimes w_{\boldsymbol{i}_+}$.

Proof. We will omit the proof since the second isomorphism is a special case of Lemma 6.20 and the first completely analogous. □

The following more general decomposition will be used in Section 7.

Lemma 6.20. *Let $r \in \mathbb{Z}_{\geq 1}$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$. We have an isomorphism of $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$-modules*

$$\Phi^\jmath : \bigwedge^{k_1} \mathbb{V}^\jmath \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{V}^\jmath \longrightarrow \bigoplus_{0 \leq s_i \leq k_i}\left(\bigwedge^{k_r - s_r} \mathbb{W}^\jmath \otimes \ldots \otimes \bigwedge^{k_1 - s_1} \mathbb{W}^\jmath \otimes \bigwedge^{s_1} \mathbb{W}^\jmath \otimes \ldots \otimes \bigwedge^{s_r} \mathbb{W}^\jmath\right),$$

where $\mathcal{U}_q(\mathfrak{gl}_\mathbb{N})$ acts on the left via the isomorphism with $\check{\mathcal{U}}^\jmath$ followed by the comultiplication of $\mathcal{U}^\jmath$ and the map is given by

$$\Phi^\jmath\left(v_{\boldsymbol{i}_1} \otimes \ldots \otimes v_{\boldsymbol{i}_r}\right) = w_{-\boldsymbol{i}_{r,-}} \otimes \ldots \otimes w_{-\boldsymbol{i}_{1,-}} \otimes w_{\boldsymbol{i}_{1,+}} \otimes \ldots \otimes w_{\boldsymbol{i}_{r,+}}.$$

Proof. The map is an isomorphism of vector spaces since it sends a basis to a basis and both spaces have the same dimension. We need to verify that it is $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$-linear. We will only check the linearity on generators E_a for $a \in I^{\lrcorner,++}$, since the arguments for the others are parallel. On the left hand side, E_a acts via the isomorphism, which maps E_a to B_a followed by successive application of the comultiplication. Hence E_a acts by multiplication with

$$\sum_{l=0}^{r-1}(K_aK_{-a}^{-1})^{\otimes l} \otimes E_aK_{-a}^{-1} \otimes (K_{-a}^{-1})^{\otimes(r-1-l)} \quad + \quad \sum_{l=0}^{r-1} 1^{\otimes l} \otimes F_{-a} \otimes (K_{-a}^{-1})^{\otimes(r-1-l)}.$$

If we denote by $d(\mathbf{i},a)$ and $d(\mathbf{i},-a)$ the integers such that

$$K_av_{\mathbf{i}} = q^{d(\mathbf{i},a)}v_{\mathbf{i}}, \qquad K_{-a}v_{\mathbf{i}} = q^{-d(\mathbf{i},-a)}v_{\mathbf{i}}.$$

for a vector $v = v_{\mathbf{i}} = v_{\mathbf{i}_1} \otimes \ldots \otimes v_{\mathbf{i}_r}$, then we get

$$\begin{aligned} E_a.v \quad &= \quad \sum_{l=0}^{r-1} 1^{\otimes l} \otimes E_a \otimes 1^{\otimes(r-1-l)}.v \cdot q^{d(\mathbf{i}_1,-a)+\ldots+d(\mathbf{i}_r,-a)}q^{d(\mathbf{i}_1,a)+\ldots+d(\mathbf{i}_l,a)} \\ &\quad + \quad \sum_{l=0}^{r-1} 1^{\otimes l} \otimes F_{-a} \otimes 1^{\otimes(r-1-l)}.v \cdot q^{d(\mathbf{i}_{l+2},-a)+\ldots+d(\mathbf{i}_r,-a)}. \end{aligned}$$

On the other hand $w := \Phi^{\lrcorner}(v) = w_{-\mathbf{i}_{r,-}} \otimes \ldots \otimes w_{-\mathbf{i}_{1,-}} \otimes w_{\mathbf{i}_{1,+}} \otimes \ldots \otimes w_{\mathbf{i}_{r,+}}$. Since

$$K_aw_{\mathbf{i}_+} = q^{d(\mathbf{i},a)}w_{\mathbf{i}_+} \qquad K_aw_{-\mathbf{i}_i} = q^{d(\mathbf{i},-a)}w_{-\mathbf{i}_-}$$

for $K_a \in \mathcal{U}_q(\mathfrak{gl}_{\mathbb{Z}})$ and any sequence $\mathbf{i}$ we obtain

$$\begin{aligned} E_a.w \quad &= \quad \sum_{l=0}^{r-1} 1^{\otimes l} \otimes E_a \otimes 1^{\otimes(2r-1-l)}.w \cdot q^{d(\mathbf{i}_r,-a)+\ldots+d(\mathbf{i}_{r-l+1},-a)} \\ &\quad + \quad \sum_{l=r}^{2r-1} 1^{\otimes l} \otimes E_a \otimes 1^{\otimes(2r-1-l)}.w \cdot q^{d(\mathbf{i}_r,-a)+\ldots+d(\mathbf{i}_1,-a)}q^{d(\mathbf{i}_1,a)+\ldots+d(\mathbf{i}_{l-r},a)} \\ &= \quad \sum_{l=0}^{r-1} 1^{\otimes(r-1-l)} \otimes E_a \otimes 1^{\otimes(r+l)}.w \cdot q^{d(\mathbf{i}_r,-a)+\ldots+d(\mathbf{i}_{l+2},-a)} \\ &\quad + \quad \sum_{l=0}^{r-1} 1^{\otimes(r+l)} \otimes E_a \otimes 1^{\otimes(r-1-l)}.w \cdot q^{d(\mathbf{i}_r,-a)+\ldots+d(\mathbf{i}_1,-a)}q^{d(\mathbf{i}_1,a)+\ldots+d(\mathbf{i}_l,a)}. \end{aligned}$$

In the last step we just reordered the first sum and shifted the indexing of the second. In particular, $\Phi^{\lrcorner}$ commutes with the action of E_a. Similar calculations for the other generators imply that $\Phi^{\lrcorner}$ is a $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$-module homomorphism. □

Using the identification of $K_0\left(\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)\right)$, resp. $K_0\left(\hat{\mathcal{O}}_{\lrcorner}^{\mathfrak{p}}(n)\right)$, with the $\mathbb{Q}(q)$-vector spaces having $\mathbb{X}_n$, resp. $\mathbb{X}_n^{\lrcorner}$, as basis we can identify our modules with these spaces.

Integer case: Consider the $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$-module $\bigwedge^n \mathbb{V}$. To a standard basis vector $v_{\underline{i}}$ we can assign a diagrammatic weight in $\mathbb{X}_n$ via defining the sets

$P_\vee(\underline{i}) = \{i_j \mid i_j > 0\}$, $P_\wedge(\underline{i}) = \{-i_j \mid i_j < 0\}$ and $P_\diamond(\underline{i}) = \{i_j \mid i_j = 0\}$. The map

$$\Gamma : \quad \bigwedge^n \mathbb{V} \longrightarrow K_0\left(\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)\right),$$

defined on bases vectors by

$$\Gamma(v_{\underline{i}})_l := \begin{cases} \vee & \text{if } l \in P_\vee(\underline{i}) \setminus P_\wedge(\underline{i}), \\ \wedge & \text{if } l \in P_\wedge(\underline{i}) \setminus P_\vee(\underline{i}), \\ \diamond & \text{if } l \in P_\diamond(\underline{i}), \\ \times & \text{if } l \in P_\wedge(\underline{i}) \cap P_\vee(\underline{i}), \\ \circ & \text{if } l \notin P_\wedge(\underline{i}) \cup P_\vee(\underline{i}) \cup P_\diamond(\underline{i}). \end{cases}$$

yields an isomorphism of vector spaces. For a diagrammatic weight λ we denote the corresponding standard basis vector by v_λ.

Half-integer case: Consider the $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{N}})$-module $\bigwedge^n \mathbb{V}^{\lrcorner}$. To a standard basis vector $v_{\underline{i}}$ we again assign a diagrammatic weight, this time in $\mathbb{X}_n^{\lrcorner}$ via defining the sets $P_\vee(\underline{i}) = \{i_j \mid i_j > 0\}$ and $P_\wedge(\underline{i}) = \{-i_j \mid i_j < 0\}$. The map

$$\Gamma^{\lrcorner} : \bigwedge^n \mathbb{V}^{\lrcorner} \longrightarrow K_0\left(\hat{\mathcal{O}}_{\lrcorner}^{\mathfrak{p}}(n)\right),$$

defined on basis vectors by

$$\Gamma^{\lrcorner}(v_{\underline{i}})_l := \begin{cases} \vee & \text{if } l \in P_\vee(\underline{i}) \setminus P_\wedge(\underline{i}), \\ \wedge & \text{if } l \in P_\wedge(\underline{i}) \setminus P_\vee(\underline{i}), \\ \times & \text{if } l \in P_\wedge(\underline{i}) \cap P_\vee(\underline{i}), \\ \circ & \text{if } l \notin P_\wedge(\underline{i}) \cup P_\vee(\underline{i}). \end{cases}$$

yields an isomorphism of vector spaces. Again we denote the preimage of a diagrammatic weight λ by v_λ.

Using these two isomorphisms Γ and $\Gamma^{\lrcorner}$ we can transfer the action of $\mathcal{U}$ and $\mathcal{U}^{\lrcorner}$ to the vector spaces of diagrammatic weights. In general the generators of $\mathcal{U}$ and $\mathcal{U}^{\lrcorner}$ will not act in a way that is compatible with the structure of the categories underlying these vector spaces, but we will see that the generators of $\mathcal{H}$ and $\mathcal{H}^{\lrcorner}$ act in a way that can be interpreted categorically as the induced action of translation functors. The following Lemmas are all proved by a straightforward case by case calculation.

Lemma 6.21. *Let λ be a diagrammatic weight, then*

$$D_j D_{-j} \lambda = \begin{cases} \lambda & \text{if } \lambda_j = \circ, \\ q\lambda & \text{if } \lambda_j \in \{\vee, \wedge\}, \\ q^2 \lambda & \text{if } \lambda_j = \times. \end{cases}$$

Lemma 6.22. *Let λ be a diagrammatic weight in $\mathbb{X}_n$ and $i \in I^{++}$. For symbols $x, y \in \{\circ, \wedge, \vee, \times\}$ we write λ_{xy} for the diagrammatic weight obtained from λ by relabelling the $(i - \frac{1}{2})$th entry to x and the $(i + \frac{1}{2})$th entry to y.*

(1) We first consider the action of B_i:
- *(i) If $\lambda = \lambda_{\circ\vee}$, then $B_i.\lambda = \lambda_{\vee\circ}$.*
- *(ii) If $\lambda = \lambda_{\circ\wedge}$, then $B_i.\lambda = \lambda_{\wedge\circ}$.*
- *(iii) If $\lambda = \lambda_{\wedge\times}$, then $B_i.\lambda = \lambda_{\times\wedge}$.*
- *(iv) If $\lambda = \lambda_{\vee\times}$, then $B_i.\lambda = \lambda_{\times\vee}$.*

(v) *If* $\lambda = \lambda_{\wedge\vee}$*, then* $B_i.\lambda = q\lambda_{\times\circ}$*.*
(vi) *If* $\lambda = \lambda_{\vee\wedge}$*, then* $B_i.\lambda = \lambda_{\times\circ}$*.*
(vii) *If* $\lambda = \lambda_{\circ\times}$*, then* $B_i.\lambda = q^{-1}\lambda_{\vee\wedge} + \lambda_{\wedge\vee}$*.*
(viii) *For all other* λ *we have* $B_i.\lambda = 0$*.*

(2) *We now consider the action of* B_{-i}*:*
(i) *If* $\lambda = \lambda_{\wedge\circ}$*, then* $B_{-i}.\lambda = \lambda_{\circ\wedge}$*.*
(ii) *If* $\lambda = \lambda_{\vee\circ}$*, then* $B_{-i}.\lambda = \lambda_{\circ\vee}$*.*
(iii) *If* $\lambda = \lambda_{\times\vee}$*, then* $B_{-i}.\lambda = \lambda_{\vee\times}$*.*
(iv) *If* $\lambda = \lambda_{\times\wedge}$*, then* $B_{-i}.\lambda = \lambda_{\wedge\times}$*.*
(v) *If* $\lambda = \lambda_{\wedge\vee}$*, then* $B_{-i}.\lambda = q\lambda_{\circ\times}$*.*
(vi) *If* $\lambda = \lambda_{\vee\wedge}$*, then* $B_{-i}.\lambda = \lambda_{\circ\times}$*.*
(vii) *If* $\lambda = \lambda_{\times\circ}$*, then* $B_{-i}.\lambda = q^{-1}\lambda_{\vee\wedge} + \lambda_{\wedge\vee}$*.*
(viii) *For all other* λ *we have* $B_{-i}.\lambda = 0$*.*

Remark 6.23. The analogous version of Lemma 6.22 also holds if we look at diagrammatic weights in $\mathbb{X}^{\lrcorner}$ and $i \in I^{\lrcorner,++}$, i.e replace $\mathbb{X}$ by $\mathbb{X}^{\lrcorner}$ and I^{++} by $I^{\lrcorner,++}$ in the statements.

