Math Script

Matthias Lesch

MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY
E-mail address: `ml@matthiaslesch.de`, `lesch@math.uni-bonn.de`
URL: `www.matthiaslesch.de`, `www.math.uni-bonn.de/people/lesch`

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ABSTRACT. Some notes of my lectures given at the Universität Bonn.

Disclaimer: This is work in progress. It is provided as is without any warranty neither explicit nor implied.

Student help with typing/making corrections is very welcome.

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Part 1

Random Notes

ABSTRACT. This is a drawer for dropping random math notes. You could also call it a math blog....

Part 2

Globale Analysis I (V3B3)

Abstract. Vorlesung gehalten im SS 2004 an der Universität zu Köln und im WS 2005/2006, SS 2009, WS 2013/2014 and der Universität Bonn. Handschriftliches Skript wird auf der Homepage http://www.math.uni-bonn.de/ag/lesch/lesch.htmpl publiziert.

Global Analysis I covers the vector and tensor analysis on manifolds. Our main source of inspiration is the book [**AMR88**].

KAPITEL 1

Differentiable manifolds

Status: Pre-Alpha; it is partly typed by a TA, one round of corrections up to Section 3

1. From submanifolds to differentiable manifolds

We recall from Analysis II/III the notion of a differentiable submanifold of $\mathbb{R}^n$:

$M \subset \mathbb{R}^n$ is called a differentiable submanifold if it is locally of the form $M \cap U = \{F = 0\}$, where $F : U \longrightarrow \mathbb{R}^{n-k}$ is a submersion.

Our first goal is to define manifolds independently of an embedding into an ambient $\mathbb{R}^n$.

1.1. Benötigte Begriffe der mengentheoretischen Topologie.

Definition 1.1. 1. *Ein topologischer Raum X heißt* separiert *(synonym* hausdorffsch*), falls für zu je zwei verschiedenen Punkten $x, y \in X, x \neq y$ disjunkte offene Umgebungen $U_x \ni x, U_y \ni y$ existieren (d. h. $U_x \cap U_y = \emptyset$.* TODO picture

2. *X heißt* parakompakt, *falls jede offene Überdeckung $(U_i)_{i\in I}$ von X eine* lokal-endliche Verfeinerung *$(V_j)_{j\in J}$ besitzt.*

$(V_j)_{j\in J}$ ist eine Verfeinerung *von U_i, falls*

$$\forall j \in J\ \exists i \in I :\ V_j \subset U_i.$$

$(V_j)_{j\in J}$ ist lokal-endlich, *falls*

$$\forall x \in X\ \exists U \textit{ offen} : x \in U \textit{ und} \left\{ j \in \,\middle|\, U \cap U_j \neq \emptyset \right\} \textit{ endlich.}$$

Es ist schwer, nicht parakompakte Räume zu finden. Für uns genügt es zu wissen, dass folgende Räume stets parakompakt sind: Verweis zu ... unten

(i) kompakte Räume
(ii) lokal kompakte Hausdorffräume mit abzählbarer Basis der Topologie.

{defatlas}

Definition 1.2. *Sei X ein separierter, parakompakter topologischer Raum.*

1. *Eine* Karte *ist ein Paar (U, φ), wobei $U \subset X$ offen und $\varphi : U \longrightarrow V \subset \mathbb{R}^n$ ein Homöomorphismus auf eine offene Teilmenge $V \in \mathbb{R}^n$ ist.*

2. *$\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ heißt* Atlas, *falls*

$$X = \bigcup_{\alpha \in I} U_\alpha.$$

3. *Der Atlas $\mathcal{A}$ heißt C^∞ (C^k, analytisch, stetig, algebraisch, linear,...), falls*

$$\forall \alpha, \beta \in I: \ \varphi_\alpha \circ \varphi_\beta^{-1} C^\infty \ (\dots).$$

Dabei ist $\varphi_\alpha \circ \varphi_\beta^{-1}$ eine Abbildung von $\varphi_\beta(U_\alpha \cap U_\beta)$ auf $\varphi_\alpha(U_\alpha \cap U_\beta)$.

4. *Eine Karte (U, φ) ist $\mathbb{C}^\infty$ $(\dots)$–verträglich mit $\mathcal{A}$, falls* {indefatlas4}

$$\forall \alpha \in I: \ \varphi_\alpha \circ \varphi^{-1}$$

C^∞ $(\dots)$ ist.

$\mathcal{A}_{max} = \big\{(U, \varphi) \ \big| \ (U, \varphi)$ verträglich mit $\mathcal{A}\big\}$ ist der zugehörige maximale *Atlas.*

5. *Eine parakompakte C^∞ $(C^k, \dots)$ Mannigfaltigkeit ist ein Paar $(M, \mathcal{A})$ bestehend aus einem separierten parakompakten topologischen Raum M und einem maximalen C^∞ $(C^k, \dots)$ Atlas $\mathcal{A}$.*

Remark 1.3. 1. Wegen 1.2.4 genügt es, irgendeinen (möglichst kleinen) Atlas anzugeben.

2. Nach Whitney enthält jede C^k–Struktur, $k \geq 1$, bereits eine C^∞–Struktur. Deshalb betrachten wir nur glatte (C^∞) Mannigfaltigkeiten.

Der Fall C^0 ist komplizierter, aus differentialtopologischer Sicht jedoch nicht interessant.

3. Es gibt C^0–Mannigfaltigkeiten, die keine differenzierbare Struktur besitzen, z. B. in Dimension 4.

4. Eine C^0–Mannigfaltigkeit kann verschiedene nicht-äquivalente C^∞–Strukturen besitzen, d. h. es existieren C^∞–Mannigfaltigkeiten, die zwar homöomorph, nicht jedoch diffeomorph sind.

Z. B. besagt ein berühmtes Ergebnis von Milnor, dass es auf der 7–Sphäre S^7 genau 28 verschiedene C^∞–Strukturen gibt.

5. Die Bedingung der Parakompaktheit garantiert die Existenz von Zerlegungen der Eins (siehe Kapitel 2). chapter name ??

6. Die Bedingung der Separiertheit schließt pathologische Fälle aus:

BILDCHEN?!

Example 1.4. Auf der eindimensionalen Mannigfaltigkeit $\mathbb{R}$ (!) betrachten wir die Äquivalenzrelation

$$x \sim y \iff |x| = |y| > 1,$$

und bilden den Quotientenraum $X = \mathbb{R}/\sim$ versehen mit der Quotiententopologie. Äquivalenzklassen bezeichnen wir mit $[\cdot]$.

Wir definieren die Karten $U_1 = (-\infty, 1)/\sim$, $\varphi_1([x]) = x$, sowie $U_2 = (-1, \infty)/\sim$, $\varphi_2([x]) = x$, wobei $x > 0$ falls $|x| \geq 1$. The $\varphi_j, j = 1, 2$ are homeomorphisms. Es ist $X \setminus U_1 = \{[1]\}$, $X \setminus U_2 = \{[-1]\}$. Thus $X = U_1 \cup U_2$. Furthermore, $U_1 \cap U_2 = X \setminus \{[\pm 1]\}$ and

$$\varphi_2 \circ \varphi_1^{-1} : (-\infty, 1) \setminus \{-1\} \longrightarrow (-1, \infty) \setminus \{1\}, \quad x \mapsto -x,$$

showing that $\big\{(U_j, \varphi_j) \ \big| \ j = 1, 2\big\}$ is a smooth atlas. However, X is *not* separated since the two points $\{[\pm 1]\}$ do not have disjoint open neighborhoods.

D.h. $\{\{(U_j,\varphi_j)\} \mid j=1,2\}$ ist ein C^∞–Atlas der *nicht-separierten* Mannigfaltigkeit X.

Some authors deal with non-separated manifolds. We do not want to cope with such pathologies and therefore in this text X is not considered to be a manifold.

1.5. Examples of manifolds

1. $M=R^n, \mathcal{A}=(\mathbb{R}^n,\mathrm{id})$. Thus we have the reassuring fact that the Euclidean standard space with the atlas consisting only of the chart id is a smooth manifold.

2. Every C^∞–Mannigfaltigkeit(i. S. v. Analysis II) ist mit dem Atlas $\{(\varphi(U),\varphi^{-1}) \mid \varphi : U \longrightarrow M, \text{ Parametrisierung}\}$ eine C^∞–Mannigfaltigkeitmit abzählbarer Basis der Topologie, insbesondere parakompakt.

3. Projektive Räume. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and denote by $\mathbb{KP}^n$ the set of all lines in $\mathbb{K}^{n+1}$, that is $\mathbb{KP}^n = K^{n+1}/\sim$ is the quotient with respect to the equivalence relation

$$x \sim y \;:\Longleftrightarrow\; \exists \lambda \in \mathbb{K}:\; x=\lambda y.$$

Furthermore, since every line contains a point on the sphere we have

$$\mathbb{RP}^n = S^n/(x\sim -x) = S^n/(\mathbb{Z}/2\mathbb{Z}), \quad \mathbb{CP}^n = S^{2n+1}/\sim = S^{2n+1}/S^1.$$

This shows that the sphere is a two-fold covering of the real projective space and that the complex projective space is the homogeneous space obtained from the natural S^1 action on odd spheres which sit inside $\mathbb{C}^{n+1}$.

Points in $\mathbb{KP}^n$ are usually written in the form $[x]=[x_0,\dots,x_n]$ with $(x_0,\dots,x_n)\in\mathbb{K}^{n+1}\setminus\{0\}$. These are called *homogeneous coordinates* since they are unique only up to multiplication by a nonzero complex number.

We define a smooth atlas for $\mathbb{KP}^n$ as follows:

$$U_j := \{[x_0:\dots:x_n]\in\mathbb{KP}^n \mid x_j\neq 0\}$$
$$\varphi_j : U_j \longrightarrow \mathbb{K}^n, \quad [x_0:\cdots:x_n]\mapsto (\frac{x_i}{x_j})_{i=0,\dots,n;\; i\neq j}.$$

φ_j ist sicher ein Homöomorphismus. We compute the change of coordinates, w. l. o. g. le $0\le i<j\le n$. Then

$$\varphi_i(U_i\cap U_j)=\{y\in\mathbb{K}^n \mid y_j\neq 0\},$$

and

$$\begin{aligned}\varphi_j\circ\varphi_i^{-1}&(y_1,\dots,y_n)\\ &=\varphi_j([y_1:\cdots:y_{i-1}:1:y_i:\cdots:y_j:\cdots:y_n])\\ &=(\frac{y_1}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_n}{y_j}).\end{aligned}$$

Thus $\{(U_j,\varphi_j) \mid 0\le j\le n\}$ is even a rational (algebraic) atlas of $\mathbb{KP}^n$.

4. Abstrakte Tori $\omega_1,\dots,\omega_n\in\mathbb{R}^n$ eine $\mathbb{R}$–Basis. Ü

$$\Gamma := \mathbb{Z}\omega_1+\cdots+\mathbb{Z}\omega_n \qquad \text{“Gitter”}$$

$\mathbb{R}^n/\Gamma$ Quotientenraum.
Übungsaufgabe: Atlas $\approx T^n$.

5. $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Topologically this is the Alexandroff(=one-point) compactification of $\mathbb{R}^n$. We define an atlas consisting of two charts: $U_\infty = \widehat{\mathbb{R}^n} \setminus \{\infty\} = \mathbb{R}^n$, $\varphi_\infty(x) = x$, $U_0 = \widehat{\mathbb{R}^n} \setminus \{0\}$, $\varphi_0 = \begin{cases} \frac{x}{|x|^n}, & x \neq \infty \\ 0, & x = \infty \end{cases}$.

We have $U_0 \cap U_\infty = \mathbb{R}^n\{0\}$ and $\varphi_0 \circ \varphi_\infty^{-1}(x) = \frac{x}{|x|^2}$, which is certainly smooth.

Exercise 1.6. Show that $\widehat{\mathbb{R}^n} \cong S^n$ (diffeomorphism).

Im Folgenden Bezeichnen wir (C^∞)–Mannigfaltigkeiten mit M. Der Atlas gehört stillschweigend dazu.

Definition 1.7. *$f : M \longrightarrow N$ heißt* differenzierbar, *zu jedem $p \in M$ Karten $(U_p, \varphi_p), (U_q, \varphi_q)$ [1], $q = f(p)$, existieren, sodass $\varphi_q \circ f \circ \varphi_p^{-1}$ in einer Umgebung von $\varphi_p(p)$ differenzierbar ist.*

Ist die Bedingung erfüllt, so ist für beliebige Karten (U, φ) von M und (V, ψ) von N die Verknüpfung $\psi \circ \phi^{-1}$ differenzierbar auf

$$(f \circ \varphi^{-1})^{-1}(V) = \{x \in \mathbb{R}^n \mid f(\varphi^{-1}(x)) \in V\} = \varphi(f^{-1}(V) \cap U).$$

1.8. Ergänzungen

1. Die Verknüpfung differenzierbarer Abbildungen ist differenzierbar.

2. $f \in C^\infty(M,N)$ heißt *Diffeomorphismus*, falls f bijektiv ist *und* $f^{-1} \in C^\infty(N,M)$. Nach dem Umkersatz ist dies äquivalent dazu, dass f bijektiv *und* für alle $p \in M$ invertierbar ist. Wichtige Beispiele für Diffeomorphismen sind Karten. verschieben

2. Teilung der Eins

{teilung1}

Wir diskutieren hier nochmal ausführlicher die Konsequenzen der Parakompaktheitsbedingung in der Definition einer Mannigfaltigkeit.

move upwards
{man-satz3.1}

Satz 2.1. *Sei $(M, \mathcal{A})$ eine C^∞–Mannigfaltigkeit mit abzählbarer Basis der Topologie, d. h.*

- *M ist ein Hausdorff-Raum mit abzählbarer Basis,*
- *$\mathcal{A}$ ist ein maximaler C^∞–Atlas.*

Dann ist M parakompakt.

In more detail, ist $(U_\alpha)_{\alpha\in\mathcal{A}}$ eine offene Überdeckung, so existiert eine abzählbare *lokal endliche Verfeinerung $(V_\beta)_{\beta\in\mathcal{B}}$. Weiterhin kann V_β so gewählt werden, dass Karten*

$$\psi_\beta : V_\beta \longrightarrow B(0,3) = \{c \in \mathbb{R}^n \mid |x| < 3\}$$

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[1] Diese Notation beinhaltet $p \in \varphi_p$.

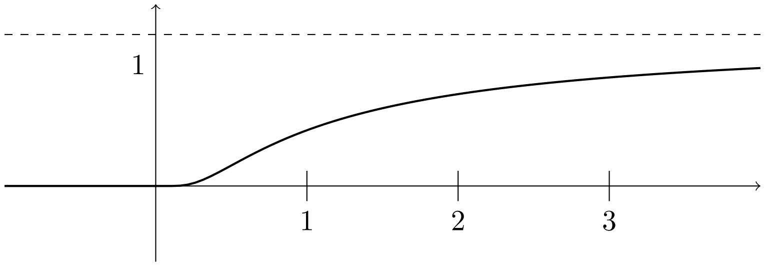

Abbildung 1. The function $f_1(x) = e^{-1/x}$ for $x \geq 0$ and 0 for $x \leq 0$.

existieren mit

$$M = \bigcup_{\beta \in \mathcal{B}} \psi_\beta^{-1}(\mathcal{B}(0,1)).$$

Beweis. M ist lokalkompakt mit abzählbarer Basis. Also existieren Kompakta $\kappa_1, \subset \mathring{\kappa}_2 \subset \kappa_2 \subset \mathring{\kappa}_3 \subset \dots$ mit

$$M = \bigcup_{j=1}^{\infty} \kappa_j.$$

$\kappa_{j+1} \smallsetminus \mathring{\kappa}_j$ ist kompakt, also existiert zu $p \in \kappa_{j+1} \smallsetminus \mathring{\kappa}_j$ eine Karte (V_p, ϕ_p) mit $\phi_p(V_p) = \mathcal{B}(0,3)$, $V_p \subset \mathring{\kappa}_{j+2} \smallsetminus \kappa_{j-1}$ und $V_p \subset U_\alpha$ für ein α.

Aus Kompaktheitsgründen gibt es $p_{j1}, \dots, p_j r_j$, sodass

$$\kappa_{j+1} \smallsetminus \mathring{\kappa}_j \subset \bigcup_{l=1}^{r_j} \phi_{jl}^{-1}(\mathcal{B}(0,1)).$$

Wegen $V_p \subset \mathring{\kappa}_{j+2} \smallsetminus \kappa_{j-1}$ ist $(V_{jl}, \phi_{jl})_{j=1,2,\dots;1 \leq l \leq v_j}$ eine abzählbare lokal-endliche Verfeinerung von $(U_\alpha)_{\alpha \in \mathcal{A}}$. ☐

Von jetzt an seien alle Mannigfaltigkeiten stillschweigend mit abzählbarer Basis vorausgesetzt

Zur Teilung der Eins verschaffen wir uns zunächst einige Hilfsfunktionen.

$$f_1(t) := \begin{cases} e^{-\frac{1}{t}}, & t > 0, \\ 0, & t \leq 0, \end{cases}.$$

One learns in Analysis I that f_1 is indeed smooth.

$$f_2(t) := \frac{f_1(t)}{f_1(t) + f_1(1-t)}, \quad f_3(t) := f_2(2+t) f_2(2-t).$$

Finally, $f_4(x) := f_3(|x|), x \in \mathbb{R}^n$.

{man-satz3.2}

Proposition 2.2 (Partition of Unity). *Sei M eine C^∞–Mannigfaltigkeit und $(U_\alpha)_\alpha$ eine offene Überdeckung. Dann existiert $\varphi_n \in C^\infty(M), n \in \mathbb{N}$, mit*

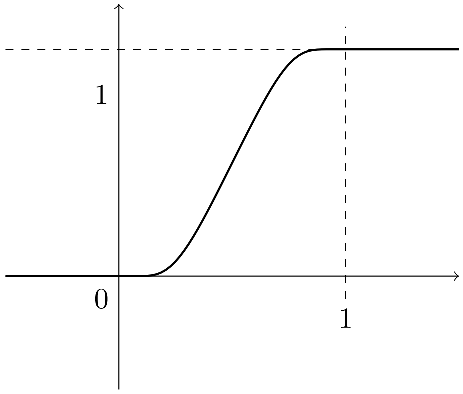

ABBILDUNG 2. The function $f_2(x) = \frac{f_1(x)}{f_1(x)+f_1(1-x)}$.

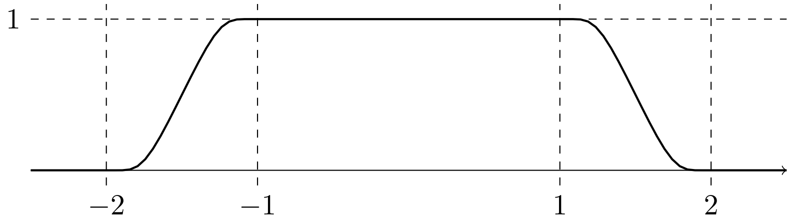

ABBILDUNG 3. The function $f_3(x) = f_2(2+x) \cdot f_2(2-x)$.

