BOOLEAN ULTRAPOWERS ①

YOUNG SET THEORISTS ~~CONFERENCE~~ WORKSHOP YSTW

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"AN INTRODUCTION TO BOOLEAN ULTRAPOWERS"

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- MAIN IDEA: GENERALIZE ULTRAPOWER CONCEPT FROM ULTRAFILTERS ON A POWER SET TO ULTRAFILTERS ON AN ARBITRARY COMPLETE BOOLEAN ALGEBRA.

CLASSICAL ULTRAPOWER:
- USE ULTRAFILTER U ON A SET I, MEASURING ELEMENTS OF $P(I)$.
- BUILD ULTRAPOWER V^I/U. ULTRAPOWER MAP $j: V \to V^I/U$.
- ULTRAPOWER MAP

BOOLEAN ULTRAPOWER:
- HAVE ULTRAFILTER U ON COMPLETE BOOLEAN ALGEBRA $\mathbb{B}$.
- CONSTRUCT BOOLEAN ULTRAPOWER $\check{V}^{\mathbb{B}}/U$
- BOOLEAN ULTRAPOWER MAP $j: V \to \check{V}^{\mathbb{B}}/U$

SEE ALSO:
FORTHCOMING ARTICLE (53 PAGES)
"WELL-FOUNDED BOOLEAN ULTRAPOWERS AS LARGE CARDINAL EMBEDDINGS," BY HAMKINS, AND DANIEL SEABOLD.

- TWO EQUIVALENT PRESENTATIONS (BUT FUNDAMENTALLY THE SAME)
  - ALGEBRAIC / MODEL-THEORETIC APPROACH
    - SIMILAR TO POWER SET ULTRAPOWERS
    - NO FORCING INVOLVED
    - MANSFIELD, OUWEHAND, ROSE, CANJAR
  - FORCING THEORETIC APPROACH
    - CONNECTED TO BOOLEAN-VALUED MODELS IN FORCING
    - VOPĚNKA, SOLOVAY, SCOTT, BELL
    - SHOWS THAT EVERY BOOLEAN ULTRAPOWER IS ACCOMPANIED BY A CANONICAL GENERIC OBJECT

REMARK
- BOOLEAN-VALUED MODELS ARE WIDELY KNOWN
- BUT THE BOOLEAN ULTRAPOWER MUCH LESS SO.
- (DIFFICULT TO SPEAK ON ~~OCCASSIONALLY AWKWARD CONVERSATIONS WHERE A RESEARCH~~ RESEARCHERS ASSUME THEY KNOW ALL ABOUT THE BOOLEAN ULTRAPOWER, WHEN IN FACT THEY KNOW ONLY BOOLEAN-VALUED MODEL APPROACH TO FORCING.)

- GOALS
  1. PRESENT THE BASIC THEORY OF BOOLEAN ULTRAPOWERS
  2. ALONG THE WAY DEVELOP THE BOOLEAN-VALUED MODEL APPROACH TO FORCING INCLUDING THE NATURALIST ACCOUNT OF FORCING.
  3. FOCUS PARTICULARLY ON THE WELL-FOUNDED BOOLEAN ULTRAPOWERS

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- ~~THEME~~ THE BOOLEAN ULTRAPOWER UNIFIES TWO CENTRAL CONCERNS IN SET THEORY: REVEALS FORCING AND LARGE CARDINALS TO BE TWO ASPECTS OF A SINGLE UNDERLYING CONSTRUCTION.

- FORESHADOWING

ONE PROBLEM WITH FORCING OVER V: FOR NONTRIVIAL $\mathbb{P}$, THERE ARE NO V-GENERIC FILTERS!

NEVERTHELESS:

THEOREM FOR EVERY FORCING NOTION $\mathbb{P}$, THERE IS A DEFINABLE CLASS MODEL SATISFYING ZFC + EVERY STATEMENT FORCED OVER V BY $\mathbb{P}$.

INDEED, THERE IS $V \preceq \bar{V} \subseteq \bar{V}[G]$, G IS $\bar{V}$-GENERIC FOR $\bar{\mathbb{P}}$

$\underbrace{\bar{V}[G]}_{\text{EXISTS IN } V.}$

LET'S DIVE IN!

START WITH

- BOOLEAN-VALUED MODELS

A MODEL-THEORETIC IDEA AT BOTTOM

~~BOOLEAN VALUED MODELS HAVE NOTHING ESSENTIALLY TO DO WITH FORCING OR SET THEORY~~

NOT NECESSARILY INVOLVING FORCING OR SET THEORY

ONE CAN HAVE BOOLEAN-VALUED GRAPHS, GROUPS, RINGS, PARTIAL ORDERS, ETC.

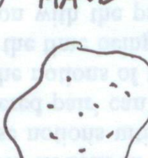
HAVE A COMPLETE BOOLEAN ALGEBRA $\mathbb{B}$

MAIN IDEA: REPLACE TRUE/FALSE WITH VALUES IN $\mathbb{B}$.

LANGUAGE $\mathcal{L}$

NO LONGER HAVE CRISP CONCEPT OF $=$

SO HAVE DOMAIN OF POSSIBLE OBJECTS M, CALLED "NAMES"



$\mathbb{B}$-VALUED STRUCTURE $\mathcal{M}$ IS DETERMINED BY BOOLEAN ASSIGNMENTS $[\![s=t]\!]$, $[\![R(\vec{t})]\!]$, $[\![y=f(\vec{t})]\!]$ OF ATOMIC FORMULAS.

- INSIST ON LAWS OF EQUALITY:

$$[\![s=s]\!] = 1$$
$$[\![s=t]\!] = [\![t=s]\!]$$
$$[\![s=t]\!] \wedge [\![t=u]\!] \leq [\![s=u]\!]$$
$$[\![\vec{s}=\vec{t}]\!] \wedge [\![R(\vec{s})]\!] \leq [\![R(\vec{t})]\!]$$

A COMPLICATION WHEN THERE ARE FUNCTION SYMBOLS:

$$[\![s=t]\!] \wedge [\![y=f(s)]\!] \leq [\![y=f(t)]\!]$$
$$\bigvee_{t \in M} [\![t=f(s)]\!] = 1$$
$$[\![t_0=f(s)]\!] \wedge [\![t_1=f(s)]\!] \leq [\![t_0=t_1]\!].$$

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- GIVEN THE ATOMIC BOOLEAN VALUES, EXTEND TO ALL FORMULAS:

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg\varphi]\!] = \neg [\![\varphi]\!]$$

$$[\![\exists x\, \varphi(x,s)]\!] = \bigvee_{t \in M} [\![\varphi(t,s)]\!]$$

(USE COMPLETENESS OF $\mathbb{B}$ HERE)

(DERIVE FULL EQUALITY AXIOMS $[\![s=t]\!] \wedge [\![\varphi(s)]\!] \le [\![\varphi(t)]\!]$.)

NOTE SET / CLASS ISSUE

DEF $\mathcal{M}$ IS FULL IF $\exists t$ s.t. $[\![\exists x\, \varphi(x,s)]\!] = [\![\varphi(t,s)]\!]$.

CHOICE-LIKE PRINCIPLE SATURATION.

- FORCING VIA BOOLEAN-VALUED MODELS

FIX $\mathbb{B}$ A COMPLETE BOOLEAN ALGEBRA

DEFINE THE $\mathbb{B}$-NAMES: τ IS A $\mathbb{B}$-NAME IF τ IS A SET OF PAIRS $\langle \sigma, b\rangle$

$V^{\mathbb{B}}$ WHERE σ IS A $\mathbb{B}$-NAME AND $b \in \mathbb{B}$.

THE IDEA: $\langle \sigma, b\rangle$ IN τ MEANS WE PUT THE OBJECT NAMED BY σ INTO THE SET NAMED BY τ TO BOOLEAN VALUE AT LEAST b.

(IF $G \subseteq \mathbb{B}$ IS ANY FILTER, CAN DEFINE $\mathrm{VAL}(\tau, G) = \{\mathrm{VAL}(\sigma, G) \mid \langle\sigma,b\rangle \in \tau,\ b \in G\}$.)