With Lemmas 6.21 and 6.22 we know the action of the generators $\check{E}_i$, $\check{F}_i$, and $\check{D}_i^{\pm 1}$ of $\mathcal{H}$ and $\mathcal{H}^{\lrcorner}$ induced on $K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}}(n)\right)$ and $K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}}_{\lrcorner}(n)\right)$ for $i \in I^{++}$ and $i \in I^{\lrcorner,++}$. We consider the action of the special generators B_0, $B_{-\frac{1}{2}}$.

Lemma 6.24. (1) *The element* B_0 *acts by zero on a diagrammatic weight* $\lambda \in \mathbb{X}_n^{\lrcorner}$ *unless* $\lambda_{\frac{1}{2}} \in \{\wedge, \vee\}$*. If* $\lambda = \lambda_{\vee}$ *then* $B_0.\lambda = \lambda_{\wedge}$ *and if* $\lambda = \lambda_{\wedge}$ *then* $B_0.\lambda = \lambda_{\vee}$*.*

(2) *Let* $\lambda \in \mathbb{X}_n$*. For symbols* $x, y \in \{\circ, \wedge, \vee, \times, \diamond\}$ *we write* λ_{xy} *for the diagrammatic weight obtained from* λ *by relabelling the* 0*th entry to* x *and the* 1*st entry to* y*.*
(i+) *If* $\lambda = \lambda_{\circ\vee}$ *then* $B_{\frac{1}{2}}.\lambda = \lambda_{\diamond\circ}$*.*
(ii+) *If* $\lambda = \lambda_{\circ\wedge}$ *then* $B_{\frac{1}{2}}.\lambda = \lambda_{\diamond\circ}$*.*
(iii+) *If* $\lambda = \lambda_{\circ\times}$ *then* $B_{\frac{1}{2}}.\lambda = q^{-1}\lambda_{\diamond\wedge} + \lambda_{\diamond\vee}$*.*
(iv+) *For all other* λ *we have* $B_{\frac{1}{2}}.\lambda = 0$*.*
(i-) *If* $\lambda = \lambda_{\diamond\circ}$ *then* $B_{-\frac{1}{2}}.\lambda = \lambda_{\circ\wedge} + \lambda_{\circ\vee}$*.*
(ii-) *If* $\lambda = \lambda_{\diamond\vee}$ *then* $B_{-\frac{1}{2}}.\lambda = q\lambda_{\circ\times}$*.*
(iii-) *If* $\lambda = \lambda_{\diamond\wedge}$ *then* $B_{-\frac{1}{2}}.\lambda = \lambda_{\circ\times}$*.*
(iv-) *For all other* λ *we have* $B_{-\frac{1}{2}}.\lambda = 0$*.*

By comparing the actions of B_i on diagrammatic weights with the induced action of translation functors we obtain the following result.

Proposition 6.25. *On* $K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}}_1(n)\right)$*, the induced action of* $B_{\pm i}$ *and* $B_{\pm\frac{1}{2}}$ *on* K_0 *via* Γ *coincides with the action of* $[\mathcal{F}_{i,\pm}]$ *and* $[\mathcal{F}_{\frac{1}{2},\pm}]$ *respectively. On* $K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}}_{\lrcorner}(n)\right)$*, the induced action of* $B_{\pm i}$ *and* B_0 *on* K_0 *via* $\Gamma^{\lrcorner}$ *coincides with the action of* $[\mathcal{F}_{i,\pm}]$ *and* $[\mathcal{F}_{\frac{1}{2},\pm}]$ *respectively.*

Proof. This follows by comparing the action as described in Lemma 6.22 and Lemma 6.24 with the description of the action of the special translation functors $\mathcal{F}_{i,\pm}$ on Verma modules in Lemma 3.8. □

6.4. **Bar involution and dual canonical basis.** We start by defining an involution on $\mathcal{H}$ and $\mathcal{H}^{\text{\textbackprime}}$. In generalizes Lusztig's bar involution to coideals.

Lemma 6.26. *There is a unique $\mathbb{Q}(q)$-antilinear bar-involution $- : \mathcal{H} \to \mathcal{H}$ such that $\overline{B_i} = B_i$ and $\overline{D_j D_{-j}} = D_j^{-1} D_{-j}^{-1}$; similar for $\mathcal{H}^{\text{\textbackprime}}$.*

Proof. Using Proposition 6.17 one checks easily that such an assignment extends to an algebra involution. □

Definition 6.27. By a *compatible bar-involution* on an $\mathcal{H}$-module M we mean an anti-linear involution $- : M \to M$ such that $\overline{uv} = \overline{u}\,\overline{v}$ for each $u \in \mathcal{H}, v \in M$ and analogously for a $\mathcal{H}^{\text{\textbackprime}}$-module.

Note that as a $\mathcal{U}$-module $\mathbb{V}$ has one-dimensional weight spaces with all standard basis vectors being weight vectors. The next lemma shows that the module $\bigwedge^n \mathbb{V}$ possesses a compatible bar-involution. As our identification of the action of $\mathcal{H}$ and the translation functors already indicates, we will need to focus on vectors w_λ that correspond to a diagrammatic weight λ. Recall that for two diagrammatic weights in the same block $\Lambda_\Theta^{\overline{\epsilon}}$ we have the Bruhat order which gives a unique minimal element in each block. As usual, we start with the case of diagrammatic weights supported on the integers.

Lemma 6.28. *There is a unique compatible bar-involution on the $\mathcal{H}$-module $\bigwedge^n \mathbb{V}$ such that $\overline{w_\lambda} = w_\lambda$ for each diagrammatic weight $\lambda \in \mathbb{X}$ that is minimal in its block with respect to the Bruhat order. Moreover:*

$$\overline{w_\lambda} \in w_\lambda + \sum_{\mu < \lambda} q^{-1}\mathbb{Z}[q^{-1}] w_\mu,$$

for any $\lambda \in \mathbb{X}$.

Proof. We prove uniqueness, the existence follows below from Lemma 6.30. For any diagrammatic weight λ let $l(\lambda)$ denote the height of the weight in the reversed Bruhat order, i.e. the minimal number of changes that need to made to obtain the minimal element as described in [ES13b, Lemma 2.3].

Enough to show: $\overline{w_\lambda}$ is uniquely determined by the properties. For λ minimal $\overline{w_\lambda} = w_\lambda$, hence it is determined. So let us fix λ non-minimal and assume that $\overline{w_\mu}$ is uniquely determined for all μ with $l(\mu) < l(\lambda)$. Note that this includes diagrammatic weights in other blocks.

Case I: Assume there exists $i < j$ such that $\lambda_i = \wedge$ and $\lambda_j = \vee$ and $\lambda_s \in \{\circ, \times\}$ for $i < s < j$. Let λ' be the diagrammatic weight that coincides with λ except for $\lambda'_{i+1} = \vee$ and $\lambda'_s = \lambda_{s-1}$ for $i+1 < s \leq j$, i.e. we have moved the $\vee$ at position j next to the $\wedge$ at position i. By Lemma 6.22 it follows that there exists $B \in \mathcal{H}$ such that $Bw_{\lambda'} = w_\lambda$. Furthermore let μ be the diagrammatic weight that coincides with λ' except the positions i and $i+1$

are swapped, then $l(\mu) < l(\lambda') = l(\lambda)$. Then it follows again by Lemma 6.22 that

$$B_{-i-\frac{1}{2}}B_{i+\frac{1}{2}}.w_\mu = q^{-1}w_\mu + w_{\lambda'}.$$

This implies

$$\overline{w_\lambda} = BB_{-i-\frac{1}{2}}B_{i+\frac{1}{2}}.\overline{w_\mu} - qB\overline{w_\mu}.$$

Hence $\overline{w_\lambda}$ is uniquely determined.

Case II: Assume there is no pair $i < j$ as in Case I, then there exists $i < j$ such that $\lambda_i \in \{\vee, \diamond\}$ and $\lambda_j = \vee$ and choose i and j minimal with this property. As before we can always assume that $j = i + 1$ since there can be no $\wedge$'s between i and j by assumption.

If $i = 0$ let μ be the weight sequence coinciding with λ except $\mu_1 = \wedge$. Then by Lemma 2

$$B_{\frac{1}{2}}B_{-\frac{1}{2}}(w_\mu) = q^{-1}w_\mu + w_\lambda.$$

Again this implies

$$\overline{w_\lambda} = B_{\frac{1}{2}}B_{-\frac{1}{2}}(\overline{w_\mu}) - q\overline{w_\mu}.$$

If on the other hand $i \neq 0$ then move the symbols at position i and $i+1$ to position 0 and 1, denote the corresponding weight by η. By the calculation above we already know

$$\overline{w_\eta} = B_{\frac{1}{2}}B_{-\frac{1}{2}}(\overline{w_\mu}) - q\overline{w_\mu},$$

where μ coincides with η except that $\mu_1 = \wedge$. We can now use the element $B' \in \mathcal{H}$ that first moves the symbols at position 1 back to position $i+1$ and afterwards the symbol at position 0 back to position i and obtain

$$B'\overline{w_\eta} = B'B_{\frac{1}{2}}B_{-\frac{1}{2}}(\overline{w_\mu}) - qB'\overline{w_\mu},$$

since $l(\mu) < l(\lambda)$ the right hand side is known, but

$$B'\overline{w_\eta} = \overline{w_\lambda} + \overline{w_{\lambda'}},$$

with λ' coinciding with λ except $\lambda'_i = \wedge$. Since $l(\lambda') = l(\lambda)$ we can use Case I to deduce that $\overline{w_{\lambda'}}$ is already determined and hence $\overline{w_\lambda}$ is uniquely determined. □

Lemma 6.29. *There is a unique compatible bar-involution on the $\mathcal{H}^\daleth$-module $\bigwedge^n \mathbb{V}^\daleth$ such that $\overline{w_\lambda} = w_\lambda$ for each diagrammatic weight $\lambda \in \mathbb{X}^\daleth$ that is minimal in its block with respect to the Bruhat order. Moreover:*

$$\overline{w_\lambda} \in w_\lambda + \sum_{\mu<\lambda} q^{-1}\mathbb{Z}[q^{-1}]w_\mu,$$

for any $\lambda \in \mathbb{X}^\daleth$.

Proof. The uniqueness of a bar-involution on the $\mathcal{H}^\daleth$-module $\bigwedge^n \mathbb{V}^\daleth$ follows with similar arguments to Lemma 6.28, although they are slightly simpler because the special generator B_0 behaves nicer. The details are left to the reader. □

For the existence of a bar-involution with these properties we use that $\bigwedge^n \mathbb{V} \cong K_0\left(\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)\right)$ and that there is a graded duality on $\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)$ satisfying these properties.

Proposition 6.30. *The graded duality $\boldsymbol{d}$ on $\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)$, sending a module M to $M^{\circledast}$, induces a compatible bar-involution on $\bigwedge^n \mathbb{V}$, satisfying the properties of Lemma 6.28. The same also holds for the graded duality on $\hat{\mathcal{O}}_{\lrcorner}^{\mathfrak{p}}(n)$ and the induced involution on $\bigwedge^n \mathbb{V}^{\lrcorner}$.*

Proof. Let $M(\lambda)$ be a parabolic Verma module corresponding to a diagrammatic weight being minimal in its block. Then $M(\lambda) = L(\lambda)$ and hence

$$\mathbf{d}M(\lambda) = \mathbf{d}L(\lambda) = L(\lambda) = M(\lambda).$$

If on the other hand $M(\lambda)$ is a Verma module corresponding to an arbitrary diagrammatic weight, we have that $\mathbf{d}M(\lambda) = M(\lambda)^{\circledast} = \nabla(\lambda)$, the dual parabolic Verma module by definition. It is known, see [Str03] that

$$[\nabla(\lambda)] \in [M(\lambda)] + \sum_{\mu<\lambda} q^{-1}\mathbb{Z}[q^{-1}][M(\mu)].$$

It remains to show that $\mathbf{d}$ is compatible with the action of $\mathcal{H}$.