(1) $0 \leq \varphi_n \leq 1$,
(2) $(\operatorname{supp} \varphi_n)_n$ *ist lokal-endlich,*
(3) $\forall n \; \exists \alpha \in \mathcal{A} \qquad \operatorname{supp}(\varphi_n) \subset U_\alpha$, `{tlg1satzteil}`
(4) $\sum_{n=1}^{\infty} \phi_n(p) = 1$ *for all* $p \in M$.

Zusatz: *Ist* $\mathcal{A} \in \mathbb{N}$, *so kann man entsprechend* $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ *finden mit* $\operatorname{supp}(\varphi_\alpha) \subset U_\alpha$. *Wegen (3) heißt* (φ_n) *der Überdeckung subordinierte Zerlegung der Eins.*

BEWEIS. Seien $V_n, \psi_n : V_n \longrightarrow \mathcal{B}(0,3)$ wie in Satz 2.1. Setze

$$\widetilde{\phi}(x) := \begin{cases} f_4(\psi_n(x)), & x \in V_n, \\ 0, & x \notin V_n, \end{cases}$$

sowie

$$\widetilde{\phi} := \sum_{n=1}^{\infty} \widetilde{\phi_n} \in C^\infty(M),$$

$$\phi_n := \frac{\widetilde{\phi_n}}{\widetilde{\phi}}.$$

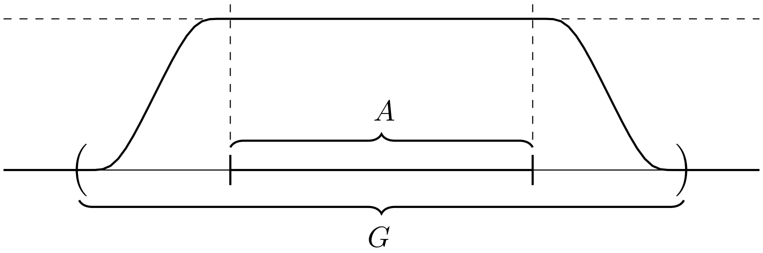

fig:CutOffClosed

ABBILDUNG 4. Given a closed set $A \subset M$ and an open neighborhood $G \supset A$ there is a smooth function which is identically 1 on A and identically 0 outside G.

Beachte, dass $\forall p \in M\ \widetilde{\phi}(p) > 0$.
Ist $\mathcal{A} \subset \mathbb{N}$, so setze induktiv

$$J_0 = \Phi, \qquad \phi_0 = 0,$$

$$J_k = \left\{ i \in \mathcal{A} \smallsetminus J_{k-1} \;\middle|\; \operatorname{supp} \phi_i \subset U_k \right\},$$

$$\phi_k = \sum_{i \in J_k} \frac{\widetilde{\phi}_i}{\widetilde{\phi}}.$$

□

{man-satz3.3}

Satz 2.3. *Sei $A \subset M$ abgeschlossen und $G \subset M$ offen mit $A \subset G$. Dann existiert $f \in C^\infty(M)$ mit $0 \le f \le 1$ und*

$$f(p) = \begin{cases} 1, & p \in A, \\ 0 & p \notin G. \end{cases}$$

See Figure 4.

BEWEIS. $\{M \smallsetminus A,\ G\}$ ist offene Überdeckung. Sei $\{\phi_0, \phi_1\}$ subordinierte Zerlegung der 1. $f = \phi_1$ leistet das Gewünschte. □

3. Der Tangentialraum

3.1. Motivation. Sei $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ eine Submersion und $M = \{F = 0\}$. Der Tangentialraum an M im Punkt p ist gegeben durch

$$\begin{aligned} T_pM &= \left\{ \gamma'(0) \in \mathbb{R}^n \;\middle|\; \gamma : I \longrightarrow M \subset \mathbb{R}^n \text{ Kurve}, \gamma(0) = p \right\} \\ &= \operatorname{grad} F(p)^\perp \end{aligned}$$

Diese Definition benutzt die Einbettung $M \subset \mathbb{R}^n$ und ist daher so nicht auf Mannigfaltigkeiten übertragbar.

Jedes $v \in T_pM$ definiert eine *Richtungsableitung* $v : C^\infty(M) \longrightarrow \mathbb{R}$ wie folgt: ist $\gamma : I \longrightarrow M$ eine Kurve mit $\gamma(0) = p$, $\gamma'(0) = v$, so setzt man

$$vf := \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)).$$

vf ist unabhängig von der Wahl von γ. Des Weiteren ist v eine *Derivation*, d.h.

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g).$$

`{man-def4.1}`

Definition 3.1. *Sei M eine C^∞–Mannigfaltigkeit.*

$$X_p : C^\infty(M) \longrightarrow \mathbb{R}$$

heißt Derivation bei p, *falls*

- X_p *linear*
- $X_{\text{ß}(f\cdot g)=f(p)X_pg+g(p)X_pf.}$

$T_pM = \{$*Derivationen bei p*$\}$ *heißt* Tangentialraum *an M in p.*
T_pM *ist in der Tat ein Vektorraum.*

`{man-4.2}`

3.2. Einfache Eigenschaften.

`{man-bem4.2.1}`

(1) Sei $\phi \in C^\infty(M)$ konstant in der Nähe von p. Dann gilt $X_p\phi = 0$.
Beweis: O.B.d.A. sei $\phi \equiv 1$ nahe p. Wähle $\Psi \in C^\infty(M)$ mit

- $\Psi \equiv 1$ nahe p
- $\operatorname{supp}\Psi \subset \{q \mid \phi(q) = 1\}$.

Es gilt dann $\Psi\phi = \Psi$ und folglich

$$X(\Psi) = X(\Psi\phi) = X(\Psi) + X(gf),$$

also $X\phi = 0$.

(2) Ist $X \in T_pM \cap T_qM$, so ist $X = 0$ oder $p = q$, d.h. $X \neq 0$ bestimmt p eindeutig.
Beweis: Übungsaufgabe/später.

(3) Sei (U, ϕ) eine Karte mit Koordinatenfunktionen $x_1, \ldots, x_n$. Dann ist

$$\frac{\partial}{\partial x_i}\bigg|_p f := \frac{\partial}{\partial r_i}\bigg|_{\phi(p)} f \circ \phi^{-1}(r) = D(f \circ \phi^{-1})(\phi(p))[e_i].$$

`{man-lemma4.3}`

Lemma 3.2. ***Korrektur?*** *Sei (U, ϕ) eine zentrierte Karte mit Koordinatenfunktionen $x_1, \ldots, x_n$. Dann existieren zu $f \in C^\infty(M)$ Funktionen $f_1, \ldots, f_n \in C^\infty(M)$ mit*

$$f|U = \sum_{j=1}^{n} f_j \cdot x_j + f(p).$$

Beweis. ? □

{man-satz4.4}

Satz 3.3. *Sei M eine C^∞–Mannigfaltigkeitund (U, ϕ) eine n–dimensionale Karte um p. Dann ist* $\dim T_pM = n$.

$$\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$$

{man-satz4.4Gl1}

Beweis. **KORREKTUR?** Original verwendet Bemerkung 1 und Lemma 3.2 ☐

Beispiel 3.4. *Beispiele:*

(1) $\mathbb{R}^n$. *In jedem* $p \in \mathbb{R}^n$ *sind die üblichen partiellen Ableitungen* $\left.\frac{\partial}{\partial x_j}\right|_p$ *eine Basis von* $Tp\mathbb{R}^n$.

(2) *Polarkoordinaten in* $\mathbb{R}^n$.

$$\Phi : (0, \infty) \times (0, 2\pi) \times (0, \pi) \longrightarrow \mathbb{R}^3,$$

$$(r, phi, \theta) \mapsto (r \cos\phi \sin\theta, r \sin\phi \sin\theta, r \cos\theta).$$

Für $r = 1$ *ist* $\Phi(1, \dots, \dots)$ *gleichzeitig eine Parametrisierung von* S^2.
Die Kettenregel liefert

$$\left.\frac{\partial}{\partial r}\right|_p f = \frac{\partial}{\partial r} f(r \cos\phi \sin\theta, \dots, \dots)\big|_{r(p),\, \phi(o),\, \theta(p)}$$

$$= \cos\phi \sin\theta\big|_p \left.\frac{\partial}{\partial x}\right|_p f + \sin\phi \sin\theta\big|_p \left.\frac{\partial}{\partial y}\right|_p f + \cos\theta(p) \left.\frac{\partial}{\partial z}\right|_p f.$$

Übung: $\frac{\partial}{\partial v}, \frac{\partial}{\partial \phi}$

(3) *Allgemein sei* $\Phi = (y_1, \dots, y_n)$ *eine Karte einer offenen Teilmenge* U *des* $\mathbb{R}^n$.

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \frac{\partial}{\partial x_i}(f \circ \Phi^{-1} \circ \Phi)(p)$$

$$= \sum_{j=1}^n \partial_j (f \circ \Phi^{-1})(\Phi(p)) \; \frac{\partial \Phi_j}{\partial x_i}(p)$$

$$= \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p) \; \left.\frac{\partial}{\partial y_j}\right|_p f.$$

(4) *Übungsaufgabe:* TM *als Äquivalenzklassen von Kurven*

{man-def4.6}

Definition 3.5 (Tangentialabbildung). *Seien M und NC^∞–Mannigfaltigkeiten, $f \in C^\infty(M,N)$, $p \in M$.*

$$T_pf = D_pf : T_pM \longrightarrow T_{f(p)}N,$$

$$T_pf(X)\cdot\phi = X(\phi\circ f), \qquad \phi \in C^\infty(N)$$

heißt Tangentialabbildung oder Ableitung von f im Punkt p.

hier ist was FAUL:f**_oder einfach nur** f**??**

Bemerkung 3.6. *Bemerkungen:*

(1) *Man mache sich klar, dass für offene Teilmengen $M \subset \mathbb{R}^m, N \subset \mathbb{R}^n, T_pf$ in der Standardbasis durch die Jacobimatrix gegeben ist.*

(2) *Kettenregel: $T_p(f\circ g) = T_{g(p)}f \cdot T_pf$.*

(3) *Jede Karte (U,ϕ) liefert einen Isomorphismus*

$$T_p\phi : T_pM \longrightarrow T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Ist ψ eine weitere Karte, so folgt

$$T_p\psi = T_p(\psi\circ\phi^{-1}\circ\phi) = D_{\phi(p)}(\psi\circ\phi^{-1})T_p\phi.$$

3.3. Bem.: Physikerverständnis von Tangentialvektoren. Ein Tangentialvektor ist eine Familie $(\xi\phi)_\phi$, wobei $\xi \in \mathbb{R}^n$ und ϕ sämtliche Karten durchläuft, mit dem *Transformationsverhalten*

$$\xi^\psi = D_{\phi(p)}(\psi\circ\phi^{-1})[\xi^\phi].$$

Kurven: Sei $I \in \mathbb{R}$ ein Intervall. Eine *Kurve in M* ist eine C^∞–Abbildung

$$c : I \longrightarrow M.$$

$$\dot c(t_0) := T_{t_0}c\left(\frac{d}{dt}\bigg|_{t_0}\right)$$

heißt *Geschwindigkeitsvektor.*
Jedes $v \in T_pM$ ist Geschwindigkeitsvektor einer Kurve durch p:
dazu sei ϕ eine bei p zentrierte Karte, $v = \sum_{j=1}^n v_j \frac{\partial}{\partial x_j}\big|_p$ (siehe Satz 3.3),
und

$$c(t) := \phi^{-1}(tv_1,\dots,tv_n).$$

Für $f \in C^\infty(M)$ gilt

$$\begin{aligned}\dot c(0)f &= T_0c\left(\frac{d}{dt}\bigg|_0\right)f = \frac{d}{dt}\bigg|_{t=0} f(c(t))\\ &= \sum_{j=1}^n v_j\,\frac{\partial f\circ\phi^{-1}}{\partial x_j}(0) = vf,\end{aligned}$$

also $\dot{c}(0) = v$.
Weiterhin gilt für $h \in C^\infty(\mathbb{R}^m)$ nach Definition 3.5, dass

$$T_p\phi(v)h = v(h \circ \phi) = \sum_{j=1}^{n} v_j \frac{\partial\, h \circ \phi \circ \phi^{-1}}{\partial x_j},$$

d.h.

$$T_p\phi(v) = (v_1, \ldots, v_m)^t.$$

4. Lokale Eigenschaften differenzierbarer Abbildungen, Untermannigfaltigkeiten

Im Folgenden betrachten wir nur Mannigfaltigkeiten “reiner” Dimension, Schreibweise M^m. Das ist keine Einschränkung der Allgemeinheit. Jede Mannigfaltigkeit zerfällt in diese disjunkte Vereinigung von Zusammenhangskomponenten reiner Dimension.

Definition 4.1. *Seien $M^m, N^n C^\infty$–Mannigfaltigkeiten und $f : M \longrightarrow N$ eine C^∞ – Abbildung.*

(1) *$p \in M$ heißt* kritischer Punkt *von f, falls* $\operatorname{rang} D_p f < n = \dim N. f(p)$ *heißt dann* kritischer Wert.

(2) *$q \in M$ heißt* regulärer Wert *von f, falls*
$$\forall p \in f^{-1}(\{q\}) \ \operatorname{rang} D_p f = n.$$
q ist auch dann regulärer Wert, wenn $f^{-1}(\{q\}) = \emptyset$!

(3) *f heißt* Submersion, *falls*
$$\forall p \in M \ \operatorname{rang} D_p f = n.$$

(4) *f heißt* Immersion, *falls*
$$\forall p \in M \ \operatorname{rang} D_p f = m = \dim M.$$

(5) *f heißt* Subimmersion, *falls $M \ni p \mapsto \operatorname{rang} D_p f$ konstant ist.*

Alle lokalen Struktursätze über differenzierbare Abbildungen (Umkehrsatz, implizite Funktionen, lokale Trivialität von Submersionen, Rangsatz[2]) gelten verbatim für Abbildungen zwischen Mannigfaltigkeiten da sie über Karten auf den $\mathbb{R}^n$ – Fall zurückgeführt werden können. Wir werden diese Sätze deshalb hier nicht nochmals alle für Mannigfaltigkeiten formulieren.

---

[2] vgl. auch Kapitel 1

{man-def5.2}

Definition 4.2. *$N \subset M^m$ heißt (C^∞)–Untermannigfaltigkeit von M, falls für jede Karte (U, ϕ) von M das Bild*

$$\phi(N \cap U) \subset \mathbb{R}^m$$

eine Untermannigfaltigkeit des $\mathbb{R}^m$ ist. N ist dann selbst wieder eine Mannigfaltigkeit.
Selbstverständlich gelten die bekannten äquivalenten Charakterisierungen von Untermannigfaltigkeiten aus Analysis II auch hier.

{man-def5.3}

Definition 4.3. *$f \in C^\infty(M, N)$ heißt* Einbettung, *falls*

- *f eine injektive*

und

- *$f : M \longrightarrow f(M) \subset N$ ein Homöomorphismus ist.*

Wegen des Immersionssatzes ist $F(M) \subset N$ dann tatsächlich ein Untermannigfaltigkeit.

{man-satz5.4}

Satz 4.4. *Seien $M^m, N^n C^\infty$–Mannigfaltigkeiten und $F : M \longrightarrow N$ eine Subimmersion vom Rang k. Dann gilt*

(1) *Für $f \in N$ ist $f^{-1}(\{q\}) \subset M$ eine Untermannigfaltigkeit der Dimension $m - k$ oder $\emptyset$.*

(2) *Zu $p \in M, q = f(p)$ gibt es Umgebungen U von p, V von q so, dass*

$$S = f(U) \cap V$$

eine Untermannigfaltigkeit von N der Dimension k ist.
Insbesondere ist für einen regulären Wert q $f^{-1}(\{q\})$ eine Untermannigfaltigkeit der Dimension $m - n$ oder $\emptyset$.

Beweis. Sei $f(p) = q$. Nach dem Rangsatz[3] können wir um p, bzw. q zentrierte Karten $(U, (x_1, \dots, x_m)), (V, (y_1, \dots, y_n))$ so wählen, dass
DIAGRAMM

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0).$$

Dann ist

$$\phi(f^{-1}(\{q\}) \cap U) = \big\{(x_1 \dots, x_m) \in U' \;\big|\; x_1 = \dots = x_k = 0\big\}$$

$$= \{0\} \times \mathbb{R}^{m-k} \cap U'.$$

□

---

[3] vgl. **??**

5. Der Satz von Morse-Sard

Von jetzt an: Mannigfaltigkeiten haben abzählbare Basis der Topologie.

Wir untersuchen zunächst die Existenz regulärer Werte.

`{man-def6.1}`

Definition 5.1 (Nullmenge). *$A \subset M^m$ heißt* Nullmenge, *falls für jede Karte (U, ϕ)*

$$\phi(A \cap U) \in \mathbb{R}^m$$

eine λ^m–Nullmenge ist.

`{man-bem6.2}`

Bemerkung 5.2. *Bemerkungen:*

(1) *$\forall p \in M$ $\{p\}$ ist eine Nullmenge.*

`{man-bem6.2.2}`

(2) *Abzählbare Vereinigungen von Nullmengen sind Nullmengen.*

(3) *Ist $A \subset M$ Nullmenge, so ist $M \setminus A$ dicht in M. Dies gilt auch, wenn* $\dim M = 0$.

`{man-satz6.3}`

Satz 5.3 (Morse-Sard). *Sei $f : M^m \longrightarrow N^n, n \geq 1$, eine C^∞–Abbildung. Dann ist die Menge der kritischen Werte von f eine Nullmenge in M.*
Bemerkung: Es gilt ein entsprechender Satz für C^l – Abbildungen zwischen C^k–Mannigfaltigkeiten. k und l hängen dann jedoch für die Gültigkeit des Satzes von m, n ab.

Beweis. Sei $C_f := \{p \in M \mid p \text{kritischer Punkt von } f\}$. Wegen Bemerkung 2 und der Abzählbarkeit der Topologie genügt es zu zeigen, dass jedes $p \in M$ eine Umgebung $U \subset M$ besitzt, sodass

$$f(C_f \cap U) \subset N$$

eine Nullmenge ist. Daher dürfen wir uns ohne Einschränkung auf den Fall $M = U \subset \mathbb{R}^m$ offen, $N = \mathbb{R}^n$, beschränken.
Wir führen Induktion über m.
$m = 0$: Bild f ist endlich, also eine Nullmenge.
Die Behauptung sei nun für alle Dimensionen $m' < m$ bewiesen.
Setze für $1 \leq l < \infty$:

$$C_l := \{x \in U \mid \forall |\alpha| \leq l \ \frac{\partial^{|\alpha|}}{\partial x^\alpha} f|_x = 0\}$$

$$= \{c \in U \mid \text{alle Ableitungen von } f \text{ bis zur Ordnung } l \text{ verschwinden in } x\}.$$

(1) $f(C \setminus C_1)$ ist eine Nullmenge. Sei $\xi \in C \setminus C_1$. Damm ost für geeignete i, j

$$\frac{\partial f_j}{\partial x_i}(\xi) \neq 0.$$

Ohne Einschränkung sei $i = j = 1$. Setze $h : U \longrightarrow \mathbb{R}^m$, mit $h(x) = (f_1(x), x_2, \dots, x_m)$.

$$Dh(\xi) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\xi) \neq 0 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Nach dem Umkehrsatz[4] existieren Umgebungen V von ξ und V' von $h(\xi)$, sodass $h|_V : V \longrightarrow V'$ ein Diffeomorphismus ist.

$$g := f \circ h^{-1} : V' \longrightarrow \mathbb{R}^n,$$
$$g(t,x) = (t, \widetilde{g}(t,x)).$$

Die kritischen Werte von g sind genau die Elemente von $h(C_q \cap V)$.
Weiterhin ist wegen

$$Dg(t,x) = \begin{pmatrix} 1 & 0 \dots\ 0 \\ * & D\widetilde{g}(t,\cdot)\big|_x \end{pmatrix}$$

(t,x) genau dann kritisch für g, wenn x kritisch für $\widetilde{g}_t = \widetilde{g}(t,\cdot)$ ist. Insbesondere gilt: ist $n = 1$, so hat g keine kritischen Punkte.
Nach Fubini ist nun[5]

$$\lambda^n(f(C_f \cap V)) = \lambda^n(\{g(t,x) \mid (t,x) \in V'\})$$
$$= \int_I \lambda^{n-1}(\{x \in W' \mid x \text{ kritisch für } \widetilde{g}_t\})dt = 0,$$

da $\lambda^{n-1}(\{x \in W' \mid x \text{ kritisch für } \widetilde{g}_t\}) = 0$ nach Induktionsvoraussetzung.