DEFINE THE ATOMIC VALUES:

$$[\![\tau \in \sigma]\!] = \bigvee_{\langle \eta, b\rangle \in \sigma} [\![\tau = \eta]\!] \wedge b$$

$$[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$$

$$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\eta \in \mathrm{dom}(\tau)} \left([\![\eta \in \tau]\!] \Rightarrow [\![\eta \in \sigma]\!]\right)$$

DEFINED BY RECURSION ON RANK OF NAMES

CHECK THE EQUALITY AXIOMS

$$[\![\sigma = \sigma]\!] = 1$$

$$[\![\sigma = \tau]\!] = [\![\tau = \sigma]\!]$$

$$[\![\sigma = \tau]\!] \wedge [\![\tau = \eta]\!] \le [\![\sigma = \eta]\!]$$

$$[\![\tau = \eta]\!] \wedge [\![\eta \in \sigma]\!] \le [\![\tau \in \sigma]\!]$$

THEN EXTEND THE $\mathbb{B}$-VALUED MODEL $V^{\mathbb{B}}$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg\varphi]\!] = \neg [\![\varphi]\!]$$

$$[\![\exists x\, \varphi(x)]\!] = \bigvee_{\tau \in V^{\mathbb{B}}} [\![\varphi(\tau)]\!].$$

MIXING LEMMA IF $A \subseteq \mathbb{B}$ IS AN ANTICHAIN & $\langle \tau_a \mid a \in A \rangle$ IS ANY SEQUENCE OF $\mathbb{B}$-NAMES THEN $\exists \tau$ s.t. $[\![\tau = \tau_a]\!] \geq a$ FOR ALL $a \in A$.

PROOF: LET $\tau = \{ \langle \sigma, a \wedge b \rangle \mid \langle \sigma, b \rangle \in \tau_a,\ a \in A \}$.

BELOW a, τ LOOKS EXACTLY LIKE τ_a. SO $[\![\tau = \tau_a]\!] \geq a$. $\boxtimes$

FULLNESS LEMMA $V^{\mathbb{B}}$ IS FULL. I.E. $[\![\exists x\, \varphi(x)]\!] = [\![\varphi(\tau)]\!]$ SOME τ. (SCHEME)

PROOF: LET $b = [\![\exists x\, \varphi(x)]\!] = \bigvee_{\tau \in V^{\mathbb{B}}} [\![\varphi(\tau)]\!]$. THUS, $\{ [\![\varphi(\tau)]\!] \mid \tau \in V^{\mathbb{B}} \}$ JOINS TO b.

SO $\exists$ ANTICHAIN $A \subseteq \mathbb{B}$ WITH $\bigvee A = b$ s.t. $\forall a \in A \quad a \leq [\![\varphi(\tau_a)]\!]$ SOME τ_a.

APPLY MIXING GET τ WITH $a \leq [\![\tau = \tau_a]\!]$. SO $[\![\varphi(\tau)]\!] \geq \bigvee_{a \in A} [\![\tau = \tau_a]\!] \wedge [\![\varphi(\tau_a)]\!]$

$= \bigvee A = b = [\![\exists x\, \varphi(x)]\!]$ $\boxtimes$

FORCING THEOREM EVERY ZFC AXIOM σ HAS $[\![\sigma]\!] = 1$. (SCHEME).

PROOF: (SKETCH)

$[\![\text{EXTENSIONALITY}]\!] = [\![\forall x, y \ \ x \subseteq y \wedge y \subseteq x \Rightarrow x = y]\!]$

$= \bigwedge_{\sigma, \tau} \left([\![\sigma \subseteq \tau]\!] \wedge [\![\tau \subseteq \sigma]\!] \Rightarrow [\![\sigma = \tau]\!] \right) = 1.$ ✓

$[\![\text{PAIRING}]\!] = [\![\forall x, y\ \exists z \ \ x \in z \wedge y \in z]\!] = \bigwedge_{\sigma, \tau} [\![\exists z \ \ \sigma \in z \wedge \tau \in z]\!]$

LET $\eta = \{ \langle \sigma, 1 \rangle, \langle \tau, 1 \rangle \}$ $\quad \geq [\![\sigma \in \eta]\!] \wedge [\![\tau \in \eta]\!] = 1.$ ✓

$[\![\text{SEPARATION INSTANCE}]\!] = [\![\forall x\, \exists z\, \forall y \,(y \in z \Leftrightarrow y \in x \wedge \varphi(y))]\!]$

$\dot{x}$ $\dot{z}$ GIVEN $\dot{x}$, LET $\dot{z} = \{ \langle \dot{y}, b \rangle \mid \dot{y} \in \mathrm{DOM}(\dot{x}),\ b = [\![\dot{y} \in \dot{x} \wedge \varphi(\dot{y})]\!] \}$.

$[\![\text{COLLECTION INSTANCE}]\!]$ HAVE $\dot{x}$ WANT TO COLLECT WITNESSES $\dot{z}$ FOR

$\forall y \in \dot{x}\ \exists z\ \varphi(y, z)$.

APPLY COLLECTION IN V TO ALL $y \in \mathrm{DOM}(\dot{x})$. $\boxtimes$

- CONSIDER HOW V SITS INSIDE $V^{\mathbb{B}}$.

CHECK NAMES $\check{x} = \{ \langle \check{y}, 1 \rangle \mid y \in x \}$ (NOTE: $\mathrm{VAL}(\check{x}, G) = x$ ANY FILTER G)

DEFINE $[\![\tau \in \check{V}]\!] = \bigvee_{x \in V} [\![\tau = \check{x}]\!]$

NOTE THAT WE CAN HAVE $[\![\tau \in \check{V}]\!] = 1$ EVEN IF $[\![\tau = \check{x}]\!] \neq 1$ ANY x.

"SUPERPOSITION OF VALUES"

FACT $V \models \varphi[x_0 \cdots x_n]$ IFF $[\![\varphi^{\check{V}}(\check{x}_0 \cdots \check{x}_n)]\!] = 1$

PROOF: ATOMIC φ, $[\![\check{x} = \check{y}]\!]$ IS 0 OR 1 DEPENDING ON $x = y$ } BY INDUCTION ON RANK x, y.
$[\![\check{x} \in \check{y}]\!]$ SIMILARLY

NOW APPLY BOOL. COMBINATIONS & QUANTIFIERS

$[\![(\exists x\, \varphi(x, \check{x}_0))^{\check{V}}]\!] = [\![\exists x \in \check{V}\ \varphi^{\check{V}}(x, \check{x}_0)]\!] = \bigvee_{x \in V} [\![\varphi(\check{x}, \check{x}_0)]\!]$ EITHER 0 OR 1 DEPENDING ON $V \models \varphi(x, x_0)$. $\boxtimes$

FACT $[\![\check{V}$ IS A TRANSITIVE INNER MODEL OF ZFC, CONTAINING ALL ORDINALS $]\!] = 1$.

- TRANSFORMING $\mathbb{B}$-VALUED MODELS INTO ACTUAL MODELS.

SUPPOSE $\mathcal{M}$ IS A $\mathbb{B}$-VALUED MODEL, $\mathcal{U} \subseteq \mathbb{B}$ ULTRAFILTER.

NOTE THERE IS NO NEED FOR $\mathcal{U}$ TO BE GENERIC IN ANY SENSE.

FORM THE QUOTIENT $\mathcal{M}/\mathcal{U}$

DEFINE $s =_{\mathcal{U}} t$ IFF $[\![s = t]\!] \in \mathcal{U}$.

$R_{\mathcal{U}}(\vec{s})$ IFF $[\![R(\vec{s})]\!] \in \mathcal{U}$.

(SLIGHT COMPLICATION WITH FUNCTIONAL LANGUAGES. TO GET VALUE $f_{\mathcal{U}}(\vec{s})$ NEED t S.T. $[\![t = f(\vec{s})]\!] \in \mathcal{U}$ FOLLOWS BY FULLNESS.)