Since the simple modules form a basis of the Grothendieck group it is enough to check the compatibility on those, i.e. we need to show that for a translation functor $\mathcal{F}$ corresponding to a generator of $\mathcal{H}$ and any simple module $L(\lambda)$ we have

$$[\mathcal{F}L(\lambda)] = [\mathcal{F}\mathbf{d}L(\lambda)] = [\mathbf{d}\mathcal{F}L(\lambda)],$$

where the first equality is due to the fact that simple modules are self-dual. Thus we need to show that $[\mathcal{F}L(\lambda)]$ is self-dual as well. We will focus on $\mathcal{F}_{i,+}$ and leave the other cases to the reader. Using Lemma 3.8 we know that

$$\mathcal{F}_{i,+}L(\lambda_{\circ\vee}) = L(\lambda_{\vee\circ}) \qquad \mathcal{F}_{i,+}L(\lambda_{\circ\wedge}) = 0,$$
$$\mathcal{F}_{i,+}L(\lambda_{\wedge\times}) = L(\lambda_{\times\wedge}) \qquad \mathcal{F}_{i,+}L(\lambda_{\vee\times}) = 0,$$
$$\mathcal{F}_{i,+}L(\lambda_{\wedge\vee}) = 0 \qquad \mathcal{F}_{i,+}L(\lambda_{\vee\wedge}) = L(\lambda_{\times\circ}).$$

All of these are obviously self-dual and we are left with the case where $\mathcal{F}_{i,+}$ is applied to a weight $\lambda = \lambda_{\circ\times}$. This is done by showing that

$$[\mathcal{F}_{i,+}L(\lambda_{\circ\times})] = q[L(\lambda_{\vee\wedge})] + [L(\lambda_{\wedge\vee})] + q^{-1}[L(\lambda_{\vee\wedge})],$$

which is self-dual. Let Y be a set of diagrammatic weights and r_μ and s_μ integers such that

$$[L(\lambda_{\circ\times})] = \sum_{\mu\in Y}(-1)^{r_\mu}q^{s_\mu}[M(\mu)],$$

then for all these μ we have $\mu = \mu_{\circ\times}$. Applying Lemma 6.22 we obtain

$$\begin{aligned}[\mathcal{F}_{i,+}L(\lambda_{\circ\times})] &= \sum_{\mu\in Y}(-1)^{r_\mu}q^{s_\mu-1}[M(\mu_{\vee\wedge})] + (-1)^{r_\mu}q^{s_\mu}[M(\mu_{\wedge\vee})] \\ &= q^{-1}\sum_{\mu\in Y}(-1)^{r_\mu}q^{s_\mu}[M(\mu_{\vee\wedge})]\end{aligned}$$

$$+\sum_{\mu\in Y}(-1)^{r_\mu}\left(q^{s_\mu}[M(\mu_{\wedge\vee})]-q^{s_\mu+1}[M(\mu_{\vee\wedge})]\right)$$
$$+q\sum_{\mu\in Y}(-1)^{r_\mu}q^{s_\mu}[M(\mu_{\vee\wedge})].$$

Expressing $[L(\lambda_{\vee\wedge})]$, and $[L(\lambda_{\wedge\vee})]$ in terms of Verma modules one checks that this is equal to $(q+q^{-1})[L(\lambda_{\vee\wedge})]+[L(\lambda_{\wedge\vee})]$. □

Note that the isomorphism classes of simple modules concentrated in degree zero form a *canonical basis* of $\wedge^n\mathbb{V}\cong K_0\left(\hat{\mathcal{O}}_1^{\mathfrak{p}}(n)\right)$, that is they are bar-invariant and the transformation matrix to the standard basis is lower diagonal with 1's in the diagonal and elements in $\mathbb{Z}[q]$ below the diagonal. These entries are the parabolic Kazhdan-Lusztig polynomials of type (D_n,A_{n-1}).

7. Skew Howe duality

In the following we want to look at more general versions of parabolic category $\mathcal{O}$ of type D and correspondingly categorifications of tensor products as in Lemma 6.20 for $\mathcal{U}_q(\mathfrak{gl}_{\mathbb{Z}})$ and $\mathcal{H}$. Since the quantum parameter in these cases is very technical we will work first in the classical case and consider categorifications of $\mathfrak{gl}_r\times\mathfrak{gl}_r$-modules, instead of $\mathcal{H}$-modules.

7.1. **The module $\wedge(n,m,r)$.** For a positive integer t we denote by $\mathbb{W}_t^{\natural}$ the vector space of dimension t with basis $v_{\frac{1}{2}},\dots,v_{t-\frac{1}{2}}$, with basis elements labelled by half-integers. Similarly we denote by $\mathbb{V}_{2k}^{\natural}$ the vector space of dimension $2t$ with basis $v_{-t+\frac{1}{2}},\dots,v_{t-\frac{1}{2}}$. Finally denote by $\mathbb{M}$ the 2-dimensional vector space with basis v_+ and v_-. Depending on the situation we will regard these as vector spaces over $\mathbb{C}$ or over $\mathbb{Q}(q)$, in which case they are truncation of the spaces $\mathbb{W}$ and $\mathbb{V}$ we had in Section 6.

In the following fix m and r positive integers. We want to consider module structures on the vector space

$$\bigwedge(n,m,r):=\bigwedge^n\left(\mathbb{W}_m^{\natural}\otimes\mathbb{M}\otimes\mathbb{W}_r^{\natural}\right).$$

Since one can identify $\mathbb{W}_m^{\natural}\otimes\mathbb{M}\simeq\mathbb{V}_{2m}^{\natural}$ and $\mathbb{M}\otimes\mathbb{W}_r^{\natural}\simeq\mathbb{V}_{2r}^{\natural}$ we can view $\wedge(n,m,r)$ natural both as a $(\mathfrak{gl}_{2m},\mathfrak{gl}_r)$-bimodule and a $(\mathfrak{gl}_m,\mathfrak{gl}_{2r})$-bimodule.

7.2. **Fixed point subalgebra.** We first identify the fixed point subalgebra of $\mathfrak{gl}_{2m}$ which is the non-quantized analogue of $\mathcal{H}$. The classical analogue of the involution from Lemma 6.7 is $\Theta:\mathfrak{gl}_{2m}\to\mathfrak{gl}_{2m}$ defined as $A\mapsto\widetilde{J}A\widetilde{J}$, where $\widetilde{J}=\begin{pmatrix}0&J\\J&0\end{pmatrix}$ with J as in Section 2. i.e has 1's on the anti-diagonal and zeros elsewhere. Then

$$\mathfrak{g}^\Theta=\left\{X\in\mathfrak{gl}_{2m}\mid\widetilde{J}X\widetilde{J}=X\right\}\text{ with }\widetilde{J}=\begin{pmatrix}0&J\\J&0\end{pmatrix},$$

We have $\mathfrak{g}^\Theta \cong \mathfrak{gl}_m \times \mathfrak{gl}_m$, since it is the image of the involution

$$T_A : \mathfrak{gl}_{2m} \longrightarrow \mathfrak{gl}_{2m},\ X \mapsto AXA^{-1} \text{ with } A = \begin{pmatrix} \mathbf{1} & J \\ J & -\mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ is the $m\times m$ identity matrix. Restricted to $\mathfrak{gl}_m\times\mathfrak{gl}_m$, let $\mathfrak{gl}_m^+$ and $\mathfrak{gl}_m^-$ denote the images of the first respectively second factor one easily verifies, using $\widetilde{J}A = A\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ and $A\widetilde{J} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}A$, that

$$\mathfrak{gl}_m^+ = \left\{ \begin{pmatrix} X & XJ \\ JX & JXJ \end{pmatrix} \middle| X \in \mathfrak{gl}_m \right\} \text{ and} \tag{44}$$

$$\mathfrak{gl}_m^- = \left\{ \begin{pmatrix} X & -XJ \\ -JX & JXJ \end{pmatrix} \middle| X \in \mathfrak{gl}_m \right\}. \tag{45}$$

Fix the identification $\mathbb{W}_m^\lrcorner \otimes \mathbb{M} \simeq \mathbb{V}_{2m}^\lrcorner$, $v_i \otimes v_+ \mapsto v_i$ and $v_i \otimes v_- \mapsto v_{-i}$ as vector spaces. Twisting with T_A gives the decomposition

$$\mathbb{V}_{2m}^\lrcorner = \mathbb{W}_m^\lrcorner \otimes \langle v_+ + v_- \rangle \oplus \mathbb{W}_m^\lrcorner \otimes \langle v_+ - v_- \rangle, \tag{46}$$

as a $\mathfrak{g}^\Theta = \mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$-module. One should note that $\mathfrak{gl}_m^+$ acts as zero on the second summand, while $\mathfrak{gl}_m^-$ acts as zero on the first summand. The *diagonally embedded* $\mathfrak{gl}_m$ in $\mathfrak{gl}_m \times \mathfrak{gl}_m$ with its Chevalley generators normalized by $\frac{1}{2}$ maps via T_A to an isomorphic subalgebra of $\mathfrak{gl}_{2m}$ with generators $E_i := E_{i+1,i} + E_{-i-1,-i}$ and $F_i := E_{i,i+1} + E_{-i,-i-1}$ for $0 < i < m$. These are exactly the specialization of the generators B_i and B_{-i} from Definition 6.9. Together with the specialization of $\mathcal{R}$ they generate the *diagonally embedded* $\mathfrak{gl}_m$ in $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$. The special generator B_0 specializes to the element $G = E_{1,-1} + E_{-1,1}$ and generates together with the diagonal embedded Lie algebra the whole fixed point Lie algebra $\mathfrak{g}^\Theta$.

7.3. **The classical and quantum skew Howe duality.** All the identifications made above can of course also be done for $\mathfrak{gl}_{2r}$ and we will use the same notations for them. Using the decomposition from (46) we obtain

$$\begin{aligned} \bigwedge(n,m,r) &= \bigwedge^n \left(\mathbb{W}_m^\lrcorner \otimes \langle v_+ + v_- \rangle \otimes \mathbb{W}_r^\lrcorner \oplus \mathbb{W}_m^\lrcorner \otimes \langle v_+ - v_- \rangle \otimes \mathbb{W}_r^\lrcorner \right) \\ &\simeq \bigoplus_{i=0}^n \bigwedge^i \left(\mathbb{W}_m^\lrcorner \otimes \langle v_+ + v_- \rangle \otimes \mathbb{W}_r^\lrcorner \right) \otimes \bigwedge^{n-i} \left(\mathbb{W}_m^\lrcorner \otimes \langle v_+ - v_- \rangle \otimes \mathbb{W}_r^\lrcorner \right) \\ &\simeq \bigoplus_{i=0}^n \bigwedge^i \left(\mathbb{W}_m^\lrcorner \otimes \langle v_+ + v_- \rangle \otimes \mathbb{W}_r^\lrcorner \right) \boxtimes \bigwedge^{n-i} \left(\mathbb{W}_m^\lrcorner \otimes \langle v_+ - v_- \rangle \otimes \mathbb{W}_r^\lrcorner \right), \end{aligned} \tag{47}$$

where the $\boxtimes$ should denote that we regard the first factor as a $(\mathfrak{gl}_m^+, \mathfrak{gl}_r^+)$-bimodule and the second as a $(\mathfrak{gl}_m^-, \mathfrak{gl}_r^-)$-bimodule and take the outer tensor product of these two bimodules in each summand. From the usual skew Howe duality of $\mathfrak{gl}_m$ and $\mathfrak{gl}_r$ on $\bigwedge^n(\mathbb{W}_m^\lrcorner \otimes \mathbb{W}_r^\lrcorner)$ we can deduce that on each summand the $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$-endomorphisms are generated by $\mathfrak{gl}_r^+ \times \mathfrak{gl}_r^-$ and that there are no such morphisms between different summands. We obtain

Theorem 7.1. *The actions of* $U(\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-)$ *and* $U(\mathfrak{gl}_r^+ \times \mathfrak{gl}_r^-)$ *on* $\bigwedge(n,m,r)$ *commute and generate each others commutant.*

7.4. **The categorified classical skew Howe duality.** To categorify the bimodule $\bigwedge(n,m,r)$ we first use classical skew Howe duality for the pair $(\mathfrak{gl}_{2m}, \mathfrak{gl}_r)$ to obtain a decomposition as $\mathfrak{gl}_{2m}$-module

$$\bigwedge(n,m,r) \cong \bigoplus_{\underline{k}\in C(n,r)} \bigwedge^{\underline{k}} \mathbb{V}_m^{\lrcorner},$$

where $C(n,r)$ denotes the set of composition of n with r parts and $\bigwedge^{\underline{k}} \mathbb{V}_m^{\lrcorner} = \bigwedge^{k_1} \mathbb{V}_m^{\lrcorner} \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{V}_m^{\lrcorner}$. Naturally each summand is also a module for the subalgebra $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$. To identify each summand with the Grothendieck group of certain blocks of parabolic category $\mathcal{O}$ we denote by $\mathfrak{p}_{\underline{k}}$ the standard parabolic of $\mathfrak{so}_{2n}$ corresponding to the root subsystem where we omit the simple roots

$$\alpha_0, \alpha_{k_1}, \alpha_{k_1+k_2}, \alpha_{k_1+k_2+k_3}, \ldots, \alpha_{k_1+\ldots+k_{r-1}},$$

for each choice of $\underline{k} \in C(n,r)$. In this language our original parabolic $\mathfrak{p} = \mathfrak{p}_{(n)}$.

As in Section 1 we have the corresponding parabolic category $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}(n)$ inside the full category $\mathcal{O}$. For the basis of the corresponding Grothendieck groups, we have to look at $\mathfrak{p}_{\underline{k}}$-dominant weights.