(2) Sei $\xi \in C_l \setminus C_{l+1}$. Dann existieren $k_1 \dots, k_{l+1}$, sodass

$$\frac{\partial^{l+1} f}{\partial x_{k_1} \dots \partial x_{k_{l+1}}}(\xi) \neq 0.$$

Ohne Einschränkung sei $k_1 \neq 1$. Wir zuvor $h : U \to \mathbb{R}^m$ mit

$$h(x) = \left(\frac{\partial^l f}{\partial x_{k_2}, \dots, x_{k_{l+1}}}, x_2, \dots, x_m \right),$$

ebenso sieht man, dass h ein Diffeomorphismus einer Umgebung V von ξ auf eine offene Menge $I \times W \subset \mathbb{R}^m$ ist, wobei $I \times W = V'$.

$$h(C_l \cap U) \subset V' \cap (\{0\} \times \mathbb{R}^{m-1}).$$

---

[4] vgl. 1.2

[5] nach eventueller Verkleinerung sei $V' = I \times W'$ mit einem Intervall $I \subset \mathbb{R}$ und $W' \subset \mathbb{R}^{m-1}$ offen.

Setze $g := f \circ h^{-1} : V' \to \mathbb{R}^n$ und $\widetilde{g} := \big|_{V' \cap \{0\} \times \mathbb{R}^{m-1}} = g|W$.

$$\big\{x \in V' \;\big|\; x \text{ kritisch für } g\big\} = \{0\} \times \big\{\widetilde{x} \in W \;\big|\; \widetilde{x} \text{ kritisch für } \widetilde{g}\big\}.$$

$$\begin{aligned} f(C_l \cap U) &= g(\big\{x \in V' \;\big|\; x \text{ kritisch für } g\big\}) \\ &= \widetilde{g}(\{0\} \times \big\{\widetilde{x} \in W \;\big|\; \widetilde{x} \text{ kritisch für } g\big\}) = 0 \end{aligned}$$

nach Induktionsvoraussetzung.

(3) Sei $d > 0$ beliebig und W ein Würfel der Kantenlänge d. Die Taylorformel liefert für $x \in C_k \cap W, y \in W$

$$|f(x) - f(y)| \leq L|x - y|^{k+1},$$

L hängt von W, k und f ab.
Zerlege W in r^m Würfel W_j der Kantenlänge $\frac{\varepsilon}{r}$. Für $x \in C_k \cap W_j, y \in W_j$, ist

$$|x - y| \leq \sqrt{m}\frac{d}{r},$$

also

$$|f(x) - f(y)| \leq L\left(\frac{\sqrt{m}d}{r}\right)^{k+1},$$

d.h. $f(C_k \cap W_j)$ liegt in einem Würfel der Kantenlänge $2L\left(\frac{\sqrt{m}d}{r}\right)^{k+1}$.
Folglich

$$\lambda^m(f(C_k \cap W)) \leq 2^m \lambda^m(f(C_k \cap W_j))$$

(wobei wir hier den W_j mit dem größten Wert nehmen)

$$\leq 2^m \left\{2L\left(\frac{\sqrt{m}d}{r}\right)^{k+1}\right\}^n$$

$$= \text{const.} \cdot r^{m-n(k+1)} \longrightarrow 0,$$

falls $k \geq \frac{m}{n}$ und $r \to \infty$.
Folglich ist $\lambda^m(f(C_k \cap W)) = 0$ falls $k \geq \frac{m}{n}$.

(1) – (3) zusammen ergeben die Behauptung.

□

6. Anwendungen des Satzes von Sard

6.1. Einbettung.

Satz 6.1 (H. Whitney). *Sei M^m eine Mannigfaltigkeit. Dann gibt es eine Einbettung*

$$M^m \longrightarrow \mathbb{R}^{2m+1}.$$

Beweis. Wir behandeln nur den Fall, dass M kompakt ist.

(1) Es gibt eine Einbettung in $\mathbb{R}^N$ mit N genügend groß. Wähle Karten $(U_j, \phi_j)_{j=1,\ldots,r}$, mit

$$\operatorname{Bild}\phi_j \supset \mathcal{B}(0,3) = \{|x| < 3\}$$

und

$$M = \bigcup_{j=1}^{r} \phi^{-1}(\mathcal{B}(0,3));$$

dies ist möglich, da M kompakt.
Wähle nun $g \in C^\infty(\mathbb{R}^m)$ mit

$$g(x) = \begin{cases} 1, & |x| \le 4/3, \\ 0, & |x| \ge 5/3, \end{cases}$$

und setze

$$F_j(p) = \begin{cases} g(\phi_j(p)) \cdot \phi_j(p), & p \in U_j, \\ 0, & \text{sonst.} \end{cases}$$

$f_j \in C^\infty(M, \mathbb{R}^m)$ und $f_j \restriction \phi_j^{-1}(\mathcal{B}(0,1))$ sind Diffeomorphismen. Folglich ist

$$(f_1, \ldots, f_r, g \circ \phi_1, \ldots, g \circ \phi_r) : M \longrightarrow \mathbb{R}^{m+1)r}$$

eine Einbettung.

(2) Sei nun also $M \subset \mathbb{R}^N$ kompakt. Für $\omega \in S^{N-1}$ sei π_ω die orthogonale Projektion auf $\mathbb{R}^{N-1} = \langle\omega\rangle^\perp$, $\pi_\omega(x) = x - \langle\omega, x\rangle\,\omega$
π_ω ist linear, also für $p, q \in M$

$$\pi_\omega p = \pi_\omega q \Leftrightarrow \pi_\omega(p-q) = 0 \Leftrightarrow p - q \mid\mid \omega.$$

Betrachte

$$\Phi : M \times M \setminus \big\{(p,q) \bigm| p \in M\big\} \longrightarrow S^{N-1}$$
$$(p,q) \mapsto (p-q)/|p-q|.$$

Falls $2m < N-1$, so ist aus Dimensionsgründen und dem Satz von Sard (Satz 5.3) ist

$$\big\{\omega \in S^{N-1} \bigm| \exists p, 1 \in M, p \ne q\ p - q \mid\mid \omega\big\}$$
$$= \operatorname{Bild}\Phi \cup -\operatorname{Bild}\Phi$$

eine Nullmenge.
π_ω ist eine Immersion, falls für $p \in M, v \in T_pM \setminus \{0\}$ stets $\pi_\omega(v) \ne 0$. Folglich ist π_ω genau dann eine Immersion, wenn

$$\forall p \in M\ \omega \notin T_pM \Leftrightarrow \omega \notin \operatorname{Bild}\sigma,$$

wobei $\sigma : TM \setminus \{0\} \to S^{N-1}$, wobei $v \mapsto v/|v|$.
Falls $2m < N-1$, ist, wiederum nach Sard, Bild σ eine Nullmenge.

$\square$

Resultat: Falls $2m+1<N$, so existiert ein ω so, dass π_ω eine injektive Immersion und somit, da M kompakt, eine Einbettung ist.

Übungen: Morse Funktionen:
Aufgabe aus Bröckner-Jänich zu Geraden, die Bild$f$ nur in endl. vielen Punkten schneiden.

6.2. Transversalität. Seien M^n, M^n Mannigfaltigkeiten, $S \subset N$ eine Untermannigfaltigkeitund $f : M \longrightarrow N$ differenzierbar. Das schließt den Fall $M \subset N$ mit ein, wobei f die Inklusionsabbildung ist.
f heißt *transversal* zu $S, f \pitchfork S$, falls für $p \in M$ mit $f(p) \in S$ gilt

$$\operatorname{Bild} D_p f + T_{f(p)} S = T_{f(p)} N. \tag{1}$$

{eqn-transversal}

Bildchen
Dann ist $f^{-1}(S)$ eine Untermannigfaltigkeit von M der Dimension $m-n+\dim S$.
Beweis: Übungsaufgabe. bzw.
Nach Wahl von Koordinaten: $N=\mathbb{R}^n, M=\mathbb{R}^n$.
$S=\{0\}\times\mathbb{R}^k$
$f^{-1}(S)=\big\{y\in\mathbb{R}^m \bigm| f_1(y)=\cdots=f_{n-k}(y)=0\big\}$
Wegen 1 ist $(f_1,\ldots,f_{n-k})$ eine Submersion und die Behauptung folgt. □
Ist $M \subset N$ und $f = jM$ die Einbettung $M \hookrightarrow N$, so ist $f^{-1}(S)=S\cap M$.
Der folgende Satz sagt, dass Transversalität eine generische Eigenschaft ist.

{man-satz7.2}

Satz 6.2. *Unter den obigen Voraussetzungen sei $N=\mathbb{R}^n$. Dann gibt es zu $\varepsilon>0$ ein $v\in\mathbb{R}^n$ mit $||v||<\varepsilon$, sodass $f+v \pitchfork S$.*
Bemerkung: Falls $m+\dim S<n$, so ist $f(M)+v\cap S=\emptyset$.

Beweis. Betrachte

$$F : M \times S \longrightarrow \mathbb{R}^n$$

$$(x,y) \mapsto y - f(x).$$

Nach Sard (5.3) existiert ein regulärer Wert $v\in\mathbb{R}^n$ von F mit $||v||<\varepsilon$ (denn $\lambda^n(\mathcal{B}(0,\varepsilon))>0$).
Falls $F^{-1}(\{v\})=\emptyset$, so ist $f(M)+v\cap S=\emptyset$.
Falls $F(x,y)=v$, so ist $f(x)+v=y\in S$.
Weiterhin ist $D_{(x,y)}F$ surjektiv (da v regulärer Wert), d.h. zu $w\in\mathbb{R}^n$ existiert

$$(\xi,\eta)\in T_{(x,y)}M\times S\cong T_xM\oplus T_yS$$

mit $-D_xf(\xi)+\eta=w$.
Das heißt aber , dass

$$\operatorname{Bild} D_xf+T_{f(x)}S=T_{f(x)}\mathbb{R}^n\cong\mathbb{R}^n,$$

also $f+v\pitchfork S$. □

7. Vektorfelder und dynamische Systeme

Bekanntlich hat jeder Tangentialvektor v bezüglich einer Karte ϕ eine Basisdarstellung

$$v|_p = \sum_{j=1}^{m} v_j \frac{\partial}{\partial x_j}\bigg|_p .$$

Definition 7.1. *Ein C^∞–Vektorfeld auf M ist eine Abbildung $X : M \longrightarrow TM$, sodass*

(1) $\forall p \, X(p) \in T_pM$

{man-dfn8.1.2}

(2) *Für jede Karte (U, ϕ) ist*

$$X|_U = \sum_{j=1}^{m} X_j^{\phi} \frac{\partial}{\partial x_j}$$

mit C^∞–Funktionen X_j^{ϕ}.
$\Gamma(TM) :=$ *{C^∞–Vektorfelder auf M}.*

Notiz 7.2. *Teil (2) ist äquivalent dazu, dass $T\phi \circ X : U \longrightarrow T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$ für jede Karte ϕ glatt ist.*

Beispiel 7.3.

$$X : S^1 \longrightarrow TS^1, (x, y) \mapsto (-y, x)$$

ist ein glattes Vektorfeld (Übungsaufgabe).

Definition 7.4 (Integralkurve). *Eine Kurve $c : I \longrightarrow M$ heißt* Integralkurve *des Vektorfelds X, wenn gilt*

$$c'(t) = X(c(t)). \qquad (2)$$ {man-Gl8.1}

Sei $c(t_0) = p$ und (U, ϕ) eine bei p zentrierte Karte. Dann ist

$$\begin{pmatrix} \phi_1 \circ c \\ \vdots \\ \phi_m \circ c \end{pmatrix}(t) = (\phi \circ c)'(t) = T_{c(t)}\phi(c'(t))$$

$$= T_{c(t)}\phi(X(c(t))) = \begin{pmatrix} X - 1^{\phi}(c(t)) \\ \vdots \\ X_m^{\phi}(c(t)) \end{pmatrix},$$

d.h. (2) mit $c(t_0) = p$ ist äquivalent zu dem autonomen Anfangswertproblem

$$(\phi \circ c)'(t) = \begin{pmatrix} X_1^{\phi} \circ \phi^{-1} \\ \vdots \\ X_m^{\phi} \circ \phi^{-1} \end{pmatrix}(\phi \circ c(t)), \qquad \phi \circ c(t_0) = 0.$$

Satz 7.5 (Flow-Box-Theorem). *Sei M eine C^∞–Mannigfaltigkeit und $X \in \Gamma(TM)$ ein Vektorfeld. Dann gibt es zu $P \in M$ eine offene Umgebung U, ein $\varepsilon > 0$ und eine C^∞ – Abbildung*

$$F : (-\varepsilon, \varepsilon) \times U \longrightarrow M$$

mit

(1) $F(0, x) = x$ *für* $x \in U$

(2) $\partial_t F(t, x) = X(F(t, x))$.

$F(t, \cdot)$ *ist der zu* X *gehörige* lokale *Fluss.*

Der Satz folgt aus dem Existenz- und Eindeutigkeitssatz für gewöhnliche Differentialgleichungen. Dabei benötigen wir jedoch die verschärfte Variante über glatte Abhängigkeit von Parametern.

Beweisskizze. Lokale Situation:

$$f : \bar{B(y_0, r)}\, c\mathbb{R}^m \longrightarrow \mathbb{R}^m$$

$C\infty$–Abbildung. f ist sicherlich Lipschitz

$$|f(x) - f(y)| \leq Lx - y|, \qquad L > 0.$$

Für $y \in B(y_0, r)$ sei $F(t, g)$ die *maximale* Lösung des Anfangswertproblems

$$\partial_t F(T_1 y) = f(F(t, y)); \qquad F(0, y) = y$$
$$a(y) < t < b(y).$$

Sicherlich $b(y) > 0$, $a(y) < 0$.

Lemma 7.6 (Lemma von Gronwall). $[a, b] \subset \Re$, $f, g, : [a, b] \to \Re_+$ *stetig. Falls*

$$f(t) \leq c + \int\limits_a^t f(s) g(s) ds,$$

so folgt

$$f(t) \leq c e^{\int\limits_a^t g(s) ds}.$$

Bew. $\widetilde{f}(t) := c + \int\limits_a^t f(s) g(s) ds \geq f(t)$.

$h(t) := \widetilde{f}(t) \exp\left(-\int\limits_a^t g(s) ds\right)$

$h'(t) = (f(t) - \widetilde{f}(t)) g(t) \exp\left(-\int\limits_a^t g(s) ds\right) \leq 0,$

$h(0) = c.$

Also $h(t) \leq c$ für $t \geq a$, folglich

$$f(t) \leq \widetilde{f}(t) = h(t)\, e^{\int_a^t g} \leq c\, e^{\int_a^t g}$$

□

(1) Abschätzung von $b(y)$ nach unten, bzw. $a(y)$ nach oben.

$$F(t_1 y) = y + \int\limits_0^t f(F(s,y))ds.$$

Folglich für $|t| \leq 1$

$$|F(t_1 y) - y_0| = |y - y_0| + \int\limits_0^t f(F(s,y))ds|$$

$$\leq |y - y_0| + |t|\,|f(y_0)| + \int\limits_0^t |f(F(s,y)) - f(y_0)|ds$$

$$\leq |y - y_0| + |f(y_0)| + L \int\limits_0^t |F(s,y) - y_0|ds$$

Mit Gronwall folgt:

$$|F(t_1 y) - y_0| \leq (|y - y_0| + |f(y_0))\,|e^{L|t|}.$$

Setzt man für $|y- <_0| \leq \frac{r}{2}$

$$c := min\left(1, \frac{1}{L}\log\frac{r}{\frac{r}{2} + |f(y_0)|}\right)$$

so ist für $|t| < c$

(2) Wir zeigen nun die Differenzierbarkeit.
Fixiere

□

KAPITEL 2

Tensor algebra and exterior algebra

Status: close to beta; many proofs missing, but otherwise not that bad

ZUSAMMENFASSUNG. This chapter covers the multilinear algebra as we need it for vector analysis. With regard to the modern theory of determinant line bundles we will pay special attention to the various lines attached to a vector space and which structures determine a *canonical* trivialization.

1. Tensors

{TENsTensors}

In the sequel K denotes a field of characteristic 0. All vector spaces over K will be finite-dimensional. If E is such a space, E^* denotes its dual space and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between E and E^* or vice versa.

Let $L(E_1,\ldots,E_r;F)$ denote the space of r–multilinear maps from $E_1\times\ldots\times E_r$ to F. One has

$$\dim L(E_1,\ldots,E_r;F)=\dim F\cdot\prod_{j=1}^{r}\dim E_j.$$

Since $\dim E<\infty$ the canonical inclusion $E\hookrightarrow E^{**}$ into its bidual is an isomorphism. Bases and its dual bases will be denoted by $e_1,\ldots,e_n;e_1^*,\ldots,e_n^*$ etc.

Definition 1.1. *For $\varphi\in E^*,\psi\in F^*$ put $\varphi\otimes\psi(e,f):=\varphi(e)\psi(f)$.*

Then $\varphi\otimes\psi\in L(E,F;K)$. $\otimes$ is a bilinear map $E^*\times F^*\to L(E,F;K)$. In terms of the bases e_i,f_i,e_j^*,f_j^* of E,F,E^*,F^* we find[1]

$$\varphi\otimes\psi=\sum_{i,j}\varphi(e_i)\psi(f_j)e_i^*\otimes f_j^*,$$

and because of

$$e_i^*\otimes f_j^*(e_k,f_l)=\delta_{ik}\delta_{jl}$$

we see that $e_i^*\otimes f_j^*$ is a basis of $L(E,F;K)$.