IN OUR SET-THEORETIC CASE:

$\sigma =_{\mathcal{U}} \tau$ IFF $[\![\sigma = \tau]\!] \in \mathcal{U}$

WARNING THIS IS NOT THE SAME AS $\mathrm{VAL}(\sigma, \mathcal{U}) = \mathrm{VAL}(\tau, \mathcal{U})$.

$\sigma \in_{\mathcal{U}} \tau$ IFF $[\![\sigma \in \tau]\!] \in \mathcal{U}$

THE EQUALITY AXIOMS ENSURE THAT $=_{\mathcal{U}}$ IS A CONGRUENCE WRT $\in_{\mathcal{U}}$

SO ~~[illegible]~~ RELATIONS ARE WELL-DEFINED ON QUOTIENT.

EQUIVALENCE CLASS $\{\tau \mid \sigma =_{\mathcal{U}} \tau\}$ PROPER CLASS. USE SCOTT'S TRICK OF REDUCED CLASS

$[\sigma]_{\mathcal{U}} \{\tau \mid \sigma =_{\mathcal{U}} \tau\ \&\ \tau$ MINIMAL RANK$\}$. NOW A SET. (SAME ISSUE IN CLASSICAL ULTRAPOWERS)

QUOTIENT MODEL

$V^{\mathbb{B}}/\mathcal{U} = \{ [\sigma]_{\mathcal{U}} \mid \sigma \in V^{\mathbb{B}} \}$ WITH RELATION $\in_{\mathcal{U}}$.

ŁOS THEOREM $V^{\mathbb{B}}/\mathcal{U} \models \varphi([\tau_0]_{\mathcal{U}} \cdots [\tau_n]_{\mathcal{U}})$ IFF $[\![\varphi(\tau_0 \cdots \tau_n)]\!] \in \mathcal{U}$.

PROOF: JUST LIKE ŁOS. BY INDUCTION ON φ. ATOMIC CASE BY DEFINITION.

BOOLEAN COMBINATIONS FOLLOW BECAUSE $\mathcal{U}$ IS AN ULTRAFILTER.

QUANTIFIER CASE USES FULLNESS

$[\![\exists x\, \varphi(x, \tau_0)]\!] \in \mathcal{U}$ IFF $\exists \tau\ [\![\varphi(\tau, \tau_0)]\!] \in \mathcal{U}$

IFF $V^{\mathbb{B}}/\mathcal{U} \models \varphi([\tau]_{\mathcal{U}}, [\tau_0]_{\mathcal{U}})$

IFF $V^{\mathbb{B}}/\mathcal{U} \models \exists x\, \varphi(x, [\tau_0]_{\mathcal{U}})$. $\boxtimes$

THUS, WE HAVE CONSTRUCTED $V^{\mathbb{B}}/\mathcal{U}$.

BOOLEAN ULTRAPOWERS (6)

OBSERVATIONS

- FOR ANY φ WITH $[\![\varphi]\!] \neq 0$ WE CAN FIND $\mathcal{U}$ WITH $[\![\varphi]\!] \in \mathcal{U}$ SO $V^{\mathbb{B}}/\mathcal{U} \models ZFC + \varphi$.
- SO WE GET CONSISTENCY RESULTS
- THIS MAKES FORCING AN INTERNAL ZFC CONSTRUCTION
  - MAKES SENSE TO SPEAK OF FORCING OVER ANY MODEL (NOT JUST COUNTABLE OR EVEN WELL-FOUNDED MODELS)
- "FORCEABILITY" IS A DEFINABLE NOTION.
  - $\Diamond\varphi$ = "φ IS FORCEABLE" = $\exists \mathbb{B}\ [\![\varphi]\!]^{\mathbb{B}} \neq 0$ $= 1$
  - $\Box\varphi$ = "φ IS NECESSARY" = $\forall \mathbb{B}\ [\![\varphi]\!]^{\mathbb{B}} = 1$

  RELATED TOPIC MODAL LOGIC OF FORCING INVESTIGATES THESE MODAL OPERATORS.
- NOTE THAT WE CONSTRUCTED $V^{\mathbb{B}}/\mathcal{U}$ WITHOUT MENTIONING ANY OF THE FOLLOWING:
  - THE GENERIC OBJECT
  - GENERIC FILTER
  - VALUE OF A NAME
  - DENSE SETS, MEETING DENSE SETS
  - FORCING RELATION.

• ROLE OF THE GENERIC OBJECT IN $V^{\mathbb{B}}$

THE CANONICAL NAME FOR THE GENERIC OBJECT IS $\dot{G} = \{\langle \check{b}, b\rangle \mid b \in \mathbb{B}\}$.

SUPPOSE $D \subseteq \mathbb{B}$. THEN $[\![\exists b \in \check{D} \cap \dot{G}]\!] = \bigvee_{b\in D} [\![\check{b} \in \dot{G}]\!] = \bigvee_{b\in D} b = \bigvee D$.

SO IF D IS DENSE (OR PRE-DENSE), THEN $\bigvee D = 1$ SO $[\![\dot{G} \text{ MEETS } \check{D}]\!] = 1$.

THEOREM ① $[\![\dot{G}$ IS A $\check{V}$-GENERIC ULTRAFILTER ON $\check{\mathbb{B}}]\!] = 1$

② $[\![\check{\tau} = \mathrm{VAL}(\check{\tau}, \dot{G})]\!] = 1$.

③ $[\![$ THE UNIVERSE IS A FORCING EXTENSION $\check{V}[\dot{G}]\,]\!] = 1$.

PROOF: ① $[\![\check{b} \in \dot{G}]\!] = b$. $[\![\forall b \in \check{\mathbb{B}}\ b \in \dot{G} \text{ or } \neg b \in \dot{G}]\!] = \bigwedge_{b\in\mathbb{B}} \underbrace{[\![\check{b} \in \dot{G}]\!] \vee [\![\neg\check{b} \in \dot{G}]\!]}_{b \,\vee\, \neg b} = 1$.

SIMILARLY, $b \leq c \Rightarrow [\![\check{b} \in \dot{G}]\!] \leq [\![\check{c} \in \dot{G}]\!]$

$[\![\check{b} \wedge \check{c} \in \dot{G}]\!] = [\![\check{b} \in \dot{G}]\!] \wedge [\![\check{c} \in \dot{G}]\!]$.

② INDUCTION ON RANK OF τ.

③ BY ①, ②. $\boxtimes$

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IN THE QUOTIENT:

~~THEOREM~~ $\check{V}_U = \{[\tau]_U \mid [\![\tau \in \check{V}]\!] \in U\}$ = THE GROUND MODEL OF $V^{\mathbb{B}}/U$.

$[\![\tau \in \check{V}]\!] = 1$

EQUIV.

THEOREM ① $\check{V}_U$ IS TRANSITIVE IN $V^{\mathbb{B}}/U$

② IF $G = [\dot{G}]_U$ IN $V^{\mathbb{B}}/U$, THEN G IS $\check{V}_U$-GENERIC FOR $[\check{\mathbb{B}}]_U$.

③ $V^{\mathbb{B}}/U = \check{V}_U[G]$.

PROOF: ① WE SAW ALREADY THAT $[\![\check{V}$ IS TRANSITIVE$]\!] = 1$.

IF $[\tau]_U \in_U [\sigma]_U \in \check{V}_U$ THEN $[\![\tau \in \sigma \in \check{V}]\!] \in U$ SO $[\![\tau \in \check{V}]\!] \in U$.

②,③ FOLLOW FROM EARLIER THEOREM. $\boxtimes$

THE BOOLEAN ULTRAPOWER

<u>DEFINITION</u> THE <u>BOOLEAN ULTRAPOWER</u> OF V BY $U \subseteq \mathbb{B}$ IS THE STRUCTURE $\check{V}_U = \{[\tau]_U \mid [\![\tau \in \check{V}]\!] \in U\}$, WITH RELATION $\in_U$, ACCOMPANIED BY THE <u>BOOLEAN ULTRAPOWER MAP</u> $j : V \to \check{V}_U$ DEFINED BY $j : x \mapsto [\check{x}]_U$ AND BY THE CANONICAL FORCING EXTENSION $\check{V}_U[G] = V^{\mathbb{B}}/U$, $G = [\dot{G}]_U$.