Definition 7.2. An integral weight λ is $\mathfrak{p}_{\underline{k}}$-*dominant* if it is of the form

$$\lambda = (\lambda_1 \leq \ldots \leq \lambda_{k_1}, \lambda_{k_1+1} \leq \ldots \leq \lambda_{k_1+k_2}, \ldots, \lambda_{k_1+\ldots+k_{r-1}+1} \leq \ldots \leq \lambda_n).$$

Let $\Lambda_{\underline{k}}$ denote the set of all $\mathfrak{p}_{\underline{k}}$-dominant weights, decomposing as usual into $\Lambda_{\underline{k}} = \Lambda_{\underline{k}}^1 \cup \Lambda_{\underline{k}}^{\lrcorner}$, depending on whether the λ_i's are integers or half-integers.

Consider the subcategory $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$ generated by simple modules with highest weight in

$$\Lambda_{\underline{k}}^{\leq m} = \left\{ \lambda \in \Lambda_{\underline{k}}^{\lrcorner} \,\middle|\, |\lambda_i + \rho_i| \leq m - \frac{1}{2} \right\}.$$

This is obviously a finite set and stable under the dot-action of the Weyl group, thus $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$ is a finite sum of certain blocks in $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$. In complete analogy to Section 6, we associate to $\lambda \in \Lambda_{\underline{k}}^{\leq m}$ a vector $v_\lambda \in \bigwedge^{\underline{k}} \mathbb{V}_m^{\lrcorner}$ in the following way. First denote $\lambda' = \lambda + \rho$, then consider the sequence

$$\underline{i}_\lambda^j = (\lambda'_{k_1+\ldots+k_j} > \lambda'_{k_1+\ldots+k_j-1} > \ldots > \lambda'_{k_1+\ldots+k_{j-1}+1}),$$

and define the corresponding standard basis vector as $v_\lambda = v_{\underline{i}_\lambda^1} \otimes \ldots \otimes v_{\underline{i}_\lambda^r}$. Then $[M^{\mathfrak{p}_{\underline{k}}}(\lambda)] \mapsto v_\lambda$ defines an isomorphism of vector spaces

$$\Gamma_{\underline{k}}: \quad K_0\left(\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)\right) \longrightarrow \bigwedge^{\underline{k}} \mathbb{V}_m^{\lrcorner}.$$

Taking (direct) sums we obtain

$$\Gamma_m: \quad K_0\left(\bigoplus_{\underline{k}\in C(n,r)} \mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)\right) \longrightarrow \bigwedge(n,m,r). \tag{48}$$

Alternatively we could look at the decomposition as $\mathfrak{gl}_{2r}$-module and get

$$\Gamma_r: \quad K_0\left(\bigoplus_{\underline{k}\in C(n,m)} \mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq r}(n)\right) \longrightarrow \bigwedge(n,m,r). \tag{49}$$

Next, we decompose the categories $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$. Although we will refer to this as a block decomposition, this is not completely correct as the subcategories might decompose further, but it is a decomposition by central character.

Remark 7.3. Note that two $\mathfrak{p}_{\underline{k}}$-dominant weights give rise to parabolic Verma modules with the same central character iff they are in the same Weyl group orbit with respect to the dot-action. After adding ρ this corresponds to usual Weyl group orbits. Then two weights lie in the same orbit iff the multiplicity for each half-integer $\frac{1}{2},\dots,m-\frac{1}{2}$, up to sign, agrees in the two weights and the multiplicities of negative integers have the same parity.

Definition 7.4. Let $(\underline{k},\underline{\mu},\overline{\epsilon})$ be two compositions of n, $\underline{k}$ with at most r non-trivial parts and $\underline{\mu}$ with at most m non-trivial parts and $\overline{\epsilon}\in\{0,1\}$ a parity. Then $\mathcal{O}_{(\underline{k},\underline{\mu},\overline{\epsilon})}(n)$, will denote the subcategory of $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$ such that all weights λ of parabolic Verma modules appearing satisfy

$$\#\left\{\ j\ \middle|\ |\ \lambda_j+\rho_j\ |= i-\tfrac{1}{2}\ \right\}=\mu_i \ \text{ and }\ \overline{\#\left\{\ j\ \middle|\ \lambda_j+\rho_i<0\ \right\}}=\overline{\epsilon}.$$

Let us denote by $\mathrm{Bl}(n,r,m)$ the set of all triples $(\underline{k},\underline{\mu},\overline{\epsilon})$ as above, called *block triples*. Then we define, for a block triple Γ, the projection

$$\mathrm{pr}_\Gamma:\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}\to\mathcal{O}_\Gamma(n).$$

For $1\leq i\leq m-1$ and a block triple $\Gamma=(\underline{k},\underline{\mu},\overline{\epsilon})$ we define the triples

$$\Gamma_{i,+}=(\underline{k},\underline{\mu}',\overline{\epsilon})\quad\text{via}\quad \mu'_l=\begin{cases}\mu_l-1 & \text{if } l=i+1\\ \mu_l+1 & \text{if } l=i\\ \mu_l & \text{otherwise,}\end{cases} \tag{50}$$

$$\Gamma_{i,-}=(\underline{k},\underline{\mu}',\overline{\epsilon})\quad\text{via}\quad \mu'_l=\begin{cases}\mu_l+1 & \text{if } l=i+1\\ \mu_l-1 & \text{if } l=i\\ \mu_l & \text{otherwise,}\end{cases} \tag{51}$$

and

$$\Gamma_0=(\underline{k},\underline{\mu},\overline{\epsilon+1}). \tag{52}$$

Generalizing Definition 3.7 we can now define certain translation functors.

Definition 7.5. We have the *special projective functors* $\mathcal{F}_{i,+}$, $\mathcal{F}_{i,-}$ for $i\in\mathbb{Z}_{\geq 0}$ and $\mathcal{F}_0$ defined by

$$\mathcal{F}_{i,-}:=\bigoplus_\Gamma \mathrm{pr}_{\Gamma_{i,-}}\circ(?\otimes V)\circ\mathrm{pr}_\Gamma \quad:\quad \bigoplus_\Gamma\mathcal{O}_\Gamma(n)\to\bigoplus_\Gamma\mathcal{O}_\Gamma(n),$$
$$\mathcal{F}_{i,+}:=\bigoplus_\Gamma \mathrm{pr}_{\Gamma_{i,+}}\circ(?\otimes V)\circ\mathrm{pr}_\Gamma \quad:\quad \bigoplus_\Gamma\mathcal{O}_\Gamma(n)\to\bigoplus_\Gamma\mathcal{O}_\Gamma(n),$$

$$\mathcal{F}_0 := \bigoplus_{\Gamma} \mathrm{pr}_{\Gamma_0} \circ (? \otimes V) \circ \mathrm{pr}_{\Gamma} \quad : \quad \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n) \to \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n),$$

with all sums running over $\Gamma \in \mathrm{Bl}(n,r,m)$.

Proposition 7.6. *Under the identification*

$$K_0\left(\bigoplus_{\Gamma \in \mathrm{Bl}(n,r,m)} \mathcal{O}_{\Gamma}(n)\right) = \bigwedge(n,m,r)$$

the action of $[\mathcal{F}_{i,-}]$ *coincides with the one of* F_i*, the action of* $[\mathcal{F}_{i,+}]$ *coincides with the one of* E_i *and the action of* $[\mathcal{F}_0]$ *coincides with the action of* G*, hence gives a categorification of the action of* $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$*.*

Proof. We show the claim for $\mathcal{F}_{i,+}$ and leave the rest to the reader. Let $\Gamma = (\underline{k}, \underline{\mu}, \epsilon)$ be a block triple and $M^{\mathfrak{p}_{\underline{k}}}(\lambda)$ be a parabolic Verma module in the corresponding block. Then we can express $M^{\mathfrak{p}_{\underline{k}}}(\lambda) \otimes V$ in the Grothendieck group as

$$[M^{\mathfrak{p}_{\underline{k}}}(\lambda) \otimes V] = \sum_{l:\lambda+\epsilon_l \in \Lambda_{\underline{k}}} [M^{\mathfrak{p}_{\underline{k}}}(\lambda + \epsilon_l)] + \sum_{l:\lambda-\epsilon_l \in \Lambda_{\underline{k}}} [M^{\mathfrak{p}_{\underline{k}}}(\lambda - \epsilon_l)].$$

Checking now when a parabolic Verma module of the form $M^{\mathfrak{p}_{\underline{k}}}(\lambda - \epsilon_l)$ lies in the block corresponding to $\Gamma_{i,+}$ we see that this is the case if and only if $\lambda_l + \rho_l = i + \frac{1}{2}$ and similarly with $M^{\mathfrak{p}_{\underline{k}}}(\lambda + \epsilon_l)$. Under our identification this coincides exactly with the way E_i acts on the Grothendieck group. □

7.5. **The commuting action of the Hecke algebra.** As before we restrict to the half-integer case. Consider the special case where $\underline{k} = (1, 1, \dots, 1)$, hence $\bigwedge^{\underline{k}} \mathbb{V}_m^{\lrcorner} = {\mathbb{V}_m^{\lrcorner}}^{\otimes r}$.

Definition 7.7. The *Hecke algebra* $\mathbb{H}_r$ corresponding to the Weyl group $W_r = \langle s_{\alpha_i} \mid 0 \le i \le r \rangle$ of $\mathfrak{so}_{2r}$ is the unital $\mathbb{Q}(q)$-algebra with generators H_i, $0 \le i \le r$ subject to the quadratic relation $H_i^2 = 1 + (q^{-1} - q)H_i$, and the type D braid relations $H_i H_j = H_j H_i$ if $s_{\alpha_i} s_{\alpha_j} = s_{\alpha_j} s_{\alpha_i}$ and $H_i H_j H_i = H_j H_i H_j$ if $s_{\alpha_i} s_{\alpha_j} s_{\alpha_i} = s_{\alpha_j} s_{\alpha_i} s_{\alpha_j}$ with $1 \le i, j \le r$.

Lemma 7.8. *There is a right action of the Hecke algebra* $\mathbb{H}_r$ *on* ${\mathbb{V}_m^{\lrcorner}}^{\otimes r}$ *commuting with the action of* $\mathcal{H}^{\lrcorner}$*. The generator* H_i *for* $1 \le i \le r$ *acts on the* i*th and* $i+1$*th tensor factor by the formula*

$$v_a \otimes v_b . H_i \quad = \quad \begin{cases} v_b \otimes v_a & \text{if } a < b \\ v_b \otimes v_a + (q^{-1} - q) v_a \otimes v_b & \text{if } a > b \\ q^{-1} v_b \otimes v_a & \text{if } a = b. \end{cases} \tag{53}$$

and H_0 *acts on the first and second tensor factor by*

$$v_a \otimes v_b . H_0 \quad = \quad \begin{cases} v_{-b} \otimes v_{-a} & \text{if (55) holds,} \\ v_{-b} \otimes v_{-a} + (q^{-1} - q) v_a \otimes v_b & \text{if (56) holds} \\ q^{-1} v_{-b} \otimes v_{-a} & \text{if } a = -b, \end{cases} \tag{54}$$

$$\text{for } a, b > 0, \quad \text{or} \quad (|a| > |b| \text{ and } b < 0 < a), \quad \text{or } (|a| < |b| \text{ and } a < 0 < b); \tag{55}$$

$$\text{for } a,b<0, \quad \text{or} \quad (|a|>|b| \text{ and } a<0<b), \;\text{or } (|a|<|b| \text{ and } b<0<a). \tag{56}$$

Proof. One can directly verify that the rules are compatible with the relations in the Hecke algebra, hence indeed define an action. The verification that it commutes with the action of the coideal subalgebra is a lengthy straight forward case by case calculation showing that $(B_j.v).H_i = B_j.(v.H_i)$ for any choice of j,i and basis vector v. □

Analogously we can identify $\bigwedge(n,m,r)$ with $K_0\left(\bigoplus_{\Gamma\in\mathrm{Bl}(n,m,r)}\mathcal{O}_\Gamma(n)\right)$ and with the analogous functors obtain an action of $\mathfrak{gl}_r^+\times\mathfrak{gl}_r^-$.

Consider the graded version $\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$ of $\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)$ as described in Section 5. We again have the corresponding identification of $\mathbb{Q}(q)$-vector spaces

$$\hat{\Gamma}_{\underline{k}}: \qquad K_0\left(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)\right)\longrightarrow \bigwedge^{\underline{k}}\mathbb{V}^{\jmath}_m. \tag{57}$$

by sending the class $[\hat{M}^{\mathfrak{p}_{\underline{k}}}(\lambda)]$ of the standard graded lift of the parabolic Verma module $M^{\mathfrak{p}_{\underline{k}}}(\lambda)$ to v_λ with head concentrated in degree zero. (The multiplication with q corresponds to the grading shift $\langle 1\rangle$).