One extends the definition of $\otimes$ to a bilinear map

$$L(E_1,\ldots,E_r;K)\times L(F_1,\ldots,F_s;K\longrightarrow L(E_1,\ldots,E_r,F_1,\ldots,F_s;K)$$

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[1] Sometimes we will adopt the physicists summation convention. That means that without explicit $\sum$ sign one sums over all indices which appear twice.

by putting

$$\varphi \otimes \psi(e_1, \dots, e_r; f_1, \dots, f_s) := \varphi(e_1, \dots, e_r)\psi(f_1, \dots, f_s)$$

It is straightforward to see that the so defined $\otimes$ is associative, that is $(\varphi \otimes \psi) \otimes \chi = \varphi \otimes (\psi \otimes \chi)$.

Definition 1.2. *$E^* \otimes F^* := L(E, F; K)$. Using the identifications $E \simeq E^{**}, F \simeq F^{**}$ one therefore defines $E \otimes F := L(E^*, F^*; K)$.*

{TENpUnivTensor}

Proposition 1.3 (Universal property of the tensor product)**.** *The bilinear map $\otimes : E \times F \to E \otimes F$ solves the following universal problem:*

$$\begin{tikzcd}
E \times F \arrow[r, "h\ bilin."] \arrow[d, "\otimes", "bilinear"'] & G \\
E \otimes F \arrow[ur, "\exists!\,\overline{h}\ linear"'] &
\end{tikzcd}$$

That is for any bilinear map $h : E \times F \to G$ into a vector space G there is a unique linear map $\overline{h} : E \otimes F \to G$ such that $h = \overline{h} \circ \otimes$. The correspondence $h \leftrightarrow \overline{h}$ is a canonical isomorphism $L(E, F; G) \leftrightarrow L(E \otimes F; G)$.

TODO

Beweis. □

Definition and Proposition 1.4. *$T(E) := \oplus_{r \geq 0} T^r E, T^r E := \otimes^r E$ is called the tensor algebra over E. Indeed, with $\otimes$ as product, it is a $\mathbb{Z}_+$–graded algebra. A basis of $T^r E$ is*

$$\big(e_{i_1} \otimes \dots \otimes e_{i_r}\big)_{1 \leq i_1, \dots, i_r \leq \dim E}.$$

In particular, $\dim T^r E = (\dim E)^r$.

More generally, one puts $T^r_s E := T^r E \otimes T^s E^*$.

Remark 1.5. 1. There is a natural dual pairing between $T^r E$ and $T^r E^*$ by

$$\langle v_1 \otimes \dots \otimes v_r, v_1^* \otimes \dots \otimes v_r^* \rangle = \prod_{j=1}^{r} \langle v_j, v_j^* \rangle,$$

hence $(e_{i_1} \otimes \dots \otimes e_{i_r})_{\dots}$ and $(e^*_{i_1} \otimes \dots \otimes e^*_{i_r})_{\dots}$ are dual bases for $T^r E$, $T^r E^*$.

Similarly, there is a dual pairing between $T^r_s E$ and $T^s_r E$.

2. Given $i, j, 1 \leq i \leq r, 1 \leq j \leq s$ there is a natural map, called *contraction of the indices i, j* $T^r_s E \to T^{r-1}_{s-1} E$ which on elementary tensors is given by

$$\begin{aligned} &v_1 \otimes \dots v_r \otimes w_1^* \otimes \dots \otimes w_s \\ &\qquad \longmapsto \langle v_i, w_j^* \rangle v_1 \otimes \dots \widehat{v_i} \dots \otimes v_r \otimes w_1 \otimes \dots \widehat{w_j} \dots \otimes w_s. \end{aligned}$$

Here we use a notation which will be used repeatedly below: $v_1 \otimes \dots \widehat{v_i} \dots \otimes v_r$ stands for $v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes \dots \otimes v_r$.

2. Totally antisymmetric (alternating) tensors

Denote by $\mathcal{S}_r$ the symmetric group of permutations of $\{1, \ldots, r\}$, the sign of a permutation σ will be denoted by $(-1)^\sigma$ or $\operatorname{sgn}(\sigma)$. By the universal property of the tensor product (Prop. 1.3), the map $(x_1, \ldots, x_r) \mapsto (x_{\sigma 1}, \ldots, x_{\sigma r})$ induces a linear map $T^r\sigma = \sigma : T^rE \to T^rE$.

Definition 2.1. *An r–tensor $T \in T^rE$ is called* alternating *(or* totally antisymmetric*) if for all $\sigma \in \mathcal{S}_r$ we have $T^r\sigma(T) = (-1)^\sigma T$.*

Denote by $\Lambda^rE \subset T^rE$ the vector space of alternating tensors.

Since every σ is a product of transpositions of adjacent indices, for proving that T is alternating it suffices to check that $T^r\tau(T) = -T$ for such transpositions.

The *antisymmetrization map* $A_r : T^rE \to T^rE$ is defined by

$$A_r(T) := \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} (-1)^\sigma T^r\sigma(T).$$

Lemma 2.2. *$A_r = A_r \circ A_r$ is a projection with range Λ^rE.*

Beweis. If $\tau \in \mathcal{S}_r$ then

$$\tau(A_r(T)) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} (-1)^{\sigma\tau} (\tau\sigma)(T)(-1)^\tau = (-1)^\tau A_r(T).$$

Thus $A_r(T^rE) \subset \Lambda^rE$. Furthermore, for $T \in \Lambda^rE$

$$A_r(T) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} (-1)^\sigma (-1)^\sigma T = T,$$

thus $A_r(T^rE) = \Lambda^rE$ and $A_r^2 = A_r$. □

3. Exterior algebra

Fix a sequence of numbers $\psi(n)$ with $\psi(0) = \psi(1) = 1$. We will abbreviate $p? := \psi(p)$. Put

$$\wedge_\psi : \Lambda^pE \times \Lambda^q_E \to \Lambda^{p+q}E, \quad \xi \wedge \eta := \frac{(p+q)?}{p?q?} A_{p+q}(\xi \otimes \eta).$$

Furthermore, we put

$$\Lambda E := \bigoplus_{p=0}^{\dim E} \Lambda^pE,$$

and for $\xi \in \lambda^pE$ we put $|\xi| := p$. That is, if $\xi \in \Lambda E$ is a pure degree (homogeneous) then $|\xi|$ denotes this degree.

Proposition 3.1. *$\wedge = \wedge_\psi$ is bilinear, associative and graded commutative. The latter means that for homogeneous $\xi, \eta \in \Lambda E$ we have $\xi \wedge \eta = (-1)^{|\xi||\eta|}\eta \wedge \xi$.*

With the wedge product ΛE, becomes a graded commutative algebra over K.

For $v_1, \dots, v_r \in E$ *we have*

$$v_1 \wedge \ldots \wedge r = \frac{r?}{r!} \sum_{\sigma \in \mathcal{S}_r} (-1)^\sigma v_{\sigma 1} \otimes \ldots \otimes v_{\sigma r}.$$

A basis for $\Lambda^r E$ *is given by* $e_{i_1} \wedge \ldots \wedge e_{i_r}$ *where* $1 \le i_1 < \ldots < i_r \le \dim E$ *runs through all ordered subsets of* $\{1, \dots, \dim E\}$*. In particular*

$$\dim \Lambda^r E = \binom{\dim E}{r}.$$

Finally, for $v_1, \dots, v_r \in E; v_1^*, \dots, v_r^* \in E^*$

$$\langle v_1 \wedge \ldots \wedge v_r, v_1^* \otimes \ldots \otimes v_r^* \rangle = \frac{r?}{r!} \det\bigl(\langle v_i, v_j^* \rangle_{1 \le i,j \le \dim E}\bigr), \tag{3}$$

{TENEQTensor10.5}

resp.

$$\langle v_1 \wedge \ldots \wedge v_r, v_1^* \wedge \ldots \wedge v_r^* \rangle = \frac{(r?)^2}{r!} \det\bigl(\langle v_i, v_j^* \rangle_{1 \le i,j \le \dim E}\bigr). \tag{4}$$

{TENEQTensor10.6}

Remark 3.2. We comment on the various choices for $r?$.

1. $r? = 1$. This gives the most natural wedge product making the exterior algebra canonically isomorphic with the quotient TE/JE. The disadvantage is that factorials occur in the pairings Eq. (3) and Eq. (4). This convention is adopted in [**?**], [**?**]. .

ref Kobayashi-Nomizu, Guillemin-Pollack

2. $r? = r!$. This is the convenction of [**AMR88**]. The constant in Eq. (3) is then 1.

3. $r? = \sqrt{r!}$ (ML). This is arguably the most natural convention making the constant in Eq. (4) equal to 1. It treats E and its dual space equally. Moreover, if $(e_i), (e_i^*)$ is a pair of dual bases for E and E^* then $(e_{i_1} \wedge \ldots \wedge e_{i_r})$, $(e_{i_1}^* \wedge \ldots \wedge e_{i_r}^*)$, $1 \le i_1 < \ldots < i_r \le \dim E$ is a pair of dual bases for $\Lambda^r E, \Lambda^r E^*$.

Obviously, it is impossible to *choose* $r?$ in such a way that the constants in Eq. (3) *and* Eq. (4) are equal to 1. How do we get around the problem that there is seemingly no optimal choice for these constants. In order not to have to deal with them for the rest of our mathematical lives we have to come to a compromise. We choose $r? := \sqrt{r!}$ such that the natural pairing between $\Lambda^r E$ and $\Lambda^r E^*$ is given by

$$\langle v_1 \wedge \ldots \wedge v_r, v_1^* \wedge \ldots \wedge v_r^* \rangle = \det\bigl(\langle v_i, v_j^* \rangle_{1 \le i,j \le \dim E}\bigr). \tag{5}$$

{TENEQTensor10.7}

Secondly, we *modify* the dual pairing between $T^r E$ and $T^r E^*$ as follows

$$\langle v_1 \otimes \ldots \otimes v_r, v_1^* \otimes \ldots \otimes v_r^* \rangle = \sqrt{r!} \langle v_1, v_1^* \rangle \cdot \ldots \cdot \langle v_r, v_r^* \rangle. \tag{6}$$

{TENEQTensor10.8}

For us the exterior algebra will be much more important than the tensor algebra; hence this modification does the least harm to us. Then

$$\begin{aligned}
&\langle v_1 \wedge \ldots \wedge v_r, v_1^* \otimes \ldots \otimes v_r^* \rangle_{TE,TE^*} \\
&= \frac{1}{\sqrt{r!}} \sum_{\sigma \in \mathcal{S}_r} (-1)^\sigma \langle v_1 \otimes \ldots \otimes v_r, v_1^* \otimes \ldots \otimes v_r^* \rangle_{TE,TE^*} \\
&= \langle v_1 \wedge \ldots \wedge v_r, v_1^* \wedge \ldots \wedge v_r^* \rangle_{\Lambda E, \Lambda E^*} = \det\bigl(\langle v_i, v_j^* \rangle_{1 \le i,j \le \dim E}\bigr).
\end{aligned}$$

Beweis. □ TODO

Remark 3.3. 1. $v_1, \dots, v_r \in E$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_r = 0$.
2. Cartan's Lemma..... TODO
3. $\Lambda^2\mathbb{R}$ and the Lie algebra of $SO(n)$. TODO

*4. Alternating maps, universal property

This section is optional and will be typed later.

*5. The determinant

Let $\dim E = n$ and $f \in \operatorname{End}(E) \simeq E^* \otimes E$. In Linear Algebra the determinant $\det f$ and trace $\operatorname{tr} f$ are defined as follows: Choose a basis $e_1, \dots e_n$ and let $f(e_j) = f_{ij} e_i$ be the matrix representation of f. Then

$$\det f := \det(f_{ij})_{1 \le i,j \le n} = \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma f_{1\sigma 1} \cdot \dots \cdot f_{n\sigma n}.$$

$$\operatorname{tr} f := \operatorname{tr}(f_{ij})_{1 \le i,j \le n} = \sum_{j=1}^n f_{jj}.$$

det and tr are independent of the choice of the basis.

$\Lambda^n E$ is 1–dimensional, hence $\Lambda^n f$ is multiplication by a number. Indeed

$$\begin{aligned}\Lambda^n f(e_1 \wedge \dots \wedge e_n) &= f(e_1) \wedge \dots \wedge f(e_n) \\ &= \sum_{\sigma \in \mathcal{S}_n} f_{\sigma 1,1} \cdot \dots \cdot f_{\sigma n,n}\, e_{\sigma 1} \wedge \dots \wedge e_{\sigma_n} \\ &= (\det f)\, e_1 \wedge \dots \wedge e_n,\end{aligned}$$

since $e_{\sigma 1} \wedge \dots \wedge e_{\sigma n} = (-1)^\sigma e_1 \wedge \dots \wedge e_n$.

Since the trace of a 1×1–matrix equals its entry, one could also say that $\det f = \operatorname{tr}(\Lambda^n f)$. This can be generalized considerably:

Proposition 5.1. *Let $\dim E = n, f \in \operatorname{End} E$. Then for $\lambda \in K$*

$$\det(f + \lambda \cdot \operatorname{Id}) = \sum_{r=0}^n \operatorname{tr}(\Lambda^r f)\, \lambda^{n-r}.$$

Beweis. W.l.o.g. we may assume that K is algebraically closed (otherwise do the calculation in the algebraic closure). Choose a basis $e_1, \dots e_n$ such that f is in Jordan normal form, i.e. $f(e_i) = \lambda_i e_i + \varepsilon_i e_{i+1}$ with $\varepsilon_i \in \{0, 1\}$. Then for $\sigma 1 < \dots < \sigma r$

$$f(e_{\sigma 1} \wedge \dots \wedge e_{\sigma n}) = \lambda_{\sigma 1} \cdot \dots \cdot \lambda_{\sigma r} e_{\sigma 1} \wedge \dots \wedge e_{\sigma n} + \sum_J \mu_J e_J,$$

where J runs through subsets of length r different from $\{\sigma 1, \dots, \sigma r\}$. Since the trace sums over the diagonal entries we find

$$\operatorname{tr}(\Lambda^r f) = \sum_{1 \leq \sigma 1 < \ldots < \sigma r \leq n} \lambda_{\sigma 1} \cdot \ldots \cdot \lambda_{\sigma r}.$$

On the other hand

$$\begin{aligned}\det(f + \lambda \cdot \mathrm{Id}) &= \prod_{j=1}^{n} (\lambda_i + \lambda) \\ &= \sum_{r=0}^{n} \lambda^{n-r} \sum_{1 \leq \sigma 1 < \ldots < \sigma r \leq n} \lambda_{\sigma 1} \cdot \ldots \cdot \lambda_{\sigma r} = \sum_{r=0}^{n} \lambda^{n-r} \operatorname{tr}(\lambda^r f).\end{aligned}$$

□

6. ext and int

Recall the normalization conventions from Eq. (5) and Eq. (6). For $\xi \in \Lambda^r E$ put

$$\operatorname{ext}_\xi : \Lambda E \to \Lambda E, \quad \eta \mapsto \xi \wedge \eta,$$

resp. for $u \in E$

$$\operatorname{int}_u : \Lambda E^* \to \Lambda E^*, \quad \eta \mapsto \langle \omega, u \otimes \ldots \rangle = \langle \omega, u \wedge \ldots \rangle$$

w.r.t. the dual pairing between ΛE and ΛE^*. ext_u and int_u are dual to each other, i.e. for $\eta \in \Lambda E$, $\omega \in \Lambda E^*$

$$\langle \operatorname{ext}_u \eta, \omega \rangle = \langle u \wedge \eta, \omega \rangle = \langle \eta, \operatorname{int}_u \omega \rangle.$$

By duality one could define int_ξ for any $\xi \in \Lambda E$.

Lemma 6.1. *For $u \in$ the operator int_u is an antiderivation, that is for homogeneous $\omega, \eta \in E$*

$$\operatorname{int}_u(\omega \wedge \eta) = (\operatorname{int}_u \omega) \wedge \eta + (-1)^{\omega} \omega \wedge \operatorname{int}_u \eta.$$

7. Orientations, volumes, and the Hodge ⋆–operator

In this section let $K = \mathbb{R}$ be the real number field.

7.1. Lines. In the following it will be crucial to understand the difference between *isomorphic* vector spaces and *canonically isomorphic* vector spaces. Consider a 1–dimensional $\mathbb{R}$ vector space L (a *line*). It is isomorphic to $\mathbb{R}$ but a priori there is no canonical (or preferred) choice for such an isomorphism. Hence, if two students are asked to choose such an isomorphism then, at least with a certain nonzero probability, they will come up with different solutions. If we are given a nonzero vector $e \in L$ (that is $\{e\}$ constitutes a basis) then e induces an isomorphism to $\mathbb{R}$ sending $\lambda e \to \lambda$.

Emphasizing the difference between isomorphism and canonical isomorphism may appear to be nitpicking; you may rest assured that in the sequel this difference will become important.

Let E be an n–dimensional $\mathbb{R}$ vector space.

Definition 7.1. *1. Let α be a nonzero real numer. An α–*density *on E is a map $\varrho : \Lambda^n E^* \to \mathbb{R}$ with $\varrho(\lambda\omega) = |\lambda|^\alpha \varrho(\omega)$ for $\omega \in \Lambda^n E^*$ and $\lambda \in \mathbb{R}$; recall* $\dim E = n$.

Denote by $|\Lambda E^*|^\alpha$ the vector space of all α–densities over E. A density is called positive/negative if $\varrho(\omega) > 0\,(< 0)$ for $\omega \neq 0$.

2. A signed density *on E is a map $\varrho : \Lambda^n E^* \to \mathbb{R}$ with $\varrho(\lambda\omega) =$* $\operatorname{sgn}\lambda\varrho(\omega), \operatorname{sgn}\lambda := \lambda/|\lambda|$ *for all $\lambda \in \mathbb{R} \setminus \{0\}, \omega \in \Lambda^n E^*$.*

The vector space $\mathcal{O}(E)$ of all signed densities is called the orientation line *of E.*

Remark 7.2. 1. Via the canonical map $E^*n \to \Lambda^n E^*$ one could equivalently define α–densities (signed densities) to be maps $\varrho : (E^*)^n \to \mathbb{R}$ with

$$\varrho(\varphi(v_1^*), \dots, \varphi(v_n^*)) = |\det\varphi|^\alpha \varrho(v_1^*, \dots, v_n^*)$$

(resp. $\operatorname{sgn}\det\varphi$ instead of $|\det\varphi|^\alpha$) for all isomorphisms $\varphi \in \mathrm{GL}(E^*)$.

2. $\dim|\lambda E|^\alpha = \dim\mathcal{O}(E) = 1$. Namely, given $\xi \in \Lambda^n E \setminus \{0\}$, put

$$\varrho_\xi^\alpha(\omega) := |\xi(\omega)|^\alpha, \quad \varrho_\xi^\sigma(\omega) := \begin{cases} \operatorname{sgn}(\langle \xi, \omega\rangle), & \omega \neq 0, \\ 0, & \omega = 0. \end{cases}$$

Then ϱ_ξ^α (ϱ_ξ^σ) is easily seen to be a basis of $|\Lambda E|^\alpha$ ($\mathcal{O}(E)$).

3. An α–density is either $> 0, < 0$ or $= 0$. Abstractly speaking, the line $|\Lambda E|^\alpha$ has a canonical order structure.