<u>THEOREM</u> THE BOOLEAN ULTRAPOWER MAP IS AN ELEMENTARY EMBEDDING.

PROOF: $V \models \varphi[x_0 \cdots x_n]$ IFF $[\![\varphi^{\check{V}}(\check{x}_0 \cdots \check{x}_n)]\!] = 1$

IFF $\check{V}_U \models \varphi[\underbrace{[\check{x}_0]}_{j(x_0)} \cdots \underbrace{[\check{x}_n]}_{j(x_n)}]$. $\boxtimes$

<u>THEOREM</u> THE FOLLOWING ARE EQUIVALENT FOR $U \subseteq \mathbb{B}$.

① U IS V-GENERIC.

② THE BOOLEAN ULTRAPOWER $j : V \to \check{V}_U$ IS AN ISOMORPHISM.

PROOF: $(\to)$ IF U IS V-GENERIC & $[\tau]_U \in \check{V}_U$ THEN $[\![\tau \in \check{V}]\!] = \bigvee_{x \in V} [\![\tau = \check{x}]\!] \in U$.

THIS IS A MAXIMAL AC BELOW $[\![\tau \in \check{V}]\!]$, SO $\exists x \in V$ $[\![\tau = \check{x}]\!] \in U$.

THUS, $j(x) = [\tau]_U$.

$(\leftarrow)$ SUPPOSE j IS ONTO. LET $A \subseteq \mathbb{B}$ BE MAXIMAL AC. BY MIXING, FIND τ S.T. $[\![\tau = \check{a}]\!] \geq a$ FOR $a \in A$. SO $[\![\tau \in \check{V}]\!] \geq \bigvee_{a \in A} [\![\tau = \check{a}]\!] = \bigvee A = 1$. SO $[\tau]_U \in \check{V}_U$.

SO $\exists x$ $j(x) = [\tau]_U$ I.E. $[\![\check{x} = \tau]\!] \in U$. ~~x must~~ $[\![\check{x} = \tau]\!] = \begin{cases} x & \text{FOR } x \in A \\ 0 & \text{O.W.} \end{cases}$ $\boxtimes$

CONCLUSION NON-GENERIC IS THE RIGHT GENERALIZATION OF NON-PRINCIPAL.

SIMPLE / TRIVIAL ———————— COMPLEX / INTERESTING

PRINCIPAL ULTRAFILTER NONPRINCIPAL

GENERIC GENERIC? NO!

NON-GENERIC

NOTE
- BOOLEAN ULTRAPOWER BY GENERIC ULTRAFILTERS ARE TRIVIAL.
- PRINCIPAL ULTRAFILTERS ARE GENERIC ON POWER SET ALGEBRA

I.E. IF $\mathbb{B} = P(I)$, THEN $U \subseteq \mathbb{B}$ IS GENERIC IFF U IS PRINCIPAL

NON-GENERIC = NON-PRINCIPAL.

THEOREM IF $U \subseteq \mathbb{B}$ ULTRAFILTER, THEN $cp(j_U)$ = SIZE OF SMALLEST MAXIMAL ANTICHAIN MISSED BY U.

PROOF: SUPPOSE U MISSES $A = \{a_\alpha \mid \alpha < \kappa\}$ BY MIXING GET $\dot{\alpha}$ WITH $[\![\dot{\alpha} = \check{\alpha}]\!] \geq a_\alpha$. NOTE $[\![\dot{\alpha} < \check{\kappa}]\!] = \bigvee_{\alpha<\kappa} [\![\dot{\alpha} = \check{\alpha}]\!] = 1$. SO $[\dot{\alpha}]_U < j(\kappa)$.
BUT $[\dot{\alpha}]_U \neq [\check{\alpha}]_U$ ANY $\alpha < \kappa$. SO j IS DISCONTINUOUS AT κ.
CONVERSELY, SUPPOSE $[\dot{\alpha}]_U \in j(\kappa) = [\check{\kappa}]_U$ BUT $[\dot{\alpha}]_U \neq [\check{\alpha}]_U$ ANY $\alpha < \kappa$.
LET $A = \{[\![\dot{\alpha} = \check{\alpha}]\!] \mid \alpha < \kappa\}$. THIS IS A MAXIMAL AC MISSED BY U. $\boxtimes$

- NATURALIST ACCOUNT OF FORCING

SYNTACTIC VERSION

THEOREM FOR ANY $\mathbb{B}$, THE FOLLOWING THEORY HAS A CLASS MODEL:

LANGUAGE: $\in$
PREDICATE SYMBOL $\bar{V}$
CONSTANT SYMBOL FOR EVERY OBJECT
CONSTANT SYMBOL G

(i) ZFC
(ii) $\bar{V}$ IS A TRANSITIVE INNER MODEL OF ZFC, CONTAINING ALL ORDINALS
(iii) $\varphi^{\bar{V}}(a)$ IF $\varphi(a)$ HOLDS IN V.
(iv) G IS $\bar{V}$-GENERIC. FOR $\mathbb{B}$.
(v) THE UNIVERSE IS THE FORCING EXTENSION $\bar{V}[G]$.

PROOF: THIS THEORY HAS BOOLEAN VALUE 1. $\boxtimes$

THIS IS HOW WE USE FORCING.
- IN A CONTEXT V.
- SAY "LET $G \subseteq \mathbb{B}$ BE V-GENERIC. ARGUE IN $V[G]$"
- THAT INVOKES THIS THEOREM.

<u>SEMANTIC VERSION</u> FOR ANY FORCING NOTION $\mathbb{B}$, THERE IS A CLASS STRUCTURE $(\bar V, \bar\in)$ AND AN ELEMENTARY EMBEDDING $V \preceq \bar V$ AND IN V THERE IS $\bar V$-GENERIC FILTER $G \subseteq \bar{\mathbb{B}}$

$$V \preceq \bar V \subseteq \bar V[G]$$

THE ENTIRE EXTENSION $\bar V[G]$ IS A CLASS IN V.

PROOF: THIS IS PRECISELY WHAT THE BOOLEAN ULTRAPOWER PROVIDES. $\boxtimes$

THIS IS A WAY OF UNDERSTANDING FORCING OVER V WITHOUT LEAVING V.

AN INTERESTING CASE: WHEN $\bar V$ IS WELL-FOUNDED.

- CONNECTED WITH LARGE CARDINAL EMBEDDINGS.

- WELL-FOUNDED BOOLEAN ULTRAPOWERS

<u>THEOREM</u> SUPPOSE $U \subseteq \mathbb{B}$, $U \in V$. THEN TFAE

EQUIVALENT TO: $V^{\mathbb{B}}/U$ IS WELL-FOUNDED

1. THE BOOLEAN ULTRAPOWER $\check V_U$ IS WELL-FOUNDED.
2. THE BOOLEAN ULTRAPOWER $\check V_U$ IS AN ω-MODEL.
3. U MEETS ALL COUNTABLE MAXIMAL ANTICHAINS $A \subseteq \mathbb{B}$ IN V.
4. U IS COUNTABLY COMPLETE OVER V. I.E. $b_n \in U$ & $\langle b_n \mid n<\omega\rangle \in V \Rightarrow \bigwedge_n b_n \in U$.
5. U IS WEAKLY COUNTABLY COMPLETE. I.E. $b_n \in U$ & $\vec b \in V \Rightarrow \bigwedge_n b_n \neq 0$.

PROOF: 1 → 2 ✓

2 → 3 FIX $A = \{a_n \mid n<\omega\}$. BY MIXING, GET τ WITH $[\![\tau = \check n]\!] = a_n$.
SO $[\![\tau \in \check\omega]\!] = \bigvee A = 1$. SO $[\tau]_U = [\check n]_U$ SOME n. SO $a_n \in U$.