Recall from [AL03], [AS03] the right exact *twisting endofunctors* $T_i := T_{s_{\alpha_i}}$ of the integral category $\mathcal{O}$ attached to s_{α_i}. Their left derived functors define auto-equivalences of the derived category. The Hecke algebra action has a categorification given by graded lifts of these derived functors:

Proposition 7.9. *Let $\underline{k}=(1,1,\dots,1)$. Then the derived functor $\mathcal{L}T_i$, $0\leq i\leq n-1$ has a graded lift,*

$$\mathcal{L}\hat{T}_i: \quad D^b(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n))\to D^b(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)), \tag{58}$$

an automorphism of the bounded derived category $D^b(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n))$ such that the induced action of $\hat{\mathcal{L}}T_i$ on $K_0(D^b(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)))=K_0(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n))$ agrees with the action of the Hecke algebra generator H_i under the identification $\hat{\Gamma}_{\underline{k}}$.

Proof. If we forget the grading this is [AL03, 2.1]. The graded version is [FKS06, Propositions 5.1 and 5.2]. □

We like to extend the categorification from Proposition 7.6 to the quantum case. We first have to give an alternative description of the special functors. Recall from [BG80], [Hum08] the definition and classification of translation on and out of walls. Given integral weights λ,μ with corresponding block $\mathcal{O}_\lambda(n)$ and $\mathcal{O}_\mu(n)$ let $W_n(\lambda)$, $W_n(\mu)$ be their stabilizer under the dot-action of the Weyl group. Assume $W_n(\lambda)\subset W_n(\mu)$ then we have the *translation to* and the *translation out of* the walls functors

$$\begin{aligned}\theta^\mu_\lambda: &\qquad \mathcal{O}_\lambda(n)\to\mathcal{O}_\mu(n)\\ \theta^\lambda_\mu: &\qquad \mathcal{O}_\mu(n)\to\mathcal{O}_\lambda(n).\end{aligned}$$

We are in particular interested in the case, where $W_n(\mu)$ differs from $W_n(\lambda)$ by one additional simple reflection s_{α_i}. We call the corresponding functors then translation *to the ith wall* and *from the ith wall* and denote them by

$\theta_{\mathrm{on},i}$, and $\theta_{\mathrm{out},i}$ respectively. To make formulas easier we abuse notation and also write $\theta_{\mathrm{on},i}$ and $\theta_{\mathrm{out},i}$ in case $W_n(\mu) = W_n(\lambda)$ in which case the two translation functors are just inverse equivalences, [Hum08, 7.8].

For $0 \le i \le m-1$ and a block triple $\Gamma = ((1,1,\dots,1), \underline{\mu}, \overline{\epsilon})$ we define the compositions $\underline{\mu}^{\uparrow,i}$, $\underline{\mu}^{\downarrow,i}$, where

$$\underline{\mu}_l^{\uparrow,i} = \begin{cases} \mu_{l-1} & \text{if } l > i+1, \\ 0 & \text{if } l = i+1, \\ \mu_l & \text{if } l < i+1, \end{cases} \qquad \underline{\mu}_l^{\downarrow,i} = \begin{cases} \mu_{l+1} & \text{if } l \ge i+1, \\ \mu_l & \text{if } l < i+1, \end{cases}$$

and $\Gamma_{\uparrow,i} = ((1,1,\dots,1), \underline{\mu}^{\uparrow,i}, \overline{\epsilon})$, $\Gamma_{\downarrow,i} = ((1,1,\dots,1), \underline{\mu}^{\downarrow,i}, \overline{\epsilon})$ and consider the composition of functors:

$$\mathbb{B}_{i,+,\Gamma}: \quad \mathcal{O}_\Gamma(n) \xrightarrow{\eta} \mathcal{O}_{\Gamma_1} \xrightarrow{\theta_{\mathrm{out},i+1}} \mathcal{O}_{\Gamma_2} \xrightarrow{\theta_{\mathrm{on},i}} \mathcal{O}_{\Gamma_3} \xrightarrow{\eta} \mathcal{O}_{\Gamma_4}, \tag{59}$$

where $\Gamma_1 = \Gamma_{\uparrow,i}$, $\Gamma_2 = (\Gamma_1)_{i+1,+}$, $\Gamma_3 = (\Gamma_2)_{i,+}$, and $\Gamma_4 = (\Gamma_3)^{\downarrow,i} = (\Gamma_1)_{i,+}$, using Definition, (50). The η's stand for the translation functors indicated by the blocks. Since by [Hum08, 7.8] these are equivalences of categories we just denote them by η. Note that these equivalences exists, since the blocks only depends on the singularity type of the corresponding highest weights. Let

$$\mathbb{B}_{i,-,\Gamma_4}: \quad \mathcal{O}_{\Gamma_4} \xrightarrow{\eta^{-1}} \mathcal{O}_{\Gamma_3} \xrightarrow{\theta_{\mathrm{out},i}} \mathcal{O}_{\Gamma_2} \xrightarrow{\theta_{\mathrm{on},i+1}} \mathcal{O}_{\Gamma_1} \xrightarrow{\eta^{-1}} \mathcal{O}_\Gamma, \tag{60}$$

be the adjoint functor. We set $\mathbb{B}_{i,+} = \oplus_\Gamma \mathbb{B}_{i,+,\Gamma}$ and $\mathbb{B}_{i,-} = \oplus_\Gamma \mathbb{B}_{i,-,\Gamma}$ for $i > 0$ and $\mathbb{B}_0 = \oplus_\Gamma(\mathbb{B}_{0,+,\Gamma} \oplus \mathbb{B}_{0,-,\Gamma})$ in analogy to Definition 7.5.

Example 7.10. For instance if $\underline{\mu} = (0,2,2,2,2)$ and $i = 2$ then for $\mathbb{B}_i$ we start at the block $\underline{k} = (1,1,\dots,1)$ which is equivalent to $(\underline{k}, (0,2,0,2,2,2), \overline{\epsilon})$, then translate out of the third wall to get to $(\underline{k}, (0,2,1,1,2,2), \overline{\epsilon})$ followed by translating to the second wall which ends in $(\underline{k}, (0,3,0,1,2,2), \overline{\epsilon})$. Applying again an equivalence we end in $(\underline{k}, (0,3,1,2,2,2), \overline{\epsilon})$, hence we reduced the numbers of 3's and increased the number of 2's.

Lemma 7.11. *There are isomorphisms $\mathbb{B}_{i,\pm} \cong \mathcal{F}_{i,\pm}$ for $i \ge 0$ and $\mathbb{B}_0 \cong \mathcal{F}_0$.*

Proof. Recall from [BG80] that projective functors, i.e. direct summands of endofunctors on $\mathcal{O}(n)$ given by taking the tensor product with finite dimensional modules are determined up to isomorphism already by their value on the Verma modules whose weight is maximal in their dot-orbit of W_n. Hence to prove the lemma it is enough to compare the value on these Verma modules. Since such a Verma module is a projective object it is enough to compare their values in the Grothendieck group. We explain this for $\mathbb{B}_{i,+}$ and $\mathbb{B}_0$, the case $\mathbb{B}_{i,-}$ is similar. Consider a block $\Gamma = ((1,1,\dots,1), \underline{\mu}, \overline{\epsilon})$ with corresponding maximal weight λ such that

$$\lambda + \rho = (\underbrace{1,\dots,1}_{\mu_1}, \dots, \underbrace{n,\dots,n}_{\mu_n}) - (\tfrac{1}{2}, \dots, \tfrac{1}{2})$$

in case $\epsilon = 0$ and with the first occurring number negated if $\epsilon = 1$. For instance $(2,2,2,3,3,3,4,4)$ respectively $(-2,2,2,3,3,3,4,4)$ are examples. Now fix first $i > 0$ and consider $\mathbb{B}_{i,+}$. Applying the equivalence η sends $M(\lambda)$ to $M(\lambda^{\uparrow,i})$, where $\lambda^{\uparrow,i}$ is obtained from λ by changing the numbers with absolute value $\geq (i+1)$ by 1 if positive and by -1 if negative.

In our examples above we obtain for instance for $i = 2$ the highest weight $\lambda^{\uparrow,2}$ (up to the shift by ρ and $\frac{1}{2}$ in each coordinate) of Verma modules as

$$(2,2,2,4,4,4,5,5) \text{ respectively } (-2,2,2,4,4,4,5,5).$$

Translation out of the $(i+1)$th wall gives a module which has a Verma flag with Verma subquotients precisely the $M(\nu - \rho)$ for ν obtained from $\lambda^{\uparrow,i}$ by changing exactly one $i+2$ to $i+1$, [Hum08, 7.12]. In our example we get for ν exactly the weights $(2,2,2,4,4,3,5,5)$, $(2,2,2,4,3,4,5,5)$, $(2,2,2,3,4,4,5,5)$ for $\epsilon = 0$ and $(-2,2,2,4,4,3,5,5)$, $(-2,2,2,4,3,4,5,5)$, $(-2,2,2,3,4,4,5,5)$ for $\epsilon = 1$. Translation to the ith wall maps each Verma module $M(\nu - \rho)$ to a Verma module $M(\nu' - \rho)$, where ν' is obtained from ν by changing the unique number $i+1$ into i, [Hum08, 7.6]. In our example we get the three highest weights (again without the shifts)

$$(2,2,2,4,4,2,5,5),\ (2,2,2,4,2,4,5,5),\ (2,2,2,2,4,4,5,5)$$

in case $\epsilon = 0$ and

$$(-2,2,2,4,4,2,5,5),\ (-2,2,2,4,2,4,5,5),\ (-2,2,2,2,4,4,5,5)$$

in case $\epsilon = 1$. The final equivalence just rescales again and we obtain

$$(2,2,2,3,3,2,4,4),\ (2,2,2,3,2,3,4,4),\ (2,2,2,2,3,3,4,4)$$

respectively

$$(-2,2,2,3,3,2,4,4),\ (-2,2,2,3,2,3,4,4),\ (-2,2,2,2,3,3,4,4).$$

Comparing with Proposition 7.6 we see this agrees with the values for $\mathcal{F}_{i,\pm}$, hence $\mathcal{F}_{i,+} \cong \mathbb{B}_{i,+}$. In case $i = 0$ we first increase all numbers in absolute value by 1, keeping their signs. For instance $(1,1,1,2,2)$ gives $(2,2,2,3,3)$ and $(-1,1,1,2,2)$ gives $(-2,2,2,3,3)$. Then we translate out of the 1th wall if $\epsilon = 1$ and out of the 0th wall if $\epsilon = 0$. In our examples we obtain the three highest weights ν (up to ρ and shift by $\frac{1}{2}$) equal to $(2,2,1,3,3)$, $(2,1,2,3,3)$, $(1,2,2,3,3)$ respectively $(2,2,-1,3,3)$, $(2,-1,2,3,3)$, $(-1,2,2,3,3)$. Applying the translation to the 0th respectively 1th wall we obtain Vermas of highest weights $(1,1,1,3,3)$, $(1,1,1,3,3)$, $(1,1,1,3,3)$ respectively $(-1,1,1,3,3)$, $(-1,1,1,3,3)$, $(-1,1,1,3,3)$. Applying the last equivalence gives the desired result and hence $\mathbb{B}_0 \cong \mathcal{F}_0$. □

7.6. **Graded translation functors.** By [Str03], the translation functors to and out of a wall $\theta_{\mathrm{on},i} : \mathcal{O}_\lambda(n) \to \mathcal{O}_\mu(n)$, $\theta_{\mathrm{out},i} : \mathcal{O}_\lambda(n) \to \mathcal{O}_\mu(n)$ have graded lifts, that means can be lifted to graded functors[2]

$$\hat{\theta}_{\mathrm{on},i}: \quad \hat{\mathcal{O}}_\lambda(n) \longrightarrow \hat{\mathcal{O}}_\mu(n), \qquad \hat{\theta}_{\mathrm{out},i}: \quad \hat{\mathcal{O}}_\mu(n) \longrightarrow \hat{\mathcal{O}}_\lambda(n). \tag{61}$$

By [BG80], the functors $\theta_{\mathrm{on},i}$, and $\theta_{\mathrm{out},i}$ are indecomposable, hence a graded lift is unique up to isomorphism and overall shift in the grading, [Str03, Lemma 1.5]. We fix graded lifts $\theta_{\mathrm{on},i}$, $\theta_{\mathrm{out},i}$ such that the graded version of the Verma module with maximal weight in the orbit is sent to a projective module with head concentrated in degree zero. We define graded lifts of the functors in (59) and (60) by setting for $i > 0$:

$$\hat{\mathbb{B}}_{i,+,\Gamma} \quad := \quad (\eta \circ \hat{\theta}_{\mathrm{on},i} \circ \theta_{\mathrm{out}}\hat{,} i+1 \circ \eta)\langle \mu_{i-1} - \mu_i + 1\rangle \tag{62}$$

$$\hat{\mathbb{B}}_{i,-,\Gamma} \quad := \quad (\eta \circ \hat{\theta}_{\mathrm{on},i} \circ \theta_{\mathrm{out}}\hat{,} i+1 \circ \eta)\langle \mu_{i+1} - \mu_i + 1\rangle \tag{63}$$

and $\hat{\mathbb{B}}_i = \oplus_\Gamma \hat{\mathbb{B}}_{i,+,\Gamma}$ and $\hat{\mathbb{B}}_{-i} = \oplus_\Gamma \hat{\mathbb{B}}_{i,-,\Gamma}$, where $\langle r\rangle$, $r \in \mathbb{Z}$, means that the grading is shifted by adding r to the degree. For $i = 0$ we set

$$\hat{\mathbb{B}}_0 \quad := \quad \oplus_\Gamma(\eta \circ \hat{\theta}_{\mathrm{on},1} \circ \theta_{\mathrm{out}}\hat{,} 0 \circ \eta) \oplus (\eta \circ \hat{\theta}_{\mathrm{on},0} \circ \hat{\theta}_{\mathrm{out},1} \circ \eta). \tag{64}$$

These functors induce functors on $\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\le m}(n)$ and then via the identification (57) $\mathbb{Q}(q)$-linear endomorphisms on the $\mathbb{Q}(q)$-vector space $\bigwedge(n,m,r)$. Our main result here is the following categorification theorem

Theorem 7.12. *Under the identification*

$$K_0\left(\bigoplus_{\Gamma \in \mathrm{Bl}(n,r,m)} \hat{\mathcal{O}}_\Gamma(n)\right) = \bigwedge(n,m,r)$$

as $\mathbb{Q}(q)$-vector space, the action induced from the exact functor $\hat{\mathbb{B}}_i$ coincides with the action of $B_i \in \mathcal{H}^\lrcorner$ for any $i \in \mathbb{Z}$.