{TENMLL11.3}

Lemma 7.3. *The multiplication $\varrho_1 \otimes \varrho_2 \mapsto \varrho_1 \cdot \varrho_2 \ni (\omega_1, \omega_2) \mapsto \varrho(\omega_1) \cdot \varrho_2(\omega_2)$ induces natural isomorphisms*[2] $|\Lambda|^\alpha \otimes |\Lambda|^\beta \to |\Lambda|^{\alpha+\beta}, |\Lambda|^1 \otimes \mathcal{O} \to \Lambda^n, \Lambda^n \otimes \mathcal{O} \to |\Lambda|^1$, $\mathcal{O} \otimes \mathcal{O} \to \mathbb{R}$.

Furthermore, the line $\mathcal{O}(E)$ has a canonical euclidean metric, defined for $\varrho \in \mathcal{O}(E)$ by $\|\varrho| := \sqrt{\varrho(\omega)^2}$ for any $\omega \neq 0$.

Beweis. This follows immediately from the definitions. ☐

7.2. Orientations. We have seen that $\mathcal{O}(E)$ is a 1–dimensional euclidean vector space, hence there are exactly 2 vectors of length 1. A priori no one is preferred over the other. The choice of one of the two vectors is called an *orientation*

Definition 7.4. *An* orientation *of the real vector space E is given by a choice $\sigma \in \mathcal{O}$ of length 1. There are exactly two orientations of E.*

In view of Lemma 7.3 there are the following equivalent characterizations of an orientation σ:

(1) An isometry $\mathcal{O}(E) \simeq \mathbb{R}, \sigma \mapsto 1$.
(2) An isomorphism $\Lambda^n E \to |\Lambda E|^1, \varrho \mapsto \varrho \cdot \sigma$.
(3) A connected component $\Lambda_+^n E$ of $\Lambda^n E \setminus \{0\}$. This is the image of the positive cone $|\Lambda E|_+^1$ in $|\Lambda E|^1$ under the isometry in (2).

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[2] E omitted for better readability

To see that (3) indeed determines an orientation note that given a connected component $\Lambda^n_+ E$ of $\Lambda^n E \setminus \{0\}$ by Lemma 7.3 there is exactly one orientation $\sigma \in \mathcal{O}(E)$ such that $\Lambda^n_+ E \cdot \sigma = |\Lambda E|^1_+$.

If E is oriented with orientation σ we call a basis $e_1, \ldots, e_n$ of E oriented if $e_1 \wedge \ldots \wedge e_n \in \Lambda^n_+ E$, equivalently $e_1 \wedge \ldots \wedge e_n = \sigma \cdot |e_1 \wedge \ldots \wedge e_n|$.

7.3. Volume elements. Let $g \in T^0_2 E = \otimes^2 E^*$ be a symmetric bilinear form on E. g induces bilinear forms on TE and ΛE as follows:

$$\widetilde{T}^r g(v_1 \otimes \ldots \otimes v_r, w_1 \otimes \ldots \otimes w_r) := \prod_{j=1}^r g(v_j, w_j),$$

$T^r g := \sqrt{r!}\widetilde{T}^r g$. On $\Lambda E \subset TE$ we find

$$\widetilde{T}^r g(v_1 \wedge \ldots \wedge v_r, w_r \wedge \ldots \wedge w_r) = \frac{1}{r!} \sum_{\sigma,\tau \in \mathcal{S}_r} (-1)^{\sigma\tau} \prod_{j=1}^r g(v_{\sigma j}, w_{\tau j})$$
$$= \det(g(v_i, w_j))_{i,j}.$$

We call this $\Lambda^r g$. Furthermore, cf. lost page 10.9

$$\widetilde{T}^r g(v_1 \wedge \ldots \wedge v_r, w_r \otimes \ldots \otimes w_r) = \sqrt{r!} \cdot \widetilde{T}^r g(\ldots)$$
$$= \sum_{\sigma \in \mathcal{S}_r} r(-1)^{\sigma} \prod_{j=1}^r g(v_{\sigma j}, w_j) = \det(g(v_i, w_j))_{i,j}.$$

Lemma 7.5. *If g is non-degenerate, so are $Tg, \widetilde{T}g$ and Λg. $\cdot^\flat : E \to E^*, v \mapsto g(v, \cdot)$ is an isomorphism whose inverse we denote by $\sharp$.*

In particular, g induces a non-degenerate symmetric bilinear form g^* on E^* by $g^*(v^*, w^*) := g(\sharp v^*, \sharp w^*)$.

Proposition 7.6. *Let g be a non-degenerate symmetric bilinear form on E. Then g determines uniquely a* positive 1–*density* $\mathrm{vol}_g \in |\Lambda(E)|^1$. *If $e_1, \ldots, e_n$ is a basis of E then*

$$\mathrm{vol}_g = |\det(g(e_i, e_j))_{i,j}|^{-1/2} |e_1 \wedge \ldots \wedge e_n|. \tag{7}$$

{TENMLEq11.1}

In particular, Eq. (7) *is basis independent.*

Consequently, g induces for any $\alpha \neq 0$ a canonical isomorphism $|\Lambda E|^\alpha \to \mathbb{R}, \lambda |\mathrm{vol}_g|^\alpha \mapsto \lambda$, sending positive α–densities to positive real numbers.

Beweis. Let $f_j = a_{ij} e_i$[3] be another basis. Then $g(f_k, f_l) = a_{ki} a_{lj} g(e_i, e_j)$ and thus

Find good place to note $f_1 \wedge \ldots \wedge f_n = \det(\ldots) e_1 \wedge \ldots \wedge e_n$.

$$\det(g(f_k, f_l))_{k,l} = \big(\det(a_{ij})_{i,j}\big)^2 \det(g(e_i, e_j))_{i,j}.$$

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[3] In this proof we adopt summation convention over repeated indices

Hence

$$
\begin{aligned}
&|\det(g(f_k,f_l))_{k,l}|^{-1/2}|f_1\wedge\ldots\wedge f_n|\\
&\quad = |\det(a_{ij})_{i,j}|^{-1}|\det(g(e_i,e_j))_{i,j}|^{-1/2}|f_1\wedge\ldots\wedge f_n|\\
&\quad = |\det(g(e_i,e_j))_{i,j}|^{-1/2}\cdot|e_1\wedge\ldots\wedge e_n|.
\end{aligned}
$$

This shows that vol_g is well-defined.

By the very construction vol_g is positive and the remaining claim is now clear. □

7.4. The Hodge $\star$–operator. As in the previous Subsection let g be non-degenerate and let $\mathrm{vol}_g \in |\Lambda(E)|^1$ be the volume element.

{TENThmHodgeStar}

Theorem 7.7. *The map $(\omega,\eta)\mapsto \omega\wedge\eta/\mathrm{vol}_g$ induces gives dual pairings $\Lambda^pE\times\Lambda^{n-p}E\otimes\mathcal{O}\to\mathbb{R}$ and $\Lambda^pE\otimes\mathcal{O}\times\Lambda^{n-p}E\to\mathbb{R}$.*

In particular, to $\eta\in\Lambda^pE(\Lambda^pE\otimes\mathcal{O})$ there exists a unique element $\star\eta\in\Lambda^{n-p}E\otimes\mathcal{O}(\mathcal{L}^{n-p}E)$ such that

$$\omega\wedge\star\eta = \Lambda^p g(\omega,\eta)\cdot\mathrm{vol}_g \tag{8}$$

{TENEqHodgeStar}

for all $\omega\in\Lambda^pE(\Lambda^pE\otimes\mathcal{O})$.

Here, we identify according to Lemma 7.3 $\Lambda^nE\otimes\mathcal{O}\simeq|\Lambda(E)|^1$. Furthermore, Λ^pg denotes the symmetric bilinear form induced by g on Λ^pE and on $\Lambda^pE\otimes\mathcal{O}$.

redundant? make note earlier?

The Hodge $\star$–operator $\star_p:\Lambda^pE(\Lambda^pE\otimes\mathcal{O})\to\Lambda^{n-p}E\otimes\mathcal{O}(\Lambda^{n-p}E)$ *has the following properties:*

(1) *If $e_1,\ldots,e_n$ is a g–orthonormal basis with $c_i=g(e_i,e_i)\in\{\pm1\}$ then*

we need it in general later...

$$\star e_{\sigma1}\wedge\ldots\wedge e_{\sigma p} = c_{\sigma1}\cdot\ldots\cdot c_{\sigma n}(-1)^{\sigma}e_{\sigma(p+1)}\wedge\ldots\wedge e_{\sigma n}\otimes\frac{e_1\wedge\ldots\wedge e_n}{\mathrm{vol}_j}.$$

(2) $\star_{n-p}\star_p = (-1)^{\mathrm{ind}\,g+p(n-p)}$.
(3) $\Lambda^{n-p}g(\star\omega,\star\eta) = (-1)^{\mathrm{ind}\,g}\Lambda^pg(\omega,\eta)$ *for $\omega,\eta\in\Lambda^pE(\otimes\mathcal{O})$.*
(4) $\omega\wedge\eta = \omega\wedge\star_p\star_{n-p}\eta(-1)^{p(n-p)+\mathrm{ind}\,g} = (-1)^{p(n-p)+\mathrm{ind}\,g}\Lambda^pg(\omega,\star\eta)\,\mathrm{vol}_g$ *for $\omega\in\Lambda^pE$, $\eta\in\Lambda^{n-p}E\otimes\mathcal{O}$.*

TODO

BEWEIS. □

Remark 7.8. The best of all worlds occurs when we have an orientation *and* a metric g. Then all the lines $|\Lambda|^\alpha,\mathcal{O}$ are canonically trivial and hence we do not have to worry about them. Λ^n is then canonically identified with $|\Lambda|^1$. Hence the Hodge $\star$–operator maps Λ^pE onto $\Lambda^{n-p}E$. Furthermore, we note that in this case by slight abuse of notation one denotes by $\mathrm{vol}_g = e_1\wedge\ldots\wedge e_n\in\Lambda^nE$ for any *oriented* g–orthonormal basis. This element corresponds to the volume element $\mathrm{vol}_g\in|\Lambda(E)|^1$ under the canonical identification $\Lambda^nE\simeq|\Lambda(E)|^1$.

7.5. Lines and direct sums. Let

$$0 \longrightarrow V_0 \xhookrightarrow{\iota} V_1 \xrightarrow{\pi} V_1/V_0 \longrightarrow 0$$

be a short exact sequence of vector spaces, $\dim V_0 = k, \dim V_1/V_0 = l, \dim V_1 = n = k + l$. Fix a basis $v_1, \dots, v_n$ of V_1 such that $v_1, \dots, v_k$ is a basis of V_0. Then $\pi(v_{k+1}), \dots, \pi(v_n)$ is a basis of the quotient V_1/V_0. Define

$$\wedge : \Lambda^k V_0 \times \Lambda^l V_1/V_0 \longrightarrow \Lambda^n V_1$$

as follows: For $\xi \in \Lambda^k V_0, \eta \in \Lambda^l V_1/V_0$ choose $\widetilde{\eta} \in \Lambda^l V_1$ with $(\Lambda^l \pi)(\widetilde{\eta}) = \eta$. Then put $\xi \wedge \eta := \xi \wedge \widetilde{\eta} \in \Lambda^n V_1$.

This map is well-defined and bilinear. Namely, there are real numbers λ, μ such $\xi = \lambda v_1 \wedge \ldots \wedge v_k$ and

$$\widetilde{\eta} = \mu v_{k+1} \wedge \ldots \wedge v_n + \sum_J \mu_J v_J,$$

where J runs through the subsets of cardinality l of $\{1, \dots, n\}$ different from $\{k + 1, \dots, n\}$. Each such J contains an element $j \in \{1, \dots, k\}$ and thus

$$\eta = (\Lambda^l \pi)(\widetilde{\eta}) = \mu \, \pi(v_{k+1}) \wedge \ldots \wedge \pi(v_n),$$

showing that μ depends only on η and not on $\widetilde{\eta}$. Now

$$\xi \wedge \eta = \lambda \cdot \mu \, v_1 \wedge \ldots \wedge v_n.$$

We have also proved:

Proposition 7.9. $\wedge : \Lambda^k V_0 \times \Lambda^l V_1/V_0 \longrightarrow \Lambda^n V_1$ *is a canonical isomorphism.*

As a Corollary we obtain canonical isomorphisms

$$|\lambda V_0|^\alpha \otimes |\lambda V_1/V_0|^\alpha \longrightarrow |\lambda V_1|^\alpha,$$
$$\mathcal{O}V_0 \otimes \mathcal{O}(V_1/V_0) \longrightarrow \mathcal{O}V_1$$

by putting, e.g. for $\varrho_0 \in |\lambda V_0|^\alpha(\mathcal{O}V_0), \varrho_2 \in |\lambda V_1/V_0|^\alpha(\mathcal{O}(V_1/V_0)), \xi \in \Lambda^k V_0^*, \eta \in \Lambda^l(V_1/V_0)^*$,

$$\varrho_0 \cdot \varrho_2(\xi \wedge \eta) := \varrho_0(\xi) \cdot \varrho_2(\eta).$$

These maps are easily seen to be well-defined. As a result we have:

Proposition 7.10. *Volums forms (orientations) on two of the three vector spaces* $V_0, V_1, V_1/V_0$ *determine a volume form (orientation) on the third.*

Note that for orientations the order of the product matters. Namely, for ξ, η above we have $\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi$ and thus the isomorphism $\Lambda^k V_0 \otimes \Lambda^l(V_0/V_1) \to \Lambda^n V_1 \to \Lambda^l(V_0/V_1) \otimes \Lambda^k V_0$ is only up the sign $(-1)^{kl}$ the map which interchanges the two factors in the tensor product! The same remark therefore applies to the isomorphism $\mathcal{O}V_0 \otimes \mathcal{O}(V_1/V_0) \to \mathcal{O}V_1 \to \mathcal{O}(V_1/V_0) \otimes \mathcal{O}V_0$.

*8. The Pfaffian

Status: beta

ZUSAMMENFASSUNG. We give a coordinate free definition of the Pfaffian as an element of the determinant line. This allows to easily derive the well–known properties and formulas.

Let V be a real vector space of dimension $2n$, V^* its dual space. A homomorphism $T \in \operatorname{Hom}(V, V^*)$ is called skew-adjoint if

$$\langle Tv, w\rangle = -\langle Tw, v\rangle. \tag{9}$$

{EqPfaff1}

In other words the dual homomorphism $T^t : V^{**} = V \to V^*$ is given by $-T$.

{LemPfaff1}

Lemma 8.1. *The map* $\Phi : \operatorname{Hom}_{\mathrm{skew}}(V, V^*) \longrightarrow \Lambda^2 V^*$, $\Phi(T)(v, w) := \langle Tv, w\rangle$ *is an isomorphism.*

BEWEIS. Since T is skew-adjoint, $\Phi(T)$ is indeed an alternating bilinear form on V and hence an element of $\Lambda^2 V^*$. Moreover, Φ is obviously injective. Since the dimensions of the spaces $\operatorname{Hom}_{\mathrm{skew}}(V, V^*)$ and $\Lambda^2 V^*$ are both $\binom{2n}{n}$ the claim follows. $\square$

{DefPfaffian}

Definition 8.2. *The* Pfaffian *of* $T \in \operatorname{Hom}_{\mathrm{skew}}(V, V^*)$ *is defined by*

$$\operatorname{Pf}(T) := \frac{1}{n!}\Phi(T)^n \in \Lambda^{2n} V^*. \tag{10}$$

{EqPfaff2}

8.1. Coordinate representation. Let $v_1, \ldots, v_{2n}; v_1^*, \ldots, v_{2n}^*$ be a pair of dual bases of V and V^*. Our conventions for the normalizing constants in the wedge product are such that for $i < j, \alpha < \beta$ we have

$$v_i^* \wedge v_j^*(v_\alpha, v_\beta) = \det\begin{pmatrix} \langle v_i^*, v_\alpha\rangle & \langle v_i^*, v_\beta\rangle \\ \langle v_j^*, v_\alpha\rangle & \langle v_j^*, v_\beta\rangle \end{pmatrix} = \delta_{i\alpha}\delta_{j\beta}, \tag{11}$$

and hence

$$\Phi(T) = \sum_{i<j} \Phi(T)(v_i, v_j) v_i^* \wedge v_j^* = \frac{1}{2}\sum_{i,j} \langle Tv_i, v_j\rangle v_i^* \wedge v_j^*. \tag{12}$$

{EqPfaff3}

The right hand side is often used as a definition of $\Phi(T)$. The way we defined $\Phi(T)$ shows in particular that the right hand side of (12) is independent of the choice of the basis.

In terms of the coefficient matrix $t_{ij} = \langle Tv_i, v_j\rangle$ we now find

$$\begin{aligned} \operatorname{Pf}(T) &= \frac{1}{2^n n!}\sum t_{i_1 i_2} \cdot \ldots \cdot t_{i_{2n-1} i_{2n}} v_{i_1}^* \wedge \ldots \wedge v_{i_{2n}}^* \\ &= \frac{1}{2^n n!}\sum_{\sigma \in S_{2n}} (-1)^\sigma \, t_{\sigma(1)\sigma(2)} \cdot \ldots \cdot t_{\sigma(2n-1)\sigma(2n)} \, v_1^* \wedge \ldots \wedge v_{2n}^*; \end{aligned} \tag{13}$$

{EqPfaff4}

8.2. Properties.

8.2.1. *Behavior under congruence transformations.*

{LemPfaff2}

Lemma 8.3. *If* $S \in \operatorname{End}(V)$ *then* $\Phi(S^tTS) = \Lambda^2 S^t\big(\Phi(T)\big)$ *and hence* $\operatorname{Pf}(S^tTS) = \det S \cdot \operatorname{Pf}(T)$.

Beweis. For $v, w \in V$ we have

$$\begin{aligned}\Phi(S^tTS)(v,w) &= \langle S^tTSv, w\rangle = \langle TSv, Sw\rangle \\ &= \Phi(T)(Sv, Sw) = \Big(\Lambda^2 S^t\big(\Phi(T)\big)\Big)(v,w),\end{aligned} \tag{14}$$

thus

$$\begin{aligned}\operatorname{Pf}(S^tTS) &= \frac{1}{n!}\Big(\Lambda^2 S^t\big(\Phi(T)\big)\Big)^n \\ &= \Lambda^{2n} S^t \frac{1}{n!}\Phi(T)^n = \Lambda^{2n} S^t\big(\operatorname{Pf}(T)\big) = \det S \cdot \operatorname{Pf}(T).\end{aligned}$$

□

8.2.2. *Euclidean vector spaces.* Now let V be equipped with a scalar product. This gives us a canonical isomorphism $\flat : V \to V^*$ and hence the Pfaffian of a skew–adjoint endomorphism of V may therefore be viewed as an element of $\Lambda^2 V$.

Lemma 8.3 shows that $\operatorname{Pf}(T)$ is *invariant under conjugations by the special orthogonal group* $\mathrm{SO}(V)$, however $\operatorname{Pf}(T)$ changes sign if T is conjugated by an element of the orthogonal group with determinant -1.