3 → 4 WLOG $b_0 \geq b_1 \geq \dots$. LET $a_n = b_n - b_{n+1}$. $\{a_n \mid n<\omega\} \cup \{\bigwedge b_n\}$ IS MAX AC BELOW b_0. SO $\bigwedge b_n \in U$.

4 → 5 ✓

5 → 1 SUPPOSE ~~[X]~~ $[\tau_{n+1}]_U \in_U [\tau_n]_U$. LET $b_n = [\![\tau_{n+1} \in \tau_n]\!] \in U$.
SO $b = \bigwedge b_n \neq 0$. LET $\dot B = \{\langle \tau_n, 1\rangle \mid n\in\omega\}$. $b \leq [\![\dot B$ HAS NO $\in$-MINIMAL ELT$]\!]$.
$\rightarrow\leftarrow$. $\boxtimes$

BOOLEAN ULTRAPOWERS (10)

THEOREM SUPPOSE $\mathbb{B}$ IS CBA WITH ULTRAFILTER $\mathcal{U} \subseteq \mathbb{B}$. (NOT NECESSARILY IN V)
THEN TFAE

1. THE BOOLEAN ULTRAPOWER $\check{V}_{\mathcal{U}}$ IS WELL-FOUNDED.
2. $\exists j: V \to M$ TRANSITIVE CLASS M $\exists G \subseteq j(\mathbb{B})$ M-GENERIC WITH $j''\mathcal{U} \subseteq G$.

PROOF: 1 $\to$ 2 USE THE BOOL. ULTRAPOWER $j: V \to \check{V}_{\mathcal{U}} \subseteq \check{V}_{\mathcal{U}}[G]$. $j''\mathcal{U} \subseteq G$

SINCE $b \in \mathcal{U} \Rightarrow j(b) = [\check{b}]$ $[\![\check{b} \in \dot{G}]\!] = b \in \mathcal{U}$.

2 $\to$ 1 DEFINE π

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j_{\mathcal{U}}"'] & M \subseteq M[G]. \\ & \check{V}_{\mathcal{U}} \arrow[u, "\pi"'] \end{tikzcd}$$

$\pi : [\tau]_{\mathcal{U}} \mapsto \mathrm{VAL}(j(\tau), G)$.

π IS WELL-DEFINED AND ELEMENTARY.

$\check{V}_{\mathcal{U}} \models \varphi([\tau]_{\mathcal{U}})$ IFF $[\![\check{\varphi}(\tau)]\!] \in \mathcal{U}$

IFF $j([\![\check{\varphi}(\tau)]\!]) \in G$

IFF $M[G] \models \check{\varphi}(j(\tau))$

IFF $M \models \varphi(\mathrm{VAL}(\tau, G))$. $\boxtimes$

- ALGEBRAIC PRESENTATION OF THE BOOLEAN ULTRAPOWER

LET $\mathcal{M}$ BE ANY FIRST-ORDER STRUCTURE

CLASSICAL ULTRAPOWER:
- $\mathcal{U}$ ULTRAFILTER ON I, MEASURING $P(I)$.
- USE FUNCTIONS $f: I \to M$

$f =_{\mathcal{U}} g \leftrightarrow \{ i \in I \mid f(i) = g(i) \} \in \mathcal{U}$

$R([f]) \leftrightarrow \{ i \in I \mid R(f(i)) \} \in \mathcal{U}$

$f \in_{\mathcal{U}} g \leftrightarrow$ $\in$

- CONSTRUCT $\mathcal{M}^I/\mathcal{U}$ QUOTIENT

NOW, HAVE COMPLETE BOOLEAN ALGEBRA $\mathbb{B}$, ULTRAFILTER $\mathcal{U} \subseteq \mathbb{B}$.

NOTE, IF $A \subseteq \mathbb{B}$ MAXIMAL AC, THEN $P(A) \subseteq_c \mathbb{B}$ COMPLETE SUBALGEBRA

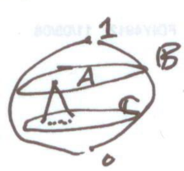
$$\begin{array}{ccc} X & \longmapsto & \bigvee X \\ \cap\!\!| & & \cap \\ A & & \mathbb{B} \end{array}$$

DEFINE SPANNING FUNCTION

$f: A \to \mathcal{M}$, $A \subseteq \mathbb{B}$ MAX AC.

IF $g: A \to \mathcal{M}$ THEN $f \equiv_{\mathcal{U}} g$ IFF $\bigvee \{ a \in A \mid f(a) = g(a) \} \in \mathcal{U}$.

IF $C \leq A$ REFINEMENT



$\forall c \in C\ \exists a \in A\ c \leq a$.

$(f \downarrow C)(c) = f(a)$

MORE GENERALLY, IF $g: B \to \mathcal{M}$ DIFFERENT ANTICHAIN $B \subseteq \mathbb{B}$

~~BA~~ FIND A COMMON REFINEMENT $C \leq A, B$

$f \equiv_u g$ IFF $(f \downarrow C) \equiv_u (g \downarrow C)$ THIS DOES NOT DEPEND ON CHOICE OF C.

$\mathcal{M}^{\downarrow \mathbb{B}}$ = COLLECTION OF SPANNING FUNCTIONS

$\mathcal{M}^{\downarrow \mathbb{B}}/u$ = QUOTIENT BY $\equiv_u$

BOOLEAN ULTRAPOWER MAP $j: \mathcal{M} \to \mathcal{M}^{\downarrow \mathbb{B}}/u$ $A = \{1\}$ MAX AC.

$C_x: 1 \mapsto x$

$x \mapsto [C_x]_u$

<u>THEOREM</u> THE TWO PRESENTATIONS OF THE BOOLEAN ULTRAPOWER ARE EQUIVALENT.

$$\begin{tikzcd} V \arrow[r, "j_u"] \arrow[d, "j"'] & \check{V}_u \\ V^{\downarrow \mathbb{B}}/u \arrow[ur, "\pi"', "\cong"] & \end{tikzcd}$$

$\pi: [f]_u \mapsto [\tau]_u$

$f: A \to V$

τ MIXES THE VALUES $\check{f}(a)$ ON A.

I.E. $[\![\tau = \check{f(a)}]\!] \geq a$.

PROOF: π IS WELL-DEFINED & $\in$-PRESERVING.

π IS ONTO: GIVEN ANY $[\tau]_u \in \check{V}_u$ GET MAX AC $A \subseteq \mathbb{B}$ OF POSSIBLE $x = f(a)$ FOR WHICH $[\![\tau = \check{x}]\!] \geq a$. $\boxtimes$

- POWERSET FACTORS.

IF $A \subseteq \mathbb{B}$ IS MAXIMAL ANTICHAIN, THEN $P(A) \subseteq_c \mathbb{B}$. ASSOCIATE $X \subseteq A$ WITH $\bigvee X \in \mathbb{B}$.

DEFINE ULTRAFILTER U_A ON $P(A)$ BY: $X \in U_A \leftrightarrow X \subseteq A$ & $\bigvee X \in U$.

<u>THEOREM</u> IN THIS SITUATION, WE HAVE

$$\begin{tikzcd} V \arrow[r, "j_u"', "\text{BOOL ULT}"] \arrow[d, "j_{U_A}"', "\text{POWERSET ULT}"] & \check{V}_u \\ V^A/U_A \arrow[ur, "k"'] & \end{tikzcd}$$

$k: [f]_{U_A} \mapsto [\tau_f]_u$ τ_f MIXES $\check{f}(a)$ ON A

$f: A \to V$ $[\![\tau_f = \check{f(a)}]\!] \geq a$.

– k IS WELL DEFINED AND ELEMENTARY.

$V^A/U_A \models \varphi[[f]_{U_A}]$ IFF $\{a \in A \mid V \models \varphi(f(a))\} \in U_A$

IFF $\bigvee \{a \in A \mid V \models \varphi(f(a))\} \in U$

IFF $[\![\check{\varphi}(\tau_f)]\!] \in U$

IFF $\check{V}_u \models \varphi[[\tau_f]]$.