We start with the following main insight:

Lemma 7.13. *Theorem 7.12 is true when restricted to ${\mathbb{V}^\lrcorner_m}^{\otimes r}$.*

Proof. It is enough to verify that the action of $[\hat{\mathbb{B}}_i]$ agrees with the action of $B_i \in \mathcal{H}^\lrcorner$ on the standard basis corresponding to isomorphism classes of graded lifts of Verma modules. Recall the commuting action of the Hecke algebra from Lemma 7.8 and note that twisting functors commute with translation functors, [AS03, Theorem 3.2]. Hence by Proposition 7.9 it is enough to compare the action for Verma modules whose highest weight λ is maximal in their dot-orbit. Moreover by linearity we can restrict ourselves to the case of standard graded lifts $\hat{M}(\lambda)$. We follow the proof of Lemma 7.11. Consider first the functors $\hat{\mathbb{B}}_i$ for $i > 0$. The equivalence η lifts to the graded setting and just renames the highest weight of $\hat{M}(\lambda)$. The translation out of the wall produces an indecomposable projective module P, [Hum08, 7.11] with a graded Verma flag. The Verma subquotients are,

[2]The arguments about the existence in [Str03] are only for the principal block, but generalize directly to arbitrary integral blocks by taking invariants of the coinvariants.

up to some grading shifts, the $\hat{M}(\nu - \rho)$ with the same weights ν as before. Amongst them let $\nu(r)$ be the weight where the rth $i+2$ from the left has been changed. By our normalization, $\hat{M}(\nu(r_{\max}) - \rho)$ with $r = r_{\max}$ maximal occurs without grading shift and hence $P \cong \hat{P}(\nu(r_{\max}) - \rho)$. By [BGS96, Theorem 3.11.4], the graded multiplicities are encoded in parabolic Kazhdan-Lusztig polynomials. Our case corresponds to the explicit formula [Soe97, Proposition 2.9] and we obtain that P has a filtration with subquotients $\hat{M}(\nu(r) - \rho)\langle r_{\max - r}\rangle$. In our example we get $[\hat{M}((2,2,2,4,4,3,5,5) - \rho)] + q[\hat{M}((2,2,2,4,3,4,5,5) - \rho)] + q^2[\hat{M}((2,2,2,3,4,4,5,5) - \rho)]$ in the Grothendieck group. Now translation to the wall just changes the numbers $i+1$ appearing in the weights to i without any extra grading shift by [Str03, Theorem 8.1]. The last step is a graded equivalence renaming the weights. In our example we obtain $[\hat{M}((2,2,2,3,3,2,4,4) - \rho)] + q[\hat{M}((2,2,2,3,2,3,4,4) - \rho)] + q^2[\hat{M}((2,2,2,2,3,3,4,4) - \rho)]$. In any case, the result agrees with the action of B_i up to an overall grading shift by $\langle \mu_{i-1} - \mu_i + 1\rangle$. In case $i = 0$ the arguments are completely parallel except of the grading shift at the end. As above, the lowest grading shift appearing for Verma modules is zero. Expressing $B_0 v_\lambda$ in the standard basis, the smallest q power appearing equals $(-1) + n_1 - n_{-1} + (n_{-1} - n_1 + 1) = 0$, where n_j is the number of j's appearing in the weight λ. Hence the claim of the Lemma follows for B_0. For B_{-i}, $i > 0$ the arguments are analogous. □

Proof of Theorem 7.12. Let $i > 0$. Recall from (40) that the action of B_i on some standard basis vector $\mathbf{v} = v_{\mathbf{i}_1} \otimes \ldots \otimes v_{\mathbf{i}_r} \in \bigwedge^{k_1} \mathbb{V}^{\text{ȷ}} \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{V}^{\text{ȷ}}$ from (41) can be obtained by viewing $\mathbf{v}$ as a vector inside $\bigotimes^{k_1} \mathbb{V}^{\text{ȷ}} \otimes \ldots \otimes \bigotimes^{k_r} \mathbb{V}^{\text{ȷ}}$ and compute the action there. Let $[\hat{M}^{\mathfrak{p}_{\underline{k}}}(\mu)]$ be the class of the standard lift of the parabolic Verma module attached to $\mathbf{v}$. Assume first that the stabilizer of μ is trivial under the dot-action and write $\mu = x \cdot \lambda$ with λ maximal in the same dot-orbit and $x \in W_n$. Recall that by definition the Levi subalgebra of $\mathfrak{p}_{\underline{k}}$ is of type A with Weyl group isomorphic to the product $S_{\underline{k}}$ of symmetric groups. Then

$$[\hat{M}^{\mathfrak{p}_{\underline{k}}}(\mu)] = [\hat{M}^{\mathfrak{p}_{\underline{k}}}(x \cdot \lambda)] = \sum_{y \in S_{\underline{k}}} (-q)^{\ell(w)} [\hat{M}(yx \cdot \lambda)] \tag{65}$$

by the formula [Soe97, Proposition 3.4] for parabolic Kazhdan-Lusztig polynomials together with [Str05, Corollary 2.5]. This however fits precisely with the formula (40) and the claim follows. Assume now that the stabilizer $W(\mu)$ of μ is not trivial under the dot-action. Then choose λ' an integral weight, maximal in its dot-orbit and with $W(\lambda')$ trivial. Following [Str03] choose a graded lift $\hat{\theta}^{\mu}_{\lambda'}$ of $\theta^{\mu}_{\lambda'}$ such that $\hat{\theta}^{\lambda}_{\lambda'} \hat{M}(\lambda') \cong \hat{M}(\lambda)$. We claim that

$$\hat{\theta}^{\lambda}_{\lambda'} \hat{M}(x \cdot \lambda') \quad \cong \quad \hat{M}(x \cdot \lambda) \tag{66}$$

for any shortest coset representative $x \in W_n / W(\lambda)$. Since this is true if we forget the grading and $M(x \cdot \lambda)$ is indecomposable we only have to figure out possible overall grading shifts, [Str03, Lemma 1.5]. If x is the identity we are done by convention. Otherwise we find a simple reflection s_{α_i} such that $\ell(s_{\alpha_i}x) < \ell(x)$ and use again that twisting functors, even graded, commute with translation functors. From Lemma 7.8 and Proposition 7.9 the result follows then by induction on $\ell(x)$. This implies $\hat{\theta}_{\lambda'}^{\lambda}\hat{M}^{\mathfrak{p}_{\underline{k}}}(x\cdot\lambda') \cong \hat{M}^{\mathfrak{p}_{\underline{k}}}(x\cdot\lambda)$, since this is again true up to a possible overall shift in the grading which is however zero by (66). Moreover, we can choose λ' such that yx is of minimal length in its coset for any $y \in S_{\underline{k}}$. (In practice we consider $\lambda+\rho$ and replace its entries by a strictly increasing sequence of positive integers. We do this by replacing first the lowest numbers from the left to right, then replace the second lowest from the left to right etc.) Now we apply $[\hat{\theta}_{\lambda'}^{\mu}]$ to (65) and obtain with (66)

$$[\hat{M}^{\mathfrak{p}_{\underline{k}}}(x\cdot\lambda)] = \sum_{y\in S_{\underline{k}}} (-q)^{\ell(w)}[\hat{M}(yx\cdot\lambda)] \tag{67}$$

The claim follows then from (40) and Lemma 7.13. □

For the block triple $\Gamma = (\underline{k},\underline{\mu},\epsilon) \in \mathrm{Bl}(n,r,m)$ we associate the *transposed block triple* $\Gamma^\vee = (\underline{\mu},\underline{k},\overline{n}\epsilon) \in \mathrm{Bl}(n,m,r)$. In other words, the parabolic type is flipped with the type of the singularity and the parity is kept if n is even and swapped if n is odd.

Theorem 7.14. *Let $\Gamma \in \mathrm{Bl}(n,r,m)$ then $\hat{\mathcal{O}}_\Gamma(n)$ is Koszul dual to $\hat{\mathcal{O}}_{\Gamma^\vee}(n)$.*

Proof. We first recall the description of the Koszul dual block in general following [Bac99]. Assume we have a block $\mathcal{O}_J^I$ of category $\mathcal{O}$ for any complex semisimple Lie algebra, where the parabolic is given by a subset I of the simple roots and the stabilizer of the highest weights of the parabolic Verma modules are generated by simple reflections given by a subset J of simple roots. Then the Koszul dual is the block $\mathcal{O}_{I'}^J$, where $I' = -w_0(I)$ with w_0 the longest element in the Weyl group, [Bac99, Theorem 1.1]. In our case of type D_n, we have $-w_0(I) = I$ if n is even and $-w_0(I)$ is the set obtained from I by applying the unique nontrivial diagram automorphism of the Dynkin diagram. Hence the Koszul dual of $\Gamma = (\underline{k},\underline{\mu},\overline{0})$ is the block $(\underline{\mu},\underline{k},\overline{0})$ if n is even and equal to $(\underline{\mu},\underline{k},\overline{1})$ in case n is odd; and the Koszul dual of $\Gamma = (\underline{k},\underline{\mu},\overline{1})$ is the block $(\underline{\mu},\underline{k},\overline{1})$ if n is even and equal to $(\underline{\mu},\underline{k},\overline{0})$ if n is odd. □

Koszul duality, [BGS96], defines an equivalence of categories

$$D^b(\hat{\mathcal{O}}_\Gamma(n)) \cong D^b(\hat{\mathcal{O}}_{\Gamma^\vee}(n))$$

sending $\hat{M}^{\mathfrak{p}_{\underline{k}}}(x\cdot\mu)$ to $\hat{M}^{\mathfrak{p}_{\underline{\mu}}}(x^{-1}w_0\cdot\lambda)$, where μ and λ are maximal weights in the corresponding dot-orbits of the Weyl group. This duality and its connection to the quantum exterior powers can be nicely encoded in terms of box diagrams.

Given $\hat{M}^{\mathfrak{p}_{\underline{k}}}(x\cdot\mu)$ consider the weight $x\cdot\mu+\rho = (\mu_1,\mu_2,\ldots,\mu_n)$. This is strictly increasing in the r parts given by $\underline{k}$. Assign to this weight a box diagram B of width r and height m and put the following symbols into row i and column j:

- the symbol $+$ if j, but not $-j$, occurs in the ith part of $\underline{k}$,
- the symbol $-$ if $-j$, but not j, occurs,
- both symbols, i.e. $\mp$ if j and $-j$ occur.

The total number of symbols $+$ or $-$ is n and the parity of the number of $-$ symbols is the parity ϵ of the block. Note that each such box diagram defines uniquely a corresponding weight, by just reading the rows from top to bottom and ordering the entries given by the row in increasing order. Let B^t be the transposed of B where we additionally also flip all symbols and let $(x \cdot \mu + \rho)^t$ be the corresponding weight.