If $e_1, \ldots, e_{2n}$ is an orthonormal basis of V (which is then of course self–dual with respect to the dual pairing given by the scalar product) we have using (12)

$$\Phi(T) = \frac{1}{2}\sum_{i,j}\langle Te_i, e_j\rangle e_i \wedge e_j, \tag{15}$$

{eq:Pfaff6}

and by (13)

$$\begin{aligned}\operatorname{Pf}(T) &= \frac{1}{n!}\Phi(T)^n \\ &= \frac{1}{2^n n!}\sum_{\sigma \in S_{2n}} \operatorname{sgn}\sigma\; t_{\sigma(1)\sigma(2)} \cdot \ldots \cdot t_{\sigma(2n-1)\sigma(2n)}\; e_1 \wedge \ldots \wedge e_{2n}.\end{aligned} \tag{16}$$

{eq:Pfaff7}

$\Lambda^{2n}V$ has a scalar product and hence a norm induced by the Euclidean structure on V. For this norm we have:

{PropPfaff1}

Proposition 8.4. $\|\operatorname{Pf}(T)\|^2 = \det T$.

Beweis. Choose an orthonormal basis $e_1, \ldots, e_{2n}$ such that $Te_{2j-1} = \lambda_j e_{2j}$, $j = 1, \ldots, n$. Then $\det T = \lambda_1^2 \cdot \ldots \cdot \lambda_n^2$,

$$\Phi(T) = \sum_{j=1}^n \lambda_j e_{2j-1} \wedge e_{2j}, \tag{17}$$

hence

$$\operatorname{Pf}(T) = \frac{1}{n!}\Phi(T)^n = \lambda_1 \cdot \ldots \cdot \lambda_n e_1 \wedge \ldots \wedge e_{2n} \tag{18}$$

and thus

$$\| \operatorname{Pf}(T)\|^2 = \lambda_1^2 \cdot \ldots \cdot \lambda_n^2 = \det T. \qquad \square$$

Finally we note that if V is an *oriented* Euclidean vector space then $\mathbb{R} \to \Lambda^{2n}V, t \mapsto t\,\mathrm{vol}$ is a canonical isomorphism. Hence in this case the Pfaffian may be viewed as a real number. This is the context in which the Pfaffian is usually presented. Proposition 8.4 then translates into the well–known formula $\operatorname{Pf}(T)^2 = \det T$. Thus Proposition 8.4 gives an interpretation of the fact that the Pfaffian is a square root of the determinant, which is independent of the choice of an orientation.

KAPITEL 3

Differential forms

1. Application: classical vector analysis

$$test \tag{19}$$

In the sequel let M^m be a smooth manifold together with a smooth symmetric non-degenerate tensor field $g \in \Gamma(T^{0,2}M)$ of index $\operatorname{ind} g$. Locally we may choose orthonormal frames such that g takes the form

$$\begin{bmatrix} -\operatorname{Id}_p & 0 \\ 0 & \operatorname{Id}_q \end{bmatrix}, \quad m = p + q.$$

Interesting cases are $p = 0$ (Riemannian metric) and $p = 1, q = 3$ (general relativity). Recall from the volume form ref

$$\begin{aligned} \mathrm{vol}_g =& |\det(g(\partial_i, \partial_j)|^{1/2} |dx_1 \wedge \ldots \wedge dx_m| \\ =& \sqrt{g} |dx_1 \wedge \ldots \wedge dx_m|. \end{aligned}$$

By slight abuse of notation the letter g is used for the metric g as well as for its Gram-determinant $|\det(g(\partial_i, \partial_j)|^{1/2}$, the meaning will be clear from the context.

Abbreviate

$$\sigma := \sigma(dx_1, \ldots, dx_m) := \frac{dx_1 \wedge \ldots \wedge dx_m}{|dx_1 \wedge \ldots \wedge dx_m|}$$

the orientation class of $dx_1, \ldots, dx_m$. Introduce

$$g_{ij} := g(\partial_i, \partial_j); g^{ij} g(dx_i, dx_j); g^{ij} g_{jk} = \delta_{ik}.$$

Lemma 1.1. $dx_i^\sharp = g^{ij} \frac{\partial}{\partial x_j}; \frac{\partial}{\partial x_j}^\sharp = g_{ij} dx_j$.

1.0.3. $\star\, dx_j$.

$$\star\, dx_j = \sum_{k=1}^{m} (-1)^{k-1} g^{kj} \sqrt{g} dx_1 \wedge \ldots \widehat{dx_k} \ldots \wedge dx_m \otimes \sigma.$$

1.1. Gradient. The gradient $\mathrm{grad}^g : \mathrm{C}^\infty(M) \to \Gamma(TM)$ is defined by the commutative diagram

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] \arrow[d, "="'] & \Omega^1(M) \arrow[d, shift left, "\flat"'] . \\
\mathrm{C}^\infty(M) \arrow[r, "\mathrm{grad}"] & \Gamma(TM) \arrow[u, shift left, "\sharp"]
\end{tikzcd}$$

Thus

$$\mathrm{grad}^g f = (df)^\sharp = (\partial_i f dx_i)^\sharp = \partial_i f g^{ij} \frac{\partial}{\partial x_j}$$

1.2. Divergence. We identify vector fields with twisted $(m-1)$-form by putting $X \mapsto \star X^\flat$.

Claim. $\star X^\flat = \mathrm{int}_X \mathrm{vol}_g$.

Beweis. Since this is a pointwise identity, it suffices to check it for $X = e_1$,

em$beingg--orthonormal\ldots.$ □

The *divergence* $\mathrm{div}^g : \Gamma(TM) \to \mathrm{C}^\infty(M)$ is defined by the commutative diagram

$$\begin{tikzcd}
\Omega_\tau^{m-1}(M) \arrow[r, "d"] & \Omega_\tau^m(M) \arrow[d, shift left, "\star"] \\
\Gamma(TM) \arrow[u, "\star(\cdot)^\flat"'] \arrow[r, "\mathrm{div}"] & \mathrm{C}^\infty(M) \arrow[u, shift left, "(-1)^p\star"]
\end{tikzcd}$$

Thus

$$\mathrm{div}^g(X) = (-1)^p \star d \star X^\flat.$$

In a coordinate system, for $X = X_j \frac{\partial}{\partial x_j}$ we have

$p = \mathrm{ind}\, g$?

$$\mathrm{div}^g(X) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j}(\sqrt{g} X_j).$$

1.3. Vector product and curl/rotation. Let $p = 0, q = 3$ and consider first $\mathbb{R}^3$ with its standard orientation. The vector product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ has the following properties:

1. it is bilinear,
2. it is alternating,
3. if e_1, e_2 are orthonormal, then $e_1, e_2, e_1 \times e_2$ is an *oriented* orthonormal basis.

Proposition 1.2. *Let V be an oriented 3–dimensional euclidean vector space. Then $V \times V \to V, v \times w := \star(v \wedge w)$ satisfies (1)-(3) and for $\mathbb{R}^3$ it coincides with the vector product.*

no labels here!?

If M is an oriented 3–dimensional Riemannian manifold define the *curl* (rotation) by the commutative diagram

ANHANG A

1. Structure theorems for differentiable maps

{wdh}

We recall here the structure theorems for differentiable maps from Analysis II.

1.1. Hauptsatz über implizite Funktionen. Seien $U \subset \mathbb{R}^p_x,\ V \subset \mathbb{R}^q_y$ offen und $F : U \times V \longrightarrow \mathbb{R}^q_y$ eine C^k–Abbildung, $k \geq 1$, mit $F(a,b) = 0$ für fest gegebene $a \in U,\ b \in V$. Weiterhin sei

$$D_2F(a,b) = (\frac{\partial F_i}{\partial y_j}(a,b))_{1 \leq i,j \leq q}$$

invertierbar.

Dann existieren offene Umgebungen $a \in U_1 \subset U$, $b \in V_1 \subset V$ und eine C^k–Abbildung $\varphi : U_1 \longrightarrow V_1$, so dass

$$\big\{(x, y \in U_1 \times V_1 \bigm| F(x,y) = 0\big\} = \big\{(x, \varphi(x)) \bigm| x \in U_1\big\},$$

d. h. in $U_1 \times V_1$ sind die Lösungen der Gleichung $F(x,y) = 0$ durch den Graphen der glatten Abbildung φ gegeben.

Es gilt noch $D\varphi(x) = -\big((D_2F)(x,\varphi(x)\big)^{-1}(D_1F)(x,\varphi(x)).$

{umkehrsatz}

1.2. Umkehrsatz. Sei $U \subset \mathbb{R}^p$ offen und $f : U \longrightarrow \mathbb{R}^p$ eine C^k–Abbildung, $k \geq 1$. Es sei $a \in U$ mit $Df(a)$ invertierbar.

Dann existieren offene Umgebungen $a \in U_0 \subset U, f(a) \in V$ so, dass $f\big|_{U_0} :\longrightarrow V$ ein C^k–Diffeomorphismus ist.

{rangsatz}

1.3. Rangsatz. Seien $U \subset \mathbb{R}^p_x$ offen und $f : U \longrightarrow \mathbb{R}^q$ eine C^k–Abbildung, $k \geq 1$. Es sei $a \in U$ fest gegeben und $f(a) =: b$. Dann gilt:

1. Ist $rgDf(a) = r$, so existieren lokale Diffeomorphismen $\psi : U_\psi \subset U \longrightarrow V_\psi \subset \mathbb{R}^p$ mit $\psi(a) = 0$, $\varphi : U_\varphi \subset \mathbb{R}^q \longrightarrow V_\varphi \subset \mathbb{R}^q$ mit $\varphi(b) = 0$ so, dass

$$\varphi \circ f \circ \psi^{-1}(x = (x_1, \ldots, x_p)) = (x_1, \ldots, x_r, \widetilde{f}(x)),$$

mit einer C^k–Abbildung $\widetilde{f}$.

2. Ist $rgDf(x) = r$ in einer ganzen Umgebung von a, so kann $\widetilde{f} = 0$ erreicht werden.

Teil 3

Einführung in die komplexe Analysis (V2B5)

Zusammenfassung. Vorlesung gehalten im SS 2003 an der Universität zu Köln und im SS 2013 and der Universität Bonn. Handschriftliches Skript wird auf der Homepage http://www.math.uni-bonn.de/ag/lesch/lesch.htmpl publiziert.

Status: Mostly to be typed

1. Komplexe Differenzierbarkeit, Cauchy-Riemann Differentialgleichungen

{sKAKDCR}

Wir setzen die reelle mehrdimensionale Analysis als bekannt voraus. In Analysis III wurden komplex differenzierbare Funktionen kurz diskutiert. Dies soll hier jedoch wiederholt und vertieft werden.

Der Körper $\mathbb{C}$ ist in natürlicher Weise ein 2-dimensionaler Vektorraum über $\mathbb{R}$ mit der kanonischen Basis $1, i$. Matrixdarstellungen von Endomorphismen beziehen sich im Folgenden auf diese Basis. Wir schreiben x, y für die entsprechenden kanonischen Koordinatenfunktionen bzw.

$$z = x + y \cdot i, \quad \overline{z} = x - y \cdot i.$$

2. Beispiele holomorpher Funktionen

3. Analytische Funktionen

4. Komplexe Kurvenintegrale

5. $\int_\gamma \frac{dz}{z-a}$

6. Der Cauchysche Integralsatz für Sterngebiete

7. Erste wichtige Konsequenzen des Cauchyschen Integralsatzes

7.1. Die Cauchysche Integralformel.

7.2. Potenzreihenentwicklung.

7.3. Satz von Liouville und Folgerungen.

7.4. Lokales Verhalten holomorpher Funktionen.

7.5. Maximum- und Minimumprinzip.

8. Holomorphe Logarithmen

9. Kreisringe, isolierte Singularitäten

9.1. Isolierte Singularitäten.

9.2. Meromorphe Funktionen.

10. Die Riemannsche Zahlenkugel

$\hat{\mathbb{C}} = C \cup \{\infty\}$

11. Möbius Transformationen

12. Konkrete Partialbruch- und Produktentwicklungen

12.1. Partialbruchzerlegung.

12.2. Produktentwicklung.

13. Die Gamma Funktion

14. Die Riemannsche ζ–Funktion I

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1, \tag{20}$$

heißt bekanntlich „Riemannsche ζ–Funktion“. Für $\Re s \geq s_0 > 1$ ist

$$|\frac{1}{n^s}| = \frac{1}{n^{\Re s}} \leq \frac{1}{n^{s_0}},$$

folglich ist die ζ–Reihe in jeder Halbebene $\Re s > s_0$ normal konvergent. Somit ist $\zeta \in \mathcal{O}(\{\Re s > 1\})$.

Die eindeutige Primfaktorzerlegung liefert

$$\begin{aligned}\zeta(s) &= \sum_{r_2, r_3, \ldots \geq 0} \left(2^{r_2} \cdot 3^{r_3} \cdot \ldots\right)^{-s} \\ &= \prod_{p \in \mathbb{P}} \sum_{r \geq 0} p^{-s} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}.\end{aligned}$$

Dabei bezeichnet $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ die Menge der Primzahlen. Dies ist das sogenannte *Euler-Produkt* für ζ.

Zusammenhang mit der Γ–Funktion:

Satz 14.1. $\Gamma(s) \cdot \zeta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt, \Re s > 1.$

14.1. Bernoulli-Zahlen. Nach Satz ?7.2? hat die Potenzreihe Label missing

$$\frac{z}{e^z - 1} =: \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

den Konvergenzradius 2π, da die linke Seite in $\mathbb{C} \setminus 2\pi i \mathbb{Z}$ holomorph ist. Die $(B_n)_n$ heißen *Bernoulli*-Zahlen. Sie sind rational, die ersten Werte sind

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0.$$

Da

$$\frac{z}{e^z - 1} + \frac{1}{2} z = \frac{z(e^z + 1)}{2(e^z - 1)} = \frac{z}{2} \frac{\cosh(z/2)}{\sinh(z/2)}$$

gerade ist, ist in der Tat

$$B_{2n+1} = 0, \quad n = 1, 2, 3, \ldots$$

Satz 14.2. *Die ζ–Funktion besitzt eine meromorphe Fortsetzung nach $\mathbb{C}$ mit einer einzigen Polstelle bei 1. Diese Polstelle ist einfach und es gilt*

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \quad s \to 1.$$

ζ hat einfache Nullstellen in $-2, -4, -6, \dots$.

14.2. Die geraden ζ–Werte.

15. Der globale Cauchysche Integralsatz

16. Der Residuensatz

17. Anwendungen des Residuensatzes I: Berechnung bestimmter Integrale

17.1. Trigonometrische Integrale $\int_0^{2\pi} R(\cos t, \sin t)dt$.

17.2. Uneigentliche Integrale $\int_{-\infty}^{\infty} f(x)dx$.

17.3. Uneigentliche Hauptwertintegrale $\int_{-\infty}^{\infty} f(x)dx$.

17.4. Uneigentliche Integrale $\int_0^{\infty} f(x)dx$, Mellin-Transformationen.

18. Anwendungen des Residuensatzes II: theoretische Anwendungen

19. Folgen holomorpher Funktionen

20. Der (kleine) Riemannsche Abbildungssatz

21. Die Riemannsche ζ–Funktion II

Analog zur Schleifenintegraldarstellung von Γ bezeichnet

$$I(z) := \int_{\gamma_\delta} \frac{w^{z-1}}{e^w - 1} dw, \quad 0 < \delta < 2\pi.$$

$I\mathcal{O}(\mathbb{C})$ ist eine ganze Funktion unabhängig von $0 < \delta < 2\pi$.

Satz 21.1. *Es gilt als Identität zwischen meromorphen Funktionen*

$$\zeta(z) = \frac{I(z)}{\Gamma(z)(e^{2\pi i} - 1)}, z \in \mathbb{C}.$$

Satz 21.2 (Funktionalgleichung der ζ–Funktion)**.**

$$\zeta(1-z) = 2(2\pi)^{-z} \cdot \cos(\pi/2z) \cdot \Gamma(z) \cdot \zeta(z)$$

Insbesondere ist $\zeta(-2n) = 0, n = 1, 2, 3, \dots$. Mit $\Psi(z) := \pi^{-z/2} \cdot \Gamma(z/2) \cdot \zeta(z)$ gilt

$$\Psi(1-z) = \Psi(z).$$

22. Der Weierstraßsche Produktsatz für $\mathbb{C}$

23. Kanonische Weierstraß-Produkte, der Hadamardsche Produktsatz

Ref Titchmarsh, Sec. 5.5

23.1. Verschärfung des Satzes von Liouville.

Satz 23.1. *Sei* $f : B(0, r_1) \longrightarrow B(0, r_2)$ *holomorph mit* $f(0) = 0$*. Dann ist* $|f(z)| \le \frac{r_2}{r_1}|z|$.

Satz 23.2. *Sei* f *holomorph auf der abgeschlossenen Kreisscheibe* $|z| \le R$*. Dann gilt für* $|z| < R$

$$|f(z)| \le \frac{2|z|}{R - |z|} \max_{\|\zeta| = R} \Re(f(\zeta)) + \frac{R + |z|}{R - |z|}|f(0)|.$$

23.2. Jensens' Formel. Wir erinnern an die Mittelwerteigenschaft. Sei u eine harmonische Funktion auf dem abgeschlossenen Ball $|z| \le r$.

24. Die Riemannsche ζ–Funktion III: Produktentwicklung

Erinnerung:

$$\zeta(0) = -\frac{1}{2}, \zeta'(0) = \log \sqrt{2\pi} = -\frac{1}{2} \log(2\pi)$$

$$\xi(z) := z(1 - z)\pi^{-z/2} \cdot \Gamma(z/2) \cdot \zeta(z).$$

Satz 24.1. $(1 - z)\zeta(z)$ *und* $\xi(z)$ *sind ganze Funktionen der Ordnung* $\alpha 1$.

Wir erhalten daher

$$(z - 1)\zeta(z) = e^{a+bz} \prod_{n=1}^{\infty} (1 + \frac{z}{2n})e^{-z/2n} \cdot \prod_{\varrho} (1 - \frac{z}{\varrho})e^{z/\varrho},$$

wobei ϱ die nichttrivialen Nullstellen $0 < \Re\varrho < 1$ durchläuft. Es ist

$$e^a = (z - 1)\zeta(z)|_{z=0} = -\zeta(0) = \frac{1}{2},$$

$$b = \frac{\partial_z (z - 1)\zeta(z)}{(z - 1)\zeta(z)}|_{z=0} = 2\zeta(0) - 2\zeta'(0) = -1 + \log 2\pi.$$

25. Der Primzahlsatz

26. Bloch, Schottky und Picard

27. Elliptische Funktionen

28. Miscellanea

Es folgt Material, welches ich mir zwar notiert habe, welches jedoch nicht zum Vortrag gekommen ist.

29. Anhang: Die Stirling Formel

30. Anhang: Harmonische Funktionen

Sei $f = u + iv \in \mathcal{O}(G)$ holomorph mit Realteil u und Imaginärteil v. Dann ist u bekanntlich harmonisch. Es gilt weiter

$$f' = \frac{1}{2}(\partial_x - i - \partial_y)(u + iv) = u_x - iu_y,$$

wobei wir die Cauchy-Riemann Gleichungen $v_x = -u_y, v_y = u_x$ benutzt haben. D.h. f' läßt sich alleine aus $\Re(f)$ berechnen!.