MORE GENERALLY:

THEOREM IF $\mathbb{B} \leq A$ REFINES, THEN HAVE COMMUTATIVE DIAGRAM

$$\begin{tikzcd}
V \arrow[rrr, bend left, "j_U"] \arrow[r, "j_{U_A}"] \arrow[rr, bend right, "j_{U_B}"'] & V^A/U_A \arrow[rr, "k_A"] \arrow[r, "k_{A,B}"] & V^B/U_B \arrow[r, "k_B"] & \check{V}_U
\end{tikzcd}$$

$f : A \to V$

$k_{A,B} : [f]_{U_A} \longmapsto [f \downarrow B]_{U_B}$

$k_A [f]_{U_A} \longmapsto [\tau_f]_U$

$k_B [f]_{U_B} \longmapsto [\tau_f]_U$

FURTHERMORE, EVERY ELEMENT OF $\check{V}_U$ ARISES AS $k_A(x)$ FOR SOME MAXIMAL ANTICHAIN $A \subseteq \mathbb{B}$.

THIS IS BECAUSE $[\tau]_U \in \check{V}_U$ MEANS $[\![\tau \in \check{V}]\!] \in U$

$= \bigvee_{x \in V} [\![\tau = \check{x}]\!]$ ANTICHAIN OF POSSIBLE VALUES.

CONCLUSION THE BOOLEAN ULTRAPOWER MAP $j : V \to \check{V}_U$ IS PRECISELY THE DIRECT LIMIT OF THE SYSTEM OF POWERSET ULTRAPOWERS V^A/U_A IN THE DIRECTED SET OF ANTICHAINS UNDER REFINEMENT.

- TWO EXTREME CASES

U IS V-GENERIC. IN THIS CASE, EVERY U_A IS PRINCIPAL. SO EVERY j_{U_A} IS ~~IDENTITY MAP~~ ISOMORPHISM. DIRECT LIMIT OF A SYSTEM OF ISOMORPHISMS. TRIVIAL MAP.

$\mathbb{B} = P(A)$. IN THIS CASE, THERE IS A MOST-REFINED ANTICHAIN ATOMS. SO SYSTEM HAS A TERMINAL INDEX $j_U \cong j_{U_A}$.

- EXTENDER REPRESENTATION

THEOREM IF $j : V \to \bar{V}$ IS THE BOOLEAN ULTRAPOWER BY $U \subseteq \mathbb{B}$ THEN $\bar{V} = \{ j(f)(b) \mid f : \mathbb{B} \to V,\ b \in j(\mathbb{B}) \}$.

IN FACT, $\bar{V} = \{ j(f)(b_A) \mid f : A \to V,\ A \subseteq \mathbb{B} \text{ MAX AC} \}$

b_A = UNIQUE ELT. OF $j(A) \cap G$, $G = [\dot{G}]_U$.

PROOF: $\bar{V} = V^{\downarrow \mathbb{B}}/U$ HAS $[f]_U = j(f)(b_A)$. $\boxtimes$

$\quad = k_{A\,\mathbb{B}}([id]_{U_A})$

- COMPARISON OF BOOLEAN ULTRAPOWERS VS. CLASSICAL ULTRAPOWERS.

→ DEF

<u>THEOREM</u> SUPPOSE $j : V \to \bar{V}$ IS THE BOOLEAN ULTRAPOWER BY $\mathcal{U} \subseteq \mathbb{B}$.

THEN TFAE

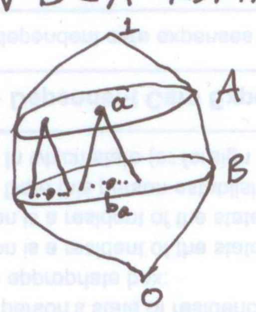
① $j : V \to \bar{V}$ IS ISOMORPHIC TO A CLASSICAL (POWERSET) ULTRAPOWER BY ULTRAFILTER $\mathcal{U}^*$ ON A SET Z.

② $\mathcal{U}$ IS V-GENERIC RELATIVE TO A MAXIMAL ANTICHAIN $A \subseteq \mathbb{B}$.

(AND IN THIS CASE, $\mathcal{U}^* = \mathcal{U}_A$, $Z = A$.)

<u>DEF</u> $\mathcal{U} \subseteq \mathbb{B}$ IS <u>V-GENERIC</u> <u>RELATIVE TO</u> MAXIMAL ANTICHAIN $A \subseteq \mathbb{B}$

IFF $\forall B \le A$ REFINING $\forall a \in A\ \exists b_a \in B\ b_a \le a$ s.t. $\bigvee_{a\in A} b_a \in \mathcal{U}$.



PROOF: ① → ② SUPPOSE

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ V^{\mathbb{B}}/\mathcal{U} \arrow[rr, "\cong", "\pi"'] & & V^Z/\mathcal{U}^* \end{tikzcd}$$

$[f]_{\mathcal{U}^*} = j_{\mathcal{U}^*}(f)(s^*)$ $\quad s^* = [id]_{\mathcal{U}^*}$

PICK s WITH $\pi(s) = s^*$.

SO $s = \pi_{A,\infty}(s_A)$ SOME $A \subseteq \mathbb{B}$ $\quad s_A \in V^A/\mathcal{U}_A$.

IF $C \le A$, $x \in V^C/\mathcal{U}_C$, $\pi_{C,\infty}(x) = j_{\mathcal{U}}(f)(s)$ SOME f.

$$= \pi_{A,\infty}(j_{\mathcal{U}_A}(f)(s_A)) = \pi_{C,\infty}(\pi_{A,C}(j_{\mathcal{U}_A}(f)(s_A)))$$

SO $x = \pi_{A,C}(j_{\mathcal{U}}(f)(s_A))$. SO $\pi_{A,C}$ IS ONTO $V^C/\mathcal{U}_C$. ~~SO U IS GEN. REL.~~

~~TO AS~~ ARGUE $\mathcal{U}$ IS GENERIC REL. TO A. $[id\restriction C]_{\mathcal{U}_C} = \pi_{A,C}([f]_{\mathcal{U}_A})$

SOME $f : A \to V$. SO $id\downarrow C \equiv_{\mathcal{U}_C} f\downarrow C$ SO $\bigvee Y \in \mathcal{U}$ WHERE $Y = \{c \in C \mid (f\downarrow C)(c) = c\}$.
$= f(a)$

LET c_a BE THIS VALUE. $\bigvee_{a\in A} c_a \ge \bigvee Y \in \mathcal{U}$.

② → ① IF $\mathcal{U}$ IS GENERIC RELATIVE TO A, ARGUE THAT $\pi_{A,C}$ = IDENTITY WHEN $C \le A$. SO $\lim_{\to} j_{\mathcal{U}_C} = j_{\mathcal{U}_A}$. I.E. j IS REALLY ULT BY $\mathcal{U}_A$. ☒

• BOOLEAN ULTRAPOWERS VIA PARTIAL ORDERS

$\mathbb{P}$ PARTIAL ORDER $\quad \mathbb{B} = RO(\mathbb{P})$ = COMPLETION OF $\mathbb{P}$ AS A COMPLETE BOOLEAN ALGEBRA

CONSISTS OF REGULAR OPEN SUBSETS OF $\mathbb{P}$.

$\mathbb{P}$ MAPS DENSELY INTO $\mathbb{B}$.

$p \mapsto$ CONE BELOW p.

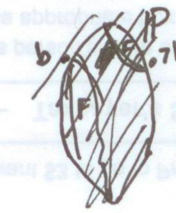
<u>GENERAL FACT</u> FORCING WITH PARTIAL ORDERS $\approxeq$ FORCING WITH COMPLETE BOOLEAN ALGEBRAS

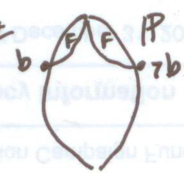
PARTLY DO TO:

<u>FACT</u> IF $U \subseteq \mathbb{P}$ IS V-GENERIC, THEN THE FILTER GENERATED BY $U \subseteq \mathbb{B}$ IS AN ULTRAFILTER (AND V-GENERIC).