Example 7.15. Here is an example for $n = 9$, $m = 5$, and $r = 4$ with the empty boxes indicated by $\circ$. Put $\underline{k} = (2,2,2,2,1)$ and look at $x \cdot \mu + \rho = (-3,2,-1,1,-4,2,-1,3,2)$ then the corresponding box diagram and its transpose are

$$B = \begin{array}{|c|c|c|c|}\hline \circ & + & - & \circ \\ \hline \mp & \circ & \circ & \circ \\ \hline \circ & + & \circ & - \\ \hline - & \circ & + & \circ \\ \hline \circ & + & \circ & \circ \\ \hline \end{array} \qquad B^t = \begin{array}{|c|c|c|c|c|}\hline \circ & \mp & \circ & + & \circ \\ \hline - & \circ & - & \circ & - \\ \hline + & \circ & \circ & - & \circ \\ \hline \circ & \circ & + & \circ & \circ \\ \hline \end{array} \tag{68}$$

and thus

$$x^{-1}w_0 \cdot \lambda + \rho = (-3,2,-1,1,-4,2,-1,3,2)^t = (-2,2,4,-5,-3,-1,-4,1,3)$$

Theorem 7.16. *The Koszul duality functor from* [BGS96],

$$\bigoplus_{\underline{k} \in C(n,r)} D^b\left(\hat{\mathcal{O}}^{\mathfrak{p}_{\underline{k}}}_{\leq m}(n)\right) \cong \bigoplus_{\underline{k} \in C(n,m)} D^b\left(\mathcal{O}^{\mathfrak{p}_{\underline{k}}}_{\leq r}(n)\right) \tag{69}$$

maps a parabolic Verma module of highest weight $\nu - \rho$ to the parabolic Verma module of highest weight $(\nu - \rho)^t$. The coideal algebra actions from Theorem 7.12 on the two sides turn into commuting actions when identifying the two categories via Koszul duality.

Proof. The first statement is an easy combinatorial check. Since translation functors are Koszul dual to Zuckerman functors, [MOS09] and Zuckermann functors commute with translation functors, the second statement follows. □

As a result we obtain a version of quantum skew Howe duality. For this denote by $\mathcal{H}^{\jmath}_m$ the coideal subalgebra corresponding to the quantum group $\mathcal{U}_q(\mathfrak{gl}_{2m})$, i.e. the truncation of $\mathcal{H}^{\jmath}$ and consider the two actions on $\bigwedge(N,m,r)$ induced by Theorem 7.16.

Theorem 7.17. *The actions of $\mathcal{H}^{\jmath}_m$ and $\mathcal{H}^{\jmath}_r$ on $\bigwedge(N,m,r)$ commute.*

We expect that a careful analysis of the structure of the occurring representations in the decomposition (47) also shows that the two actions generate each others commutant.

8. Appendix

8.1. **Proof of 2.8.** This subsection is devoted to the proof of Theorem 2.8.

Lemma 8.1. *For $1 \leq i \leq d-1$ and $j \notin \{i, i+1\}$ we have $s_i y_j = y_j s_i$.*

Proof. The case $j < i$ is obvious by definition. For the case $j > i+1$ we have for $v \in M \otimes V^{\otimes d}$:

$$\begin{aligned} s_i y_j(v) &= s_i\left(\left(\sum_{0\leq k<j} \Omega_{kj} + \frac{2n-1}{2}\right) v\right) = \sum_{0\leq k<j} s_i\left(\Omega_{kj} v\right) + \frac{2n-1}{2} s_i(v) \\ &= \left(\sum_{0\leq k<j} \Omega_{kj}\right) s_i(v) + \frac{2n-1}{2} s_i(v) = y_j s_i(v). \end{aligned}$$

The claim follows. □

The following are the Brauer algebra relations, see e.g. [GW09, Lemma 10.1.5]:

Lemma 8.2. *For $1 \leq i \leq d-2$, the relations $e_i s_i = e_i = s_i e_i$, $s_i e_{i+1} e_i = s_{i+1} e_i$, $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, and $e_{i+1} e_i e_{i+1} = e_{i+1}$ hold.*

Proof. The first equality is obvious by definition of the map τ. We will only proof the second one, the final two are shown analogously. To shorten the notations we will assume that our morphisms s_i, e_i, and e_{i+1} only live on a threefold tensor product, with the first factor being the position i, the second $i+1$, and the third $i+2$. Let $a, b, c \in I$ and $v = v_a \otimes v_b \otimes v_c$, then

$$(s_{i+1} e_i).v = \langle v_a, v_b\rangle \sum_{k\in I} v_k \otimes v_c \otimes v_k$$

and

$$(s_i e_{i+1} e_i).v = (s_i e_{i+1})\left(\langle v_a, v_b\rangle \sum_{k\in I} v_k \otimes v_k^* \otimes v_c\right) = \langle v_a, v_b\rangle \sum_{k\in I} v_k \otimes v_c \otimes v_k^*.$$

and the claim follows. □

Lemma 8.3. *For $1 \leq i \leq d-1$ we have $s_i y_i - y_{i+1} s_i = e_i - 1$.*

Proof. Since s_i is invertible it is equivalent to show $y_i - s_i y_{i+1} s_i = e_i - s_i$ by Lemma8.2. Note that $s_i y_{i+1} s_i$ equals

$$s_i\left(\sum_{0\leq k<i+1} \Omega_{ki} + \left(\frac{2n-1}{2}\,\mathrm{Id}\right)\right) s_i = \left(\sum_{0\leq k<i} \Omega_{ki} + s_i \Omega_{i(i+1)} s_i + \left(\frac{2n-1}{2}\,\mathrm{Id}\right)\right)$$

Therefore, $y_i - s_i y_{i+1} s_i = -s_i \Omega_{i(i+1)} s_i = -\Omega_{i(i+1)} = e_i - s_i$, where the last equality follows from Remark 2.6. □

The proofs of the following identities rely on the explicit form of Ω and its action on $V \otimes V$.

Lemma 8.4. *For $1 \leq i, j \leq d-1$ and $j \notin \{i, i+1\}$ we have $e_i y_j = y_j e_i$.*

Proof. As in Lemma 8.1 the case $j < i$ is obvious by definition and we are left with the case $j > i+1$. The essential part that we need to prove is

$$\left(\Omega_{ij} + \Omega_{(i+1)j}\right) e_i = e_i \left(\Omega_{ij} + \Omega_{(i+1)j}\right), \tag{70}$$

since e_i naturally commutes with all other summands of y_j. Again, we pretend that Ω_{ij}, $\Omega_{(i+1)j}$, and e_i act only on a threefold tensor product, corresponding to the factors at position i, $i+1$ and j.

Consider first the left hand side. Since we first apply e_i, the element $e_i y_j$ kills all basis vectors except the ones of the form $x = v_a \otimes v_a^* \otimes v_c$ for some $a, c \in I$, in which case $x.e_i = \sum_{k \in I} v_k \otimes v_k^* \otimes v_c$. and applying $\Omega_{ij} + \Omega_{(i+1)j}$ gives

$$\begin{aligned}
& \sum_{\alpha \in B_n} \sum_{k \in I} (X_\alpha v_k \otimes v_k^* + v_k^* \otimes X_\alpha v_k) \otimes X_\alpha^* v_c \\
= & \sum_{\alpha \in B_n} \sum_{k \in I} \sum_{l \in I} (\langle X_\alpha v_k, v_l \rangle v_l^* \otimes v_k^* + v_k^* \otimes \langle X_\alpha v_k, v_l \rangle v_l^*) \otimes X_\alpha^* v_c \\
= & \sum_{\alpha \in B_n} \sum_{k,l \in I} (\langle X_\alpha v_k, v_l \rangle v_l^* \otimes v_k^* - \langle X_\alpha v_l, v_k \rangle v_k^* \otimes v_l^*) \otimes X_\alpha^* v_c \\
= & \ 0
\end{aligned}$$

Thus $\left(\Omega_{ij} + \Omega_{(i+1)j}\right) e_i = 0$ on $M \otimes V^{\otimes d}$.

Consider now the right hand side of (70), let $x = v_a \otimes v_b \otimes v_c$ for $a, b, c \in I$, then

$$\begin{aligned}
x.(\Omega_{ij} + \Omega_{(i+1)j}) &= \sum_{\alpha \in B_n} (X_\alpha v_a \otimes v_b + v_a \otimes X_\alpha v_b) \otimes X_\alpha^* v_c \\
&= \sum_{\alpha \in B_n, k \in I} (\langle X_\alpha v_a, v_k \rangle v_k^* \otimes v_b + \langle X_\alpha v_b, v_k \rangle v_a \otimes v_k^*) \otimes X_\alpha^* v_c
\end{aligned}$$

Applying e_i kills, for fixed α, all except one summand and we obtain

$$\begin{aligned}
& \sum_{l \in I} (\langle X_\alpha v_a, v_b \rangle v_l \otimes v_l^* + \langle X_\alpha v_b, v_a \rangle v_l \otimes v_l^*) \otimes X_\alpha^* v_c \\
= & \sum_{l \in I} (\langle X_\alpha v_a, v_b \rangle v_l \otimes v_l^* - \langle X_\alpha v_a, v_b \rangle v_l \otimes v_l^*) \otimes X_\alpha^* v_c = 0.
\end{aligned}$$

Thus, $e_i \left(\Omega_{ij} + \Omega_{(i+1)j}\right) = 0$ on $M \otimes V^{\otimes d}$ and the lemma follows. □

Lemma 8.5. *For $1 \le i, j \le d$ we have $y_i y_j = y_j y_i$.*

Proof. We show this by using the following statements:

(i) $[\Omega_{ij}, \Omega_{kl}] = 0$ for pairwise different i, j, k, l,

(ii) $[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0$ for pairwise different i, j, k.

Equality (i) is obvious. To show (ii) we consider the i-th, j-th and k-th factor in $V^{\otimes d}$ and calculate

$$\begin{aligned}
[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] &= \sum_{\alpha, \beta, \gamma \in B_n} [X_\alpha \otimes X_\alpha^* \otimes 1 + 1 \otimes X_\beta \otimes X_\beta^*, X_\gamma \otimes 1 \otimes X_\gamma^*] \\
&= \sum_{\alpha, \beta, \gamma \in B_n} [X_\alpha, X_\gamma] \otimes X_\alpha^* \otimes X_\gamma^* + X_\gamma \otimes X_\beta \otimes [X_\beta^*, X_\gamma^*]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha,\beta,\gamma,\mu\in B_n} ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha^* \otimes X_\gamma^* \\
&\quad + \sum_{\alpha,\beta,\gamma,\mu\in B_n} X_\gamma \otimes X_\beta \otimes ([X_\beta^*, X_\gamma^*], X_\mu) X_\mu^* \\
&\overset{(1)}{=} \sum_{\alpha,\beta,\gamma,\mu\in B_n} ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha^* \otimes X_\gamma^* \\
&\quad + \sum_{\alpha,\beta,\gamma,\mu\in B_n} ([X_\alpha^*, X_\mu^*], X_\gamma) X_\mu \otimes X_\alpha \otimes X_\gamma^* \\
&\overset{(2)}{=} \sum_{\alpha,\beta,\gamma,\mu\in B_n} ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha^* \otimes X_\gamma^* \\
&\quad - \sum_{\alpha,\beta,\gamma,\mu\in B_n} ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha^* \otimes X_\gamma^* = 0
\end{aligned}
$$

Here equality (1) is just a relabelling in the second sum and extracting a scalar factor, while equality (2) follows from the invariance of the Killing form, i.e.

$$
\begin{aligned}
\sum_{\alpha\in B_n} ([X_\alpha^*, X_\mu^*], X_\gamma) X_\alpha = \sum_{\alpha\in B_n} (X_\alpha^*, [X_\mu^*, X_\gamma]) X_\alpha = \sum_{\alpha\in B_n} (X_\alpha, [X_\mu^*, X_\gamma]) X_\alpha^* \\
= - \sum_{\alpha\in B_n} ([X_\alpha, X_\gamma], X_\mu^*) X_\alpha^*.
\end{aligned}
$$

□

Lemma 8.6. *For $1 \leq 1 < d$ we have*

$$
e_i(y_i + y_{i+1}) = 0 \qquad \textit{and} \qquad (y_i + y_{i+1})e_i = 0.
$$

Proof. We start with the first relation and expand the left hand side.

$$
\begin{aligned}
e_i(y_i + y_{i+1}) &= e_i \left(\sum_{0\leq k<i} \left(\Omega_{ki} + \Omega_{k(i+1)}\right) + \Omega_{i(i+1)} + N - 1 \right) \\
&= e_i \left(\sum_{0\leq k<i} \left(\Omega_{ki} + \Omega_{k(i+1)}\right) + s_i - e_i + N - 1 \right) \\
&= e_i \left(\sum_{0\leq k<i} \left(\Omega_{ki} + \Omega_{k(i+1)}\right) \right) + e_i - Ne_i + (N-1)e_i \\
&= e_i \left(\sum_{0\leq k<i} \left(\Omega_{ki} + \Omega_{k(i+1)}\right) \right)
\end{aligned}
$$

We have seen in the proof of Lemma 8.4 that $e_i\left(\Omega_{ki} + \Omega_{k(i+1)}\right)$ acts as zero if $k > 0$. Thus we are left to show that $e_i\left(\Omega_{0i} + \Omega_{0(i+1)}\right)$ acts as zero. Let $x = m \otimes v_a \otimes v_b$ for $m \in M$ and $a, b \in I$, again these should be thought of positions 0, i, and $i+1$. Applying $\left(\Omega_{0i} + \Omega_{0(i+1)}\right)$ gives us

$$
x.\left(\Omega_{0i} + \Omega_{0(i+1)}\right)
$$

$$\begin{aligned}
&= \sum_{\alpha \in B_n} X_\alpha m \otimes X_\alpha^* v_a \otimes v_b + X_\alpha m \otimes v_a \otimes X_\alpha^* v_b \\
&= \sum_{\alpha \in B_n} \sum_{k \in I} \langle X_\alpha^* v_a, v_k \rangle X_\alpha m \otimes v_k^* \otimes v_b + \langle X_\alpha^* v_b, v_k \rangle X_\alpha m \otimes v_a \otimes v_k^*.
\end{aligned}$$

Now applying e_i to each summand for $\alpha \in B_n$ we obtain

$$\begin{aligned}
&\sum_{l \in I} \langle X_\alpha^* v_a, v_b \rangle X_\alpha m \otimes v_l \otimes v_l^* + \langle X_\alpha^* v_b, v_a \rangle X_\alpha m \otimes v_l \otimes v_l^* \\
&= \sum_{l \in I} \langle X_\alpha^* v_a, v_b \rangle X_\alpha m \otimes v_l \otimes v_l^* - \langle X_\alpha^* v_a, v_b \rangle X_\alpha m \otimes v_l \otimes v_l^* = 0.
\end{aligned}$$

Thus we have $e_i(y_i + y_{i+1}) = 0$ on $M \otimes V^{\otimes d}$.