Satz 30.1. *Sei G ein Elementargebiet und $u \in C^2(G)$ harmonisch. Dann existiert $f \in \mathcal{O}(G)$ mit $u = \Re(f)$.*

Beweis. $g := u_x - iu_y$ ist holomorph, denn $g \in C^2(G)$ und $\Re(g)_x = u_{xx} = -u_{yy} = \Im(g)_y, \Re(g)_y = u_{xy} = -(-u_y)_x = -\Im(g)_x$. Da G ein Elementargebiet ist, existiert $\widetilde{f} \in \mathcal{O}(G)$ mit $\widetilde{f}' = g$. Dann ist $\square$

Part 4

Globale Analysis II (V3B4)

Abstract. Vorlesung gehalten in 1997 (Lineare Partielle Differentialgleichungen) an der Humboldt-Universität zu Berlin, im SS 2001 (Mikrolokale Analysis) an der Universität zu Köln und im SS 2006, WS 2009/2010, SS 2011 and der Universität Bonn. Vielen Dank an Thomas Petig für die Hilfe bei der LaTeX-Umsetzung.

This course covers distributions, the Fourier transform and the theory of pseudodifferential operators.

Status: Pre-Alpha, typed 2012, not checked!!

CHAPTER 2

Distributions, Fourier transform, and the Schwartz kernel theorem

1. Function spaces

1.1. L^p**-spaces.** I assume that Lebesgue theory is known. We will freely use the *Banach spaces* $L^p(X,\mu), 1 \leq p \leq \infty$, for any measure space (X,μ). Most of the time $X \subset \mathbb{R}^n$ measurable (e.g. open or closed) and $\mu = \rho d^n x$ with a locally integrable density ρ.

1.2. $C_b^k(\Omega)$**.** For $\Omega \subset \mathbb{R}^n$ open we have the Banach spaces

$$\begin{aligned} C_b^k(\Omega) &= \big\{ f : \Omega \to \mathbb{C} \bigm| f\ k-\text{times continuously differentiable, } D^\alpha f \text{ bounded, } |\alpha| \leq k \big\} \\ \|f\|_{C^k} &= \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{\infty,\Omega}, \\ \|g\|_{\infty,\Omega} &= \sup_{x\in\Omega} |g(x)|, \\ C_{(0)}^k(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \subset C_b^k(\Omega). \end{aligned}$$

1.3. The Schwartz space. Put for multiindices $\alpha, \beta \in \mathbb{Z}_+^n$ and $f \in \mathrm{C}^\infty(\mathbb{R}^n)$.

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

$p_{\alpha,\beta}$ is a seminorm on the Schwartz space

$$\mathscr{S}(\mathbb{R}^n) = \big\{ f \in \mathrm{C}^\infty(\mathbb{R}^n) \bigm| \forall_{\gamma,\delta\in\mathbb{Z}_+^n} : p_{\gamma,\delta}(f) < \infty \big\}$$

of rapidly decreasing functions. $(\mathscr{S}(\mathbb{R}^n), \{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}_+^n})$ is a Fréchet space.

1.4. $\mathscr{E}(\Omega)$**.** Let $\Omega \subset \mathbb{R}^n$ open and put for $\alpha \in \mathbb{Z}_+^n$ and $K \subset \Omega$ compact

$$p_{\alpha,K} := \sup_{x\in K} |D^\alpha f(x)|.$$

$p_{\alpha,K}$ is a seminorm and we put

$$\mathscr{E}(\Omega) := \Big(\mathrm{C}^\infty(\Omega), \{p_{\alpha,K}\}_{\alpha\in\mathbb{Z}_+^n, K\subset\Omega \text{ compact}} \Big).$$

A priori the number of seminorms $\#\{p_{\alpha,K}\}$ is uncountable. However, it is very easy to see that $\mathscr{E}(\Omega)$ is a Fréchet space. Namely let $K_j \subset \Omega$ be compact sets such that

$$\Omega = \bigcup_{j=1}^{\infty} K_j, \qquad\qquad K_1 \subset \overset{\circ}{K}_2 \subset \overset{\circ}{K}_3 \subset \cdots \tag{21}$$

{I.1.4.exhaustion}

The *countable* family of seminorms $\{p_{\alpha,K_{ij}}\}_{\alpha\in\mathbb{Z}_+^n, j\in\mathbb{N}}$ induces the topology on $\mathscr{E}(\Omega)$.

1.5. $\mathscr{D}(\Omega)$**.** Put for $K \subset \Omega$ compact

$$\mathscr{D}_K(\Omega) := \{f \in \mathrm{C}_c^\infty(\Omega) \mid \operatorname{supp}(f) \subset K\}$$

$\mathscr{D}_K(\Omega)$ with seminorms $(p_{\alpha,K})_{\alpha\in\mathbb{Z}_+^n}$ is a Fréchet space. Moreover, if $K \subset K'$ then $\mathscr{D}_K(\Omega) \hookrightarrow \mathscr{D}_{K'}(\Omega)$ is a topological inclusion. Now choose an exhaustion as in equation (21). Then

$$\mathscr{D}(\Omega) := \bigcup_{n=1}^{\infty} \mathscr{D}_{K_n}(\Omega)$$

becomes an LF-space.

{p:I.1.1}

Proposition 1.1.

(1) *The topology of $\mathscr{D}(\Omega)$ is independent of the choice of an exhaustion (K_n). $U \subset \mathscr{D}(\Omega)$ is open iff for each compact set $K \subset \Omega$ the set $U \cap \mathscr{D}_K(\Omega)$ is open in $\mathscr{D}_K(\Omega)$. A seminorm p on $\mathscr{D}(\Omega)$ is continuous iff $p\big|_{\mathscr{D}_K(\Omega)}$ is continuous for each compact subset $K \subset \Omega$. Differently put $\mathscr{D}_K(\Omega) \hookrightarrow \mathscr{D}(\Omega)$ is a topological inclusion.*

(2) *A sequence $(f_n) \subset \mathscr{D}(\Omega)$ converges to $f \in \mathscr{D}(\Omega)$ iff there exits a compact subset $K \subset \Omega$ such that*

$$\forall_n \operatorname{supp} f_n \subset K \tag{22}$$

and

$$\forall_{\alpha\in\mathbb{Z}_+^n} D^\alpha f_n \xrightarrow{n\to\infty} D^\alpha f \ \textit{uniformly}\ . \tag{23}$$

Proof. (1) We prove the statement about seminorms. If $p\big|_{\mathscr{D}_K(\Omega)}$ is continuous for all compact $K \subset \Omega$ then in particular for K_n. Thus p is continuous by definition. Conversely, let p be a continuous seminorm on $\mathscr{D}(\Omega)$ and let $K \subset \Omega$ be compact. By equation (21) is $\overset{\circ}{K}_j$ an open covering of K and hence there is a $N \in \mathbb{N}$ with $K \subset \overset{\circ}{K}_N \subset K_n$. Then by definition $p\big|_{\mathscr{D}_{K_N}}(\Omega)$ is continuous. Thus

$$p \le C \max_{j=1}^{r} p_{\alpha_j,K_N} \text{ for some } \alpha_1, \ldots, \alpha_r.$$

But then

$$p\big|_{\mathscr{D}_K(\Omega)} \le C \max_{j=1}^{r} p_{\alpha_j,K}$$

hence $p\big|_{\mathscr{D}_K(\Omega)}$ is continuous.

(2) This follows from Theorem 1.5 of the appendix.

$\square$

1.6. Variants of $\mathscr{D}(\Omega)$. $C_c^k(\Omega)$ is in a natural way an LF space with defining sequence $C_K^k(\Omega)$. The case $k = 0$ is maybe of some interest.

1.7. $L^p_{\mathbf{comp}}(\Omega), L^p_{\mathbf{loc}}(\Omega)$. Put

$$L^p_{\text{comp}}(\Omega) = \{f \in L^p(\Omega) \mid f \text{ has a compactly supported representative}\}$$

We will give a rigorous definition of support for elements in $L^p(\Omega)$ later; but it is not needed here. $L^p_{\text{comp}}(\Omega)$ is an LF-space with defining sequence $L^p(K_n)$; (K_n) an exhaustion as before.

$$L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ measurable for } K \subset \Omega \text{ compact, } \int_K |f|^p < \infty\}/\{f = 0 \text{ a.e.}\}$$

$L^p_{\text{loc}}(\Omega)$ is a F-space with seminorms

$$\|f\|_{p,K_n} := \left(\int_{K_n} |f|^p\right)^{\frac{1}{p}}.$$

1.8. $C^k(\Omega)$. $C^k(\Omega)$ is a F-space with seminorms $p_{\alpha,K}$, $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha| \leq k$, $K \subset \Omega$ compact.

2. Convolution, regularization

From integration theory (e.g. Analysis III) we recall that for $\Omega \subset \mathbb{R}^n$ open the space $C_c(\Omega)$ of compactly supported continuous functions is dense in $L^p(\Omega), 1 \leq p < \infty$.

Furthermore, recall that $f, g \in Ł^1(\mathbb{R}^n)$. The function $f(.)g(x - .)$ is integrable for a.e. $x \in \mathbb{R}^n$ and that

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}^n} f(y)g(x-y)dy \\ &= \int_{\mathbb{R}^n} f(x-y)g(y)dy = (f * g)(x)\end{aligned}$$

is integrable and

$$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}.$$

Hence $(L^1(\mathbb{R}^n), *)$ is a *Banach Algebra.*

{prop:I.2.1}

Proposition 2.1. *Let $1 \leq p \leq \infty$ and $f \in Ł^1(\mathbb{R}^n)$, $g \in Ł^p(\mathbb{R}^n)$. Then $f * g$ is defined a.e. and*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$$

In other words, the convolution product, say $C_c(\mathbb{R}^n) \times C_c(\mathbb{R}^n) \to C_c(\mathbb{R}^n)$, extends by continuity to a bilinear map $L^1 \times L^p \to L^p$ of bound 1.

PROOF.

(1) $p = \infty$: So g is (essentially) bounded and thus $y \mapsto f(y)g(x-y)$ is integrable. Furthermore

$$|f * g(x)| \leq \int_{\mathbb{R}^n} |f(x)g(x-y)|dy \leq \|g\|_\infty \int |f| = \|f\|_{L^1}\|g\|_{L^\infty}.$$

(2) $p = 1$: See above (and Analysis III).

$$\begin{aligned} \int \left[\int |f(y)g(x-y)|dy \right] dx &\overset{\text{Tonelli}}{=} \int |f(y)| \int |g(x-y)|dxdy \\ &\leq \|g\|_\infty \int |f| = \|f\|_{L^1}\|g\|_{L^1} \end{aligned}$$

and the claim follows.

(3) $1 < p < \infty$: Let $q := \frac{p}{p=1}$, so we have $\frac{1}{q} + \frac{1}{p} = 1$. Then

$$\begin{aligned} |\int |f(x-y)|^{\frac{1}{p}+\frac{1}{q}}|g(y)|dy|^p &\overset{\text{Hölder}}{=} \left(\int |f(x-y)||g(y)|^p dy\right)\left(\int |f(y)|dy\right)^{\frac{p}{q}} \\ &= |f| * |g|^p(x)\|f\|_{L^1}^{\frac{p}{q}}. \end{aligned}$$

Integration over x yields to

$$\begin{aligned} \int |\int |f(x-y)||g(y)|dy|^p dx &\leq \|_{\|} L^{1^{1+\frac{p}{q}}}\|g\|_{L^p}^p, \quad 1 + \frac{p}{q} = p \\ &= (\|f\|_{L^1}\|g\|_{L^p})^p. \end{aligned}$$

Thus, by Fubini, $y \mapsto f(x-y)g(y)$ is integrable for a.e. $x \in \mathbb{R}^n$ and

$$\left(\int |\int f(x-y)g(y)dy|^p dx\right)^{\frac{1}{p}} \leq \|f\|_{L^1}\|g\|_{L^p}.$$

□

Let $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $\phi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$. Then writing

$$(f * \phi)(x) = \int \phi(x-y)f(y)dy$$

we see that we can differentiate under the integral, hence $f * \phi \in \mathrm{C}^\infty(\mathbb{R}^n)$ and

$$D^\alpha(f * \phi) = f * D^\alpha \phi, \quad \alpha \in \mathbb{Z}_+^n.$$

Similarly, if $f \in C^k(\mathbb{R}^n)$ then for $|\alpha| \leq k$ also

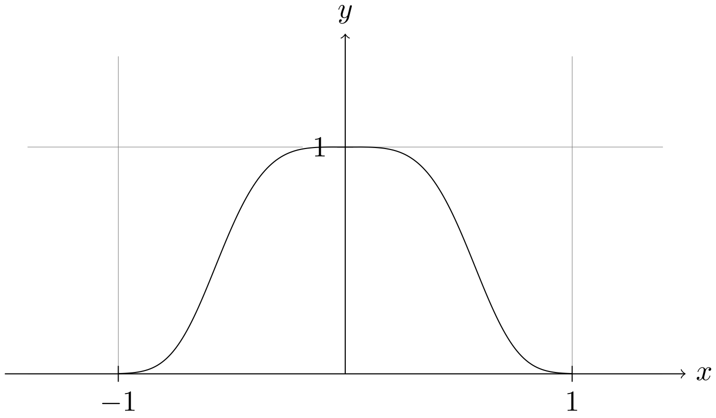
$$D^\alpha(f * \phi) = (D^\alpha f) * \phi.$$

Now fix a function $\varrho \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ with the following properties

$$0 \leq \varrho, \tag{24}$$

$$\int \varrho = 1, \tag{25}$$

$$\varrho(x) = 0 \text{ if } |x| \geq 1. \tag{26}$$



For $\lambda > 0$ put $\varrho_\lambda(x) = \lambda^{-n}\varrho\left(\frac{x}{\lambda}\right)$ certainly.

$$\int \varrho_\lambda = 1, \tag{27}$$

$$\varrho_\lambda(x) = 0 \text{ if } |x| > \lambda \tag{28}$$

We will give some important applications of convolution by ϱ_λ.

Lemma 2.2. *Given a compact set $K \subset \mathbb{R}^n$ and an open neighborhood $\Omega \subset K$ there exists $\chi \in \mathrm{C}_c^\infty(\Omega)$ such that*

1. *$0 \leq \chi(x) \leq 1$ for all $x \in \Omega$,*
2. *$\chi(x) = 1$ for all x in a neighborhood V of K.*

PROOF. ☐

{I.2.3}

Lemma 2.3. *Let $f \in C(\mathbb{R}^n)$. Then $\varrho_\lambda * f$ converges as $\lambda \to 0$ to f uniformly on compact subsets of $\mathbb{R}^n$. I.e. $\varrho_\lambda * f \to f$ in the natural Fréchet topology of $C(\mathbb{R}^n)$.*

PROOF. ☐

Remark 2.4. From Lemma 2.3 one deduces several denseness results. We do not list all of them here. E.g. $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$. We leave this as an exercise.

{thm:I.2.5}

Theorem 2.5. *Let X be one of the following spaces:*

1. $L^p(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$, $L^p_{\mathrm{comp}}(\Omega)$, $1 \leq p < \infty$
2. $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, $\mathscr{S}(\Omega)$, $C_c^k(\Omega)$, $C^k(\mathbb{R}^n)$

Then $\varrho_\lambda * f$ converges in X to f as $\lambda \to 0$.

PROOF. $\square$

{I.2.6}

Corollary 2.6. *Let $f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open. If $\int_\Omega f\varphi = 0$ for all $\varphi \in \mathscr{D}(\Omega)$ then $f = 0$.*

PROOF. $\square$

3. Distributions

Unless otherwise said, dual spaces will be equipped with the weak topology.

{def:I.3.1}

Definition 3.1.
- *$\mathscr{D}'(\Omega) := (\mathscr{D}(\Omega))'$ is the space of distributions*
- *$\mathscr{E}'(\Omega) := (\mathscr{E}(\Omega))'$ is the space of distributions with compact support*
- *$\mathscr{S}'(\mathbb{R}^n) := (\mathscr{S}(\mathbb{R}^n))'$ is the space of tempered distributions*

Remark 3.2.
- A linear functional $T : \mathscr{D}(\Omega) \longrightarrow \mathbb{C}$ is a distribution iff for each compact set $K \in \Omega$ there exists $N \in \mathbb{Z}_+$, $C \in \mathbb{R}_+$ such that:
$$|T(f)| \leq C \max_{|\alpha| \leq N} p_{\alpha,K}(f)$$
for all $f \in \mathscr{D}_K(\Omega)$.
- A linear functional $T : \mathscr{E}(\Omega) \longrightarrow \mathbb{C}$ is a distribution iff there exists a compact set $K \in \Omega$ and $N \in \mathbb{Z}_+$, $C \in \mathbb{R}_+$ such that:
$$|T(f)| \leq C \max_{|\alpha| \leq N} p_{\alpha,K}(f)$$
for all $f \in \mathscr{E}(\Omega)$.

Examples 3.3.

(1) We consider $L^1_{\mathrm{loc}}(\Omega)$[1] as a subspace of $\mathscr{D}'(\Omega)$ by putting
$$T_\bullet : L^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathscr{D}'(\Omega), \quad T_f(\varphi) = \int_\Omega f\varphi.$$
Indeed $T_\bullet$ is a continuous injection: To see that $T_f \in \mathscr{D}'$, let $K \subset \Omega$ compact, $\varphi \in \mathscr{D}_K(\Omega)$. Then
$$|T_f(\varphi)| = |\int_\Omega f\varphi| \leq \left(\int_K |f|\right) \sup_{x \in K} |\varphi(x)|. \tag{29}$$
{I.3.2.1}

So we choose $N = 0$. The injectivity follows from Corollary 2.6. Regarding Continuity: (29) shows that for *fixed* φ $|\langle T_f, \varphi\rangle| \leq c\|f\|_{L^1(K)}$ holds!

(2) δ-Distribution: For $a \in \Omega$ put $\delta_a : \mathscr{D}(\Omega) \to \mathbb{C}$, $\varphi \mapsto \varphi(a)$. Certainly $|\delta_a(\varphi)| \leq |\varphi(a)| = p_{0,a}(\varphi)$, so this is a distribution. There is no $f \in L^1_{\mathrm{loc}}(\Omega)$ with $\delta_a = T_f$. For if $\delta_a = T_f$ then for all $\varphi \in \mathscr{D}(\Omega \setminus \{a\})$
$$0 = \delta_a(\varphi) = \int_{\Omega\setminus\{a\}} f\varphi$$
and hence by Corollary 2.6 we have $f = 0$ a.e. on $\Omega \setminus \{a\}$ and thus on Ω. But then $\int_\Omega f\varphi = 0$ for all $\varphi \in \mathscr{D}(\Omega)$, which is a contradiction.

---

[1] Note that $L^p_{\mathrm{loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for $1 \leq p < \infty$

For the functions ρ_λ of Section 2, however, we have by Lemma 2.3 for $\varphi \in \mathscr{D}(\Omega)$

$$\begin{aligned} T_\rho(\varphi) &= \int_\Omega \rho_\lambda(y)\varphi(y)dy \\ &= \int_{\mathbb{R}^n} \rho_\lambda(y)\varphi(y)dy, && \text{for } \lambda \text{ small enough} \\ &= (\rho_\lambda * \varphi)(0) \xrightarrow{\lambda\to 0} \varphi(0) = \delta_0(\varphi). \end{aligned}$$

Thus we have proved that $\lim_{\lambda\to 0} T_{\rho_\lambda} = \delta_0$ in $\mathscr{D}'(\Omega)$.