<u>BUT</u> THIS IS NOT TRUE WITHOUT GENERICITY.

<u>FACT</u> $\forall\, \mathbb{P}$ INCOMPLETE BOOLEAN ALGEBRA, $\exists$ ULTRAFILTER $U \subseteq \mathbb{P}$ THAT DOES NOT GENERATE AN ULTRAFILTER ON $\mathbb{B} = RO(\mathbb{P})$.

PF: 

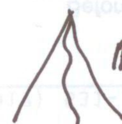
~~CBA~~ 

PICK $b \in \mathbb{B} - \mathbb{P}$.

$F = \{ p \mid b \le p \text{ or } \neg b \le p \}$. HAS FIP. SO $F \subseteq U \subseteq \mathbb{P}$ MAXIMAL FILTER.

U DOES NOT DECIDE b. $\boxtimes$

• <u>NOTE</u> IN PARTIAL ORDERS, WE ARE USED TO ULTRAFILTERS BEING GENERATED BY LINEARLY ORDERED SETS. e.g. FORCING WITH A TREE.

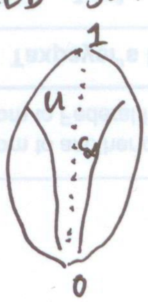


$\mathbb{P}$ A TREE

GENERIC FILTER IS GENERATED BY A BRANCH.

BUT: $\mathbb{B}$ CBA ~~[illegible]~~

<u>THEOREM</u> IF $U \subseteq \mathbb{B}$ IS (NONPRINCIPAL) ULTRAFILTER, THEN U IS NOT GENERATED BY ANY LINEARLY ORDERED SET OF ELEMENTS.

PROOF: 

PICK c_α STRICTLY DESCENDING COFINALLY IN THE LINEARLY ORDERED SET. LET $d_\alpha = c_\alpha - c_{\alpha+1}$. ANTICHAIN.

LET $a = \bigvee \{ d_\alpha \mid \alpha \text{ even} \}$ $\quad b = \bigvee \{ d_\alpha \mid \alpha \text{ odd} \}$.

$a \vee b = c_0 \in U$ BUT $a \wedge b = 0$. a, b NOT DECIDED BY ANY c_α. $\boxtimes$

- INSTRUCTIVE EXAMPLE

LET $\mathbb{P} = \mathrm{ADD}(\omega, 1) = 2^{<\omega}$.

2^{2^ω}

EVERY ULTRAFILTER $\mathcal{U} \subseteq \mathbb{P}$ IS DETERMINED BY A BRANCH b THROUGH TREE.

SUCH A $\mathcal{U}$ MEETS EVERY COUNTABLE MAXIMAL ANTICHAIN IN $\mathbb{P}$.

IS THE BOOLEAN ULTRAPOWER WELL-FOUNDED? NO, IT CANNOT BE.

$\mathbb{P}$ IS TOO SMALL. < LEAST MEASURABLE CARDINAL.

THE RESOLUTION: $\mathcal{U}$ DOES NOT GENERATE ULTRAFILTER ON $\mathbb{B}$.

$\mathcal{U}$ DOES NOT MEET EVERY CTBLE MAXIMAL AC IN $\mathbb{B}$.

- SUBALGEBRAS

<u>THEOREM</u> IF $\mathbb{B} \subseteq \mathbb{C}$ COMPLETE SUBALGEBRA, $\mathcal{U} \subseteq \mathbb{C}$ ULTRAFILTER THEN:

$$\begin{tikzcd}
V \arrow[r, "j_{\mathcal{U}}"] \arrow[d, "j_{\mathcal{U}_0}"'] & \bar{V} \subseteq \bar{V}[G \cap j_{\mathcal{U}}(\mathbb{B})] \subseteq \bar{V}[G] \\
V_0 \subseteq V_0[G_0] \arrow[ur, "k \restriction V_0"] \arrow[ur, "k"'] &
\end{tikzcd}$$

$\mathcal{U}_0 = \mathcal{U} \cap \mathbb{B}$

$G_0 = [\dot{G}]_{\mathcal{U}_0}$

$k : [\tau]^{\mathbb{B}}_{\mathcal{U}_0} \longmapsto [\tau]^{\mathbb{C}}_{\mathcal{U}}$ INTERPRETING $\mathbb{B}$-NAME τ AS A $\mathbb{C}$-NAME.

- ITERATIONS

$\mathbb{B}_0$ COMPLETE BOOLEAN ALGEBRA. $\dot{\mathbb{B}}_1$ IS $\mathbb{B}_0$-NAME FOR CBA. (ASSUME FULL NAME)

$\mathbb{B}_0 * \dot{\mathbb{B}}_1 = \mathrm{RO}(\mathbb{P})$ $\quad \mathbb{P} = \{ \langle b, \dot{c} \rangle \mid b \in \mathbb{B}_0^+,\ \dot{c} \in \mathrm{dom}(\dot{\mathbb{B}}_1),\ b \leq [\![\dot{c} \in \dot{\mathbb{B}}_1]\!] \}$.

$(b, \dot{c}) \leq (d, \dot{e})$ IFF $b \leq d$ & $b \leq [\![\dot{c} \leq \dot{e}]\!]$.

IF $\mathcal{U}_0 \subseteq \mathbb{B}_0$ ULTRAFILTER & $\mathcal{U}_1 \subseteq \mathbb{B}_1 = [\dot{\mathbb{B}}_1]_{\mathcal{U}_0}$ IN $V^{\mathbb{B}_0}/\mathcal{U}_0$.

THEN LET $\mathcal{U}_0 * \mathcal{U}_1$ = FILTER GENERATED BY $\{ (b, \dot{c}) \mid b \in \mathcal{U}_0,\ [\dot{c}]_{\mathcal{U}_0} \in \mathcal{U}_1 \}$.

<u>THEOREM</u> IN THIS SITUATION, $\mathcal{U}_0 * \mathcal{U}_1$ IS AN ULTRAFILTER AND

$$\begin{tikzcd}
V \arrow[r, "j_{\mathcal{U}_0 * \mathcal{U}_1}"] \arrow[d, "j_{\mathcal{U}_0}"'] & \bar{V} \subseteq \bar{V}[G_0] \subseteq \bar{V}[G_0 * G_1] \\
V_0 \subseteq V_0[G_0] \arrow[ur, "j_{\mathcal{U}_1} \restriction V_0"] \arrow[ur, "\dot{j}_{\mathcal{U}_1}"'] &
\end{tikzcd}$$

WHERE $V_0[G_0] = V^{\mathbb{B}_0}/\mathcal{U}_0$

$\bar{V}[G_0 * G_1] = V^{\mathbb{B}_0 * \dot{\mathbb{B}}_1} / \mathcal{U}_0 * \mathcal{U}_1$

- PRODUCTS

$\mathbb{B}_0$, $\mathbb{B}_1$ COMPLETE BOOLEAN ALGEBRAS.

Q: HOW TO TAKE THE PRODUCT?

NATURAL TO LOOK AT $\{(b,c) \mid b \in \mathbb{B}_0, \& c \in \mathbb{B}_1\}$. COORDINATE-WISE OPERATIONS.

THIS IS A COMPLETE BOOL. ALGEBRA.

BUT IT IS NOT THE CORRECT ONE!

IT HAS $\{(0,1),(1,0)\}$ MAXIMAL AC.

SO ANY ULTRAFILTER CONCENTRATES ON ONE SIDE.

SO WE CALL IT $\mathbb{B}_0 \oplus \mathbb{B}_1$ SIDE-BY-SIDE FORCING

"LOTTERY SUM".

INSTEAD, PRODUCT FORCING USES $\mathbb{B}_0 \times \mathbb{B}_1 = \mathrm{RO}(\mathbb{B}_0^+ \times \mathbb{B}_1^+)$. USUAL PRODUCT FORCING.