For the second equality with start with the same x as above and apply e_i, then the result will be either zero or

$$\sum_{l \in I} m \otimes v_l \otimes v_l^* = \sum_{l \in I} m \otimes v_l^* \otimes v_l.$$

Applying $\left(\Omega_{0i} + \Omega_{0(i+1)}\right)$ to this, we obtain

$$\begin{aligned}
&\sum_{\alpha \in B_n} \sum_{l \in I} X_\alpha m \otimes X_\alpha^* v_l^* \otimes v_l + X_\alpha m \otimes v_l \otimes X_\alpha^* v_l^* \\
&= \sum_{\alpha \in B_n} \sum_{k,l \in I} \langle X_\alpha^* v_l^*, v_k^* \rangle X_\alpha m \otimes v_k \otimes v_l + \langle X_\alpha^* v_l^*, v_k^* \rangle X_\alpha m \otimes v_l \otimes v_k \\
&= \sum_{\alpha \in B_n} \sum_{k,l \in I} \langle X_\alpha^* v_l^*, v_k^* \rangle X_\alpha m \otimes v_k \otimes v_l - \langle X_\alpha^* v_k^*, v_l^* \rangle X_\alpha m \otimes v_l \otimes v_k = 0.
\end{aligned}$$

For the last equality one just switches k and l in the second sum. Hence we also have $(y_i + y_{i+1})e_i = 0$ on $M \otimes V^{\otimes d}$. □

8.2. **Proof of Lemma 3.8.**

Proof. The statements about Verma modules follow directly from Lemma 2.16 and the definition of our functors. Moreover, the functors $\bigoplus_{i \in X_\delta}(F_{i,-} \oplus \mathcal{F}_{i,+}) \oplus F_{\frac{1}{2}}$ and $F := (? \otimes V)$ applied to a Verma module produce the same number of Verma modules, hence part (1) of the Lemma follows.

We next check (viii) and (ix) for $L(\lambda)$. In all these cases, we know already that $\mathcal{F}_{i,-} M^{\mathfrak{p}}(\lambda) = \{0\}$. As $L(\lambda)$ is a quotient of $M^{\mathfrak{p}}(\lambda)$ and $\mathcal{F}_{i,-}$ is exact, it follows immediately that $\mathcal{F}_{i,-} L(\lambda) = \{0\}$.

The proofs of (i)-(iv) for $L(\lambda)$ are not much harder. For example, if $\lambda = \lambda_{\vee\circ}$ as in (i), then $\mathcal{F}_{i,-} L(\lambda)$ is a quotient of $\mathcal{F}_{i,-} M^{\mathfrak{p}}(\lambda) \cong M^{\mathfrak{p}}(\lambda_{\circ\vee})$. Moreover it is self-dual. This is because $L(\lambda)$ is self-dual with respect to the duality on category $\mathcal{O}$, and $\mathcal{F}$ commutes with this duality and $\mathcal{F} = \mathcal{F}_{i,-}$ when applied to our modules. Hence we either have that $\mathcal{F}_{i\_} L(\lambda) = \{0\}$ or $\mathcal{F}_{i\_} L(\lambda) \cong L(\lambda_{\circ\vee})$, as these are the only self-dual quotients of $M^{\mathfrak{p}}(\lambda_{\circ\vee})$. To rule out the possibility that it is zero, consider the group homomorphism $K_0(\mathcal{F}_{i,-}) : K_0(\mathcal{O}^{\mathfrak{p}}(n,\delta)_\Gamma) \to K_0(\mathcal{O}^{\mathfrak{p}}(n,\delta)_{\Gamma_-})$ induced by $\mathcal{F}_{i,-}$. Expressed in the basis of isomorphism classes of Verma modules, it has an inverse, the morphism induced by $\mathcal{F}_{i,+}$. Hence $\mathcal{F}_{i,-}$ is non-zero on every non-zero module. This proves (i) for $L(\lambda)$, and the proofs of (ii)-(iv) are similar.

Next we check (v) and (vi) for $L(\lambda)$, i.e. we show that $\mathcal{F}_{i,-}L(\lambda_{\vee\wedge}) \cong L(\lambda_{\circ\times})$ and $\mathcal{F}_{i,-}L(\lambda_{\wedge\vee}) = \{0\}$. We know that $\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda_{\vee\wedge}) \cong \mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda_{\wedge\vee}) \cong M^{\mathfrak{p}}(\lambda_{\circ\times})$. Again by self-duality, we either have that $\mathcal{F}_{i\_}L(\lambda_{\vee\wedge}) \cong L(\lambda_{\circ\times})$ or $\mathcal{F}_{i,-}L(\lambda_{\vee\wedge}) = \{0\}$. Similarly, either $\mathcal{F}_{i,-}L(\lambda_{\wedge\vee}) \cong L(\lambda_{\circ\times})$ or $\mathcal{F}_{i,-}L(\lambda_{\wedge\vee}) = \{0\}$. As $[\mathcal{F}_{i,-}M^{\mathfrak{p}}(\lambda_{\vee\wedge}) : L(\lambda_{\circ\times})] = 1$, there must be some composition factor $L(\mu)$ of $M^{\mathfrak{p}}(\lambda_{\vee\wedge})$ such that $[\mathcal{F}_{i,-}L(\mu) : L(\lambda_{\circ\times})] = 1$. The facts proved so far imply either that $\mu = \lambda_{\vee\wedge}$ or that $\mu = \lambda_{\wedge\vee}$. But the latter case cannot occur as $\lambda_{\wedge\vee}$ is strictly bigger than $\lambda_{\vee\wedge}$ in the Bruhat ordering. Hence $\mu = \lambda_{\vee\wedge}$ and we have proved that $[\mathcal{F}_{i,-}L(\lambda_{\vee\wedge}) : L(\lambda_{\circ\times})] = 1$. This gives $\mathcal{F}_{i,-}L(\lambda_{\vee\wedge}) \cong L(\lambda_{\circ\times})$ as required for (v). It remains for (vi) to show that $\mathcal{F}_{i,-}L(\lambda_{\wedge\vee}) = \{0\}$. Suppose for a contradiction that it is non-zero, hence $\mathcal{F}_{i,-}L(\lambda_{\wedge\vee}) \cong L(\lambda_{\circ\times})$. By Lemma 3.10 and BGG-reciprocity we have $M^{\mathfrak{p}}(\lambda_{\wedge\vee})$ has both $L(\lambda_{\wedge\vee})$ and $L(\lambda_{\vee\wedge})$ as composition factors, so we deduce that $[F_i M^{\mathfrak{p}}(\lambda_{\wedge\vee}) : L(\lambda_{\circ\times})] \geq 2$, which is the desired contradiction.

In this paragraph, we check (vii)(c). Take λ with $\lambda = \lambda_{\times\circ}$. Let Γ be the block containing λ. Note that $\mathcal{F}_{i,-}$ maps the summand corresponding to the block Γ to Γ_- and then to $\Gamma' = (\Gamma_-)_-$. We know for any $\nu \in \Gamma$ that $\mathcal{F}_{i,-}^2 M^{\mathfrak{p}}(\nu) \cong M^{\mathfrak{p}}(\nu_{\circ\times}) \oplus M^{\mathfrak{p}}(\nu_{\circ\times})$. Hence $\mathcal{F}_{i,-}^2$ induces a $\mathbb{Z}$-module isomorphism between $K_0(\mathcal{O}^{\mathfrak{p}}(n,\delta)_\Gamma)$ and $2[\mathcal{O}^{\mathfrak{p}}(n,\delta)_{G'}]$. We deduce for any non-zero module $M \in \mathcal{O}^{\mathfrak{p}}(n,\delta)_\Gamma$ that $\mathcal{F}_{i,-}^2 M$ is non-zero and its class is divisible by two in $[\mathcal{O}^{\mathfrak{p}}(n,\delta)_{\Gamma'}]$. In particular, $\mathcal{F}_{i,-}^2 L(\lambda)$ is a non-zero self-dual quotient of $M^{\mathfrak{p}}(\lambda_{\circ\times}) \oplus M^{\mathfrak{p}}(\lambda_{\circ\times})$ whose class is divisible by two. This implies that

$$\mathcal{F}_{i,-}^2 L(\lambda) \cong L(\lambda_{\circ\times}) \oplus L(\lambda_{\circ\times}). \tag{71}$$

Now take any $\mu \in \Gamma_-$. We know already that $\mathcal{F}_{i,-}L(\mu) \cong L(\mu_{\circ\times})$ if $\mu = \mu_{\vee\wedge}$, and $\mathcal{F}_{i,-}L(\mu) = \{0\}$ otherwise. Assuming now that $\mu = \mu_{\vee\wedge}$, we deduce from this that $[\mathcal{F}_{i,-}L(\lambda) : L(\mu)] = [\mathcal{F}_{i,-}^2 L(\lambda) : L(\mu_{\circ\times})]$. Using (71), we conclude for $\mu = \mu_{\vee\wedge}$ that $[\mathcal{F}_{i,-}L(\lambda) : L(\mu)] = 0$ unless $\mu = \lambda_{\vee\wedge}$, and $[\mathcal{F}_{i,-}L(\lambda) : L(\lambda_{\vee\wedge})] = 2$. It is easy to see that the analogous statements hold for $\mathcal{F}_{i,+}$.

Now we deduce all the statements (i)–(ix) for $P(\lambda)$ by using the fact that $(\mathcal{F}_{i,-}, \mathcal{F}_{i,+})$ is an adjoint pair of functors. We just explain the argument in case (v), since the other cases are similar (actually, easier). As $\mathcal{F}_{i,-}$ sends projectives to projectives, $\mathcal{F}_{i,-}P(\lambda)$ is a direct sum of indecomposable projectives. To compute the multiplicity of $P(\mu)$ in this decomposition we calculate

$$\hom_{\mathfrak{g}}(\mathcal{F}_{i,-}P(\lambda), L(\mu)) \cong \hom_{\mathfrak{g}}(P(\lambda), \mathcal{F}_{i,-}L(\mu)) = [\mathcal{F}_{i,-}L(\mu) : L(\lambda)].$$

By the analogue of (vi)(c) for $\mathcal{F}_{i,+}$ this multiplicity is zero unless $\mu = \lambda_{\circ\times}$, when it is two. Hence $\mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\circ\times}) \oplus P(\lambda_{\circ\times})$.

For part (1) it just remains to deduce (vii)(d). By (vii)(a), (vii)(c) and exactness of $\mathcal{F}_{i,-}$, we get that $\mathcal{F}_{i,-}L(\lambda)$ is a non-zero quotient of $P(\lambda_{\vee\wedge})$, hence it has irreducible head isomorphic to $L(\lambda_{\vee\wedge})$. Since it is self-dual it also has irreducible socle isomorphic to $L(\lambda_{\vee\wedge})$. Now consider the case $i = 0$. The statement for the Verma modules is again clear and for the simple modules we argue as above. For the projective modules we consider again each case. Part (xiii) is proved as (i) or (ii), whereas (xv) is proved as the dual version (x) of (iii). Now for (xiv) and (xviii) each Verma module occurring in a filtration of the projective module has highest weight ν and ν equals λ at the places 0 and 1, hence the whole module get killed by the functor. Case (xvi) is proved as case (iv) whereas (xvii) is proved as the dual (x) of (vii). Hence it remains to show (xi) and (xii) which however can be proved with the arguments from (v). Finally (4) can be proved as the cases (i)-(iv). □

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