<u>THEOREM</u> IF $U \subseteq \mathbb{B}_0 \times \mathbb{B}_1$ IS AN ULTRAFILTER, THEN PROJECT TO $U_0 \subseteq \mathbb{B}_0$, $U_1 \subseteq \mathbb{B}_1$

GET:

$$\begin{tikzcd}
 & & V_1[G_1] \arrow[dr, "k_1"] & & \\
 & V_1 \arrow[ur, phantom, "\subseteq"] \arrow[dr, "k_1"] & & \bar{V}[\bar{G}_1] \arrow[dr, phantom, "\subseteq"] & \\
V \arrow[ur, "j_{U_1}"] \arrow[rr, "j_U"] \arrow[dr, "j_{U_0}"'] & & \bar{V} \arrow[ur, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq"] & & \bar{V}[\bar{G}_0 \times \bar{G}_1] \\
 & V_0 \arrow[ur, "k_0 \restriction V_0"] \arrow[dr, phantom, "\subseteq"] & & \bar{V}[\bar{G}_0] \arrow[ur, phantom, "\subseteq"] & \\
 & & V_0[G_0] \arrow[ur, "k_0"] & &
\end{tikzcd}$$

$$V_0[G_0] = V^{\mathbb{B}_0}/U_0$$

$$V_1[G_1] = V^{\mathbb{B}_1}/U_1$$

$$\bar{V}[\bar{G}_0 \times \bar{G}_1] = V^{\mathbb{B}_0 \times \mathbb{B}_1}/U$$

<u>WARNING</u> IN GENERAL, U IS NOT DETERMINED BY U_0 & U_1.

WITHOUT GENERICITY, $U_0 \times U_1$ MAY NOT GENERATE AN ULTRAFILTER IN $\mathbb{B}_0 \times \mathbb{B}_1$.

(EVEN IN POWER SETS, ONE DOESN'T ASK FOR MEASURING SUBSETS OF PLANE WITH <u>RECTANGLES</u>. BUT ALSO, IN CBA CONTEXT, THE SLICE PRODUCT $U_0 \otimes U_1$ ~~#~~ MAY NOT GENERATE AN ULTRAFILTER.)

$b \sim_I c$ IFF $b \triangle c \in I$.

• IDEALS

$I \subseteq \mathbb{B}$ IDEAL. CAN FORM QUOTIENT $\mathbb{B}/I = \{[b]_I \mid b \in \mathbb{B}\}$.

IF $G \subseteq \mathbb{B}/I$ ULTRAFILTER, THEN $\bigcup G \subseteq \mathbb{B}$ IS AN ULTRAFILTER ON $\mathbb{B}$.

DEFINITION SUPPOSE $I \subseteq \mathbb{B}$ IDEAL & $\mathbb{B}/I$ IS A COMPLETE BOOLEAN ALGEBRA.

① I IS PRECIPITOUS IF WHENEVER $G \subseteq \mathbb{B}/I$ IS V-GENERIC, THEN BOOLEAN ULTRAPOWER BY $\bigcup G \subseteq \mathbb{B}$ IS WELL-FOUNDED.

② I IS DISJOINTIFYING IF EVERY MAXIMAL ANTICHAIN IN $\mathbb{B}/I$ HAS A CHOICE OF REPRESENTATIVES FORMING AN ANTICHAIN IN $\mathbb{B}$.

THESE CONCEPTS DIRECTLY GENERALIZE THE POWERSET CONTEXT.

THEOREM IF $U_0 \subseteq \mathbb{B}/I$ & I DISJOINTIFYING THEN IF $U = \bigcup U_0 \subseteq \mathbb{B}$

$$\begin{tikzcd} V \arrow[r, "j_U"] \arrow[d, "j_{U_0}"'] & \check{V}_U \\ \check{V}_{U_0} \arrow[ur, "k"'] & \end{tikzcd}$$

THEOREM IF I IS COUNTABLY COMPLETE IDEAL ON $\mathbb{B}$ & DISJOINTIFYING & $\mathbb{B}$ IS COUNTABLY DISTRIBUTIVE, THEN I IS PRECIPITOUS.

DEFINITION SUPPOSE I J IDEALS IN $\mathbb{B}$ WITH $J \subseteq I \subseteq \mathbb{B}$, FOR WHICH $\mathbb{B}/J$, $\mathbb{B}/I$ QUOTIENTS ARE COMPLETE.

I IS PRECIPITOUS OVER J IFF WHENEVER $G \subseteq \mathbb{B}/I$ IS V-GENERIC, THEN BOOL ULTRAPOWER BY $U = \{[b]_J \mid [b]_I \in G\} \subseteq \mathbb{B}/J$ IS WELL-FOUNDED.

NOTE: I IS PRECIPITOUS IFF I IS PRECIPITOUS OVER $\{0\}$.

EVERY I IS PRECIPITOUS OVER ITSELF.

SO: BEING PRECIPITOUS OVER A STRICTLY INTERMEDIATE IDEAL IS AN INTERMEDIATE NOTION OF PRECIPITOUSNESS.

MANY OPEN QUESTIONS CONCERNING THIS CONCEPT.

• A FEW APPLICATIONS

~~THEOREM~~

DEF $\mathbb{B}$ IS $<\kappa$-FRIENDLY IFF $\forall \gamma<\kappa\ \exists b\in\mathbb{B}\ b\neq 0$, $\mathbb{B}\restriction b$ ADDS NO SUBSETS TO γ

THEOREM IF κ IS STRONGLY COMPACT & $\mathbb{B}$ IS $<\kappa$-FRIENDLY, THEN $\exists U\subseteq\mathbb{B}$ WHOSE BOOLEAN ULTRAPOWER IS WELL FOUNDED.
IN PARTICULAR, $\exists$ TRANSITIVE INNER MODEL $W \models ZFC+\varphi$ FOR ANY STATEMENT φ FORCED BY $\mathbb{B}$.

PROOF USES THE MENAS LIFTING TECHNIQUE FOR STRONGLY COMPACT κ.

COROLLARY IF THERE IS A SUPERCOMPACT CARDINAL, THEN

① THERE IS AN TRANSITIVE INNER MODEL W OF $ZFC + \exists$ S.C. κ + $2^\kappa=\kappa^+$.

② - - - - - - - - - - - - - - - - + $2^\kappa=\kappa^{++}$
κ^{+++}
ETC.
ANY DESIRED PATTERN ABOVE κ.

③ THERE IS AN TRANSITIVE INNER MODEL $W\models ZFC$ + κ IS S.C. + LAVER INDESTRUCTIBLE. ETC.

④ THERE IS TRANS $W\models\kappa$ IS STR.C. + LEAST MEASURABLE.

CAN ALSO GET: $\forall\theta\ \exists W$ AS ABOVE WITH $W^\theta\subseteq W$.

PROOF: THE FORCING TO ACHIEVE IT IS $<\kappa$-FRIENDLY. $\boxtimes$

THEME: OBTAINING LC PROPERTIES USUALLY OBTAINED BY FORCING IN AN INNER MODEL. - JOINT WORK WITH A. APTER + V. GITMAN. + H.

THM ① IF κ IS STR.C. THEN $\exists W\models ZFC + \exists$ STR.C. + $H_{\kappa^+}\subseteq HOD^W$.

~~ARGH~~

② IF κ IS SC THEN $\forall A\in H_{\kappa^+}\ \exists W$ TRANS. $W\models\exists$ S.C. CARD AND $A\in HOD^W$, $W^\theta\subseteq W$ & A IS DEFINABLE IN W WITHOUT PARAMETERS.

LASTLY: IF κ IS MEASURABLE CARDINAL μ NORMAL MEASURE ON κ.

$j: V\to M_1\to M_2\to\cdots\to M_\omega$ ITERATED ULTRAPOWER.

FACT (H + FUCHS) $j_\omega: V\to M_\omega$ IS A BOOL ULTRAPOWER MAP BY AN ULTRAFILTER U ON $\mathbb{B}=RO(\text{PRIKRY})$. G = CRIT.SEQ., WHICH IS M_ω-GEN.