(3) Distributions give us a lot of freedom. E.g. $T = \sum_{n=1}^{\infty} T_{\cos(n.)}$ converges in $\mathscr{D}'(\mathbb{R})$. Similarly $\int e^{ixy}dy$ has a meaning when interpreted in $\mathscr{D}'(\Omega)$. {ex:I.3.7.2}

Definition 3.4. *(Differentiation of distributions) For $T \in \mathscr{D}'(\Omega)$ (or $\mathscr{E}'(\Omega)$, $\mathscr{S}'(\mathbb{R}^n)$) and $\alpha \in \mathbb{Z}_+^n$ put* {def:I.3.5}

$$\langle D^\alpha T, \phi\rangle := (-1)^{|\alpha|} \langle T, D^\alpha \phi\rangle .$$

Remark 3.5. This definition is consistent with the differentiation of functions under the embeddings

$$C^k(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega) \hookrightarrow \mathscr{D}'(\Omega).$$

Namely, if $f \in C^k(\Omega)$, $\phi \in \mathscr{D}(\Omega)$ then by using integration by parts we find

$$\begin{aligned} \langle D^\alpha T_f, \phi\rangle &= (-1)^\alpha \langle T_f, D^\alpha\phi\rangle \\ &= (-1)^\alpha \int_\Omega f D^\alpha \phi \\ &= \int_\Omega (D^\alpha f)\phi \\ &= \langle T_{D^\alpha f}, \phi\rangle . \end{aligned}$$

From now on we can differentiate any L^1_{loc}-function. But what happens if it is not differentiable in the ordinary sense?

Example 3.6.

Lemma 3.7. *Let $I \subset \mathbb{R}$ be an open interval and $T \in \mathscr{D}'(I)$ with $\partial T = 0$. Then $T = c$ is a constant function.*

Proposition 3.8. *Let $I \subset \mathbb{R}$ be an open interval and $f \in L^1_{\text{loc}}(I)$. For $x_0 \in I$ put*

$$F(x) := \int_{x_0}^{x} f. \tag{30}$$ {I.3.10.1}

- *F is continuous on I and $\partial F = f$ in $\mathscr{D}'(I)$.*
- *If $T \in \mathscr{D}'(I)$ with $\partial T = f$ then there is a $c \in \mathbb{C}$ such that $T = F + c$.*

Remark 3.9. Functions F of the form (30) are called absolutely continuous. One can show that F is differentiable a.e. (in the sense of Analysis I) and that $F' = f$ a.e.. We do not need this. The identity $\partial F = f$ in $\mathscr{D}'(\Omega)$ is perfectly sufficient for our purposes.

Proof. $\square$

Theorem 3.10. (1) *Let $I \subset \mathbb{R}$ be an open interval and let $T \in \mathscr{D}'(I^n)$ with $\partial_n T = 0$. Then there is a $T_1 \in \mathscr{D}'(I^{n-1})$ such that for $\varphi \in \mathscr{D}(I)$*

$$\langle T, \varphi\rangle = \langle T_1, \int \varphi(., x_n)dx_n\rangle.$$

(2) *Let $\Omega \subset \mathbb{R}^n$ be open and $T \in \mathscr{D}'(\Omega)$ with $\partial_j T = 0$ $j = 1, \cdots, n$. Then T is a locally constant function.*

4. Fourier transform

{def:I.4.1}

Definition 4.1. *For $f \in L^1(\mathbb{R}^n)$,*

$$\hat{f}(\xi) := (\mathscr{F}f)(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx$$

is called the Fourier transform of f.

4.1. Facts and formulas.

{fact:I.4.1}

(1)

$$D_j\mathscr{F}f = -\mathscr{F}(x_j f) \qquad \text{if } f, x_j f \in L^1(\mathbb{R}^n)$$
$$\xi_j\hat{f}(\xi) = \widehat{D_j f}(\xi) \qquad \text{if } f, D_j f \in L^1(\mathbb{R}^n)$$

bessere Optik?

(2)

$$(f * g)\hat{} = \hat{f}\hat{g}$$
$$(f \cdot g) = (2\pi)^{-n}\hat{f} * \hat{g}$$

(3) $\mathscr{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is continuous, (so called Riemann-Lebesgue Lemma) more precisely

$$\|\mathscr{F}f\|_{\infty,\mathbb{R}^n} \le \|f\|_{L^1}$$
$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$$

(4)

$$\int_{\mathbb{R}^n} \hat{f}g = \int_{\mathbb{R}^n} f\hat{g}, \qquad f, g \in L^1$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dxg(\xi)d\xi = \int f(x)\int e^{-i\langle x,\xi\rangle} g(\xi)d\xi dx$$

(5)

$$\mathscr{F}(f(\lambda .)) = \lambda^{-n}(\mathscr{F}f)\left(\frac{\cdot}{\lambda}\right)$$
$$\int e^{-i\langle x,\xi\rangle} f(\lambda x)dx = \lambda^{-n}\int e^{-i\langle x,\frac{\xi}{\lambda}\rangle} f(x)dx$$

(6) $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ continuous (hence also $\mathscr{S}' \to \mathscr{S}'$) follows immediately from 1.

$$\begin{aligned}|\xi^\alpha D^\beta \mathscr{F} f(\xi)| &= |\mathscr{F}(D^\alpha(x^\beta f))(\xi)| \\ &\leq \int_{\mathbb{R}^n} D^\alpha |x^\beta f(x)| dx \\ &\leq \sup_x |(1+|x|^2)^{\frac{n_1}{2}} D^\alpha(x^\beta f)| \int (1+|x|^2)^{-\frac{n+1}{2}} dx\end{aligned}$$

(7)

$$\mathscr{F}\left(e^{-\frac{\|\cdot\|^2}{2}}\right) = (2\pi)^{\frac{n}{2}} e^{-\frac{\|\cdot\|^2}{2}}$$

Proof. $e^{-\frac{\|x\|^2}{2}} = \prod_{j=1}^n e^{-\frac{x_j^2}{2}}$ so it is sufficient to check for $n = 1$. Consider:

$$\begin{aligned}f(t) &= \widehat{\left(e^{-\frac{|.|^2}{2}}\right)}(t) \\ &= \int e^{-ixt} e^{-\frac{x^2}{2}} dx \\ f'(t) &= ti \int e^{-ixt} \underbrace{\left(-xe^{-\frac{x^2}{2}}\right)}_{=\frac{d}{dx}e^{-\frac{x^2}{2}}} dx \\ &= -i \int -ite^{-ixt} e^{-\frac{x^2}{2}} dx \\ &= -tf(t)\end{aligned}$$

Thus $f(t) = f(0)e^{-\frac{t^2}{2}}$ and $f(0) = \int e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$. □

(8) If $f, \hat{f} \in L^1$ then $f, \hat{f} \in C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and

$$\begin{aligned}f(x) &= (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\xi \\ &= \int e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\end{aligned}$$

Let $g(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}}$. $g \in L^1$, $g \geq 0$, $\int g = 1$ and hence by (a slight generalization of) Theorem 2.5, we have $\lim_{\lambda \to 0+} f * g_\lambda = f$ in L^1. And on the other hand by (7) and (8), we have $g_\lambda = (2\pi)^{-n}\mathscr{F}\left(e^{-gl^2\frac{\|.\|^2}{2}}\right)$. Hence

$$\begin{aligned} f * g_\lambda(x) &= (2\pi)^{-n}\int f(x-y)\mathscr{F}\left(e^{-\lambda^2\frac{\|.\|^2}{2}}\right)(y)dy \\ &\underset{(4)}{=} (2\pi)^2 \int \underbrace{\left(\mathscr{F}_{y\rightsquigarrow\xi} f(x-y)\right)(\xi)}_{\int e^{-i\langle\xi,y\rangle}f(x-y)dy = e^{-i\langle x,\xi\rangle}\hat{f}(\xi)} e^{-\lambda^2\frac{\|\xi\|^2}{2}} d\xi \qquad (31) \\ &\xrightarrow[\text{Lebesgue}]{\lambda\to 0} (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\hat{f}(\xi)d\xi. \end{aligned}$$

{eqn:I.4.4.1}

Hence $f * g_\lambda$ converges in L^1 to f and pointwise to the rhs(?) of (31), thus the claim follows.

Theorem 4.2. *$\mathscr{F}$ is a continuous isomorphism with (continuous) inverse $\overline{\mathscr{F}}f = \mathscr{F}(f(-.)) : \mathscr{S}^{(1)} \to \mathscr{S}^{(1)}$. Moreover it is a unitary isomorphism $L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, d\xi)$.*

PROOF. ☐

Because of its importance the definition of $\mathscr{F}$ in $\mathscr{S}'$ us singled out as a:

Definition 4.3. *For $T \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we define*

$$\langle \mathscr{F}T, \varphi\rangle = \langle T, \mathscr{F}\varphi\rangle.$$

$\mathscr{F}$ is a continuous linear bijection $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$.

Examples 4.4. (1) $a \in \mathbb{R}^n$:

$$\begin{aligned} \langle \mathscr{F}\delta_a, \varphi\rangle &= \langle \delta_a, \hat{\varphi}\rangle \\ &= \hat{\varphi}(a) \\ &= \langle e^{-i\langle a,.\rangle}, \varphi\rangle \\ \text{So:}\quad \mathscr{F}\delta_a &= e^{-i\langle a,.\rangle} \end{aligned}$$

(2) Exercise:

$$\int_{\|x\|\le R} e^{-i\langle .,x\rangle} d^n x \xrightarrow[\mathscr{S}'(\mathbb{R}^n)]{R\to\infty} (2\pi)^n\delta_0$$

(3) $H = 1_{[0,\infty)}$

$$f_\varepsilon := \begin{cases} e^{-\varepsilon x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

For $\varphi \in \mathscr{S}(\mathbb{R})$

$$\langle f_\varepsilon, \varphi\rangle = \int_0^\infty e^{-\varepsilon x}\varphi(x)dx \xrightarrow{\varepsilon\searrow 0} \int_0^\infty \varphi = \langle H, \varphi\rangle$$

thus $f_\varepsilon \xrightarrow{\varepsilon\searrow 0} H$ in $\mathscr{S}'(\mathbb{R})$ and hence $\hat{f}_\varepsilon \to \hat{H}$ in $\mathscr{S}'$.

$$\begin{aligned}\hat{f}_\varepsilon &= \int_0^\infty e^{-ix\xi-\varepsilon x}dx \\ &= \frac{-1}{i\xi+\varepsilon}\bigg|_0^\infty \\ &= \frac{1}{i\xi+\varepsilon}\end{aligned}$$

we have proved:

$$\hat{H} = \frac{1}{i\xi+0}$$

(shorthand for $\lim_{\varepsilon\searrow 0}\frac{1}{i\xi+\varepsilon}$) Note that $\xi \mapsto \frac{1}{i\xi}$ is not locally integrable. However, from example 3 we infer:

$$\int_{-\infty}^{\infty}\frac{1}{i\xi+\varepsilon}\varphi(\xi)d\xi = \int_{|\xi|\geq 1}\frac{\varphi(\xi)}{i\xi+\varepsilon}d\xi + \int_{-1}^{1}\frac{\varphi(\xi)-\varphi(0)}{i\xi+\varepsilon}d\xi + \varphi(0)\underbrace{\int_{-1}^{1}\frac{d\xi}{i\xi+\varepsilon}}_{=\frac{1}{i}\log(\xi-i\varepsilon)\big|_{-1}^{1}}$$

$$\xrightarrow{\varepsilon\to 0}\frac{1}{i}\langle\partial\log|.|,\varphi\rangle + \pi\langle\delta_0,\varphi\rangle$$

Second proof:

1.
$$\begin{aligned}
\langle \partial \log|.|, \varphi\rangle &= -\langle \log|.|, \varphi'\rangle \\
&= -\int_{\mathbb{R}^n} \log|x| \varphi'(x) dx \\
&= \lim_{\delta\to 0} -\int_{|x|\leq\delta} \log|x|\varphi'(x)dx \\
&= \lim_{\delta\to 0}\left\{ \log\delta(\varphi(\delta)-\varphi(-\delta)) + \int_{|x|\geq\delta} \frac{\varphi(x)}{x}dx \right\} \\
&= \lim_{\delta\to 0}\int_{|x|\geq\delta}\frac{\varphi(x)}{x}dx \\
&=: \text{H.W.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx
\end{aligned}$$

2.
$$\begin{aligned}
\lim_{R\to\infty} 1_{[0,R]} &= 1_{[0,\infty)} \quad \text{in } \mathscr{S}'(\mathbb{R}) \\
\widehat{1_{[0,R]}}(\xi) &= \int_0^R e^{-ix\xi}dx \\
&= \left.\frac{-R^{-ix\xi}}{i\xi}\right|_0^R \\
&= \frac{1}{i\xi}\left(1-e^{-i\xi R}\right)
\end{aligned}$$

Thus
$$\begin{aligned}
\langle \widehat{1_{[0,R]}}(\xi), \varphi\rangle &= \int_{\mathbb{R}}\frac{1}{i\xi}\left(1-e^{-i\xi R}\right)\varphi(\xi)d\xi \\
&= \text{H.W.}\int_{\mathbb{R}}\frac{1}{i\xi}\varphi(\xi)d\xi - \text{H.W.}\int_{\mathbb{R}}\frac{e^{-i\xi R}}{i\xi}\varphi(\xi)d\xi \\
\text{H.W.}\int_{\mathbb{R}}\frac{e^{-i\xi R}}{i\xi}\varphi(\xi)d\xi &= \underbrace{\int_{\mathbb{R}} e^{-i\xi R}\frac{\varphi(\xi)-\varphi(0)}{i\xi}d\xi}_{\xrightarrow[R\to\infty]{0}} + \varphi(0)\ \underbrace{\text{H.W.}\int_{\mathbb{R}}\frac{e^{-i\xi R}}{i\xi}d\xi}_{\text{independent of } R,\ \xi R \rightsquigarrow \xi} \\
&\xrightarrow{R\to\infty} \varphi(0)\int_{\mathbb{R}}\frac{-\sin\xi}{\xi}d\xi \quad (\cos \text{ even!})
\end{aligned}$$

Finally
$$\begin{aligned}
\mathscr{F}'H &= \text{H.W.}\frac{1}{i\xi} + \int_{\mathbb{R}}\frac{\sin\xi}{\xi}d\xi\delta_0 \\
&\overset{!}{=} \text{H.W.}\frac{1}{i\xi} + \pi\delta_0 \qquad \blacksquare
\end{aligned}$$

We have proved:
$$\int_{-\infty}^{\infty}\frac{\sin\xi}{\xi}d\xi = \pi.$$

Remark 4.5. Let $T \in \mathscr{E}'(\Omega)$. For any $z \in \mathbb{C}$ the function $e_z : x \mapsto e^{-i\langle x,z\rangle}$ is certainly smooth on Ω. Moreover $\mathbb{C}^n \ni z \mapsto e_z \in \mathscr{E}'(\Omega)$ is easily seen to be complex analytic. Thus the Fourier transform

$$\hat{T}(z) := \langle T, e_z\rangle$$

is an entire analytic function of $z \in \mathbb{C}^n$. Note also that $\mathscr{E}'(\Omega)$ naturally embeds into $\mathscr{S}(\mathbb{R}^n)$ by putting

$$\langle iT, \varphi\rangle := \langle T, \varphi\big|_{\Omega}\rangle, \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Since $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$ and any $\varphi \in \mathscr{D}(\Omega)$ is the restriction to Ω of a Schwartz function it follows that iT is injective. Certainly $\hat{T}(\xi) = \mathscr{F}(iT)(\xi)$. Hence $\hat{T} = 0$ iff $T = 0$.

5. (Singular) support

Distributions can be multiplied by functions. E.g. $\varphi \in \mathscr{D}(\Omega)$, $T \in \mathscr{D}'(\Omega)$. Then for $\psi \in \mathscr{E}(\Omega)$ we put

$$\langle \varphi T, \psi\rangle := \langle T, \varphi\psi\rangle$$

and hence there are bilinear continuous maps:

$$\begin{aligned} &\mathscr{D}(\Omega) \times \mathscr{D}'(\Omega) \to \mathscr{E}'(\Omega), \qquad &&\mathscr{E}(\Omega) \times \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega),\\ &\mathscr{E}(\Omega) \times \mathscr{E}'(\Omega) \to \mathscr{E}'(\Omega), \qquad &&\mathscr{S}(\Omega) \times \mathscr{S}'(\Omega) \to \mathscr{S}'(\Omega). \end{aligned}$$

etc.

Definition 5.1. *For $T \in \mathscr{D}'(\Omega)$, $U \subset \Omega$ open we put*

$$T\big|_U := T\big|_{\mathscr{D}(U)}.$$

Similarly for $\mathscr{S}$ and $\mathscr{E}$.

Like functions, distributions form a sheaf! (Needs, of course, to check the sheaf axioms)

{prop:I.5.2}

Proposition 5.2. *Let $\Omega = \bigcup_{i\in I} U_i$, U_i open and $T_i \in \mathscr{D}'(U_i)$ with*

$$\forall_{i,j\in I} T_i\big|_{U_i\cap U_j} = T_j\big|_{U_i\cap U_j}.$$

Then there is a unique $T \in \mathscr{D}'(\Omega)$ such that $\forall_{i,j\in I} T\big|_{U_i} = T_i$.

Proof. Let (ρ_i) be a partition of the unity. $T = \sum \rho_i T_i$. Details are exercise. $\square$

We see immediately

$$\begin{aligned} T \text{ vanishes on } U &:\Leftrightarrow T\big|_U = 0 \\ &\Leftrightarrow \forall_{\varphi\in\mathscr{D}(U)}\langle T,\varphi\rangle = 0 \\ T\big|_U \text{ smooth } &\Leftrightarrow \exists_{\psi\in C^\infty(U)} T\big|_U = T_\psi \\ &\Leftrightarrow \exists_{\psi\in C^\infty(U)}\forall_{\varphi\in\mathscr{D}(U)}\langle T,\varphi\rangle = \int \psi\varphi \end{aligned}$$

Definition 5.3. *Let $T \in \mathscr{D}'(\Omega)$. We define the support of T as*

$$\operatorname{supp} T := \Omega \setminus \bigcup_{\Omega'\subset\Omega \text{ open}, T|_{\Omega'}=0} \Omega'$$

and the singular support of T as

$$\text{sing supp}\, T := \Omega \setminus \bigcup_{\Omega'\subset\Omega \text{ open}, T|_{\Omega'} \text{ smooth}} \Omega'.$$

$\operatorname{supp} T$ and $\text{sing supp}\, T$ are closed subsets of Ω and by Proposition 5.2 follows $T\big|_{(\Omega\setminus\operatorname{supp} T)} = 0$ and $T\big|_{(\Omega\setminus\text{sing supp}\, T)}$ smooth. Obviously $\text{sing supp}\, T \subset \operatorname{supp} T$. One easily checks:

Proposition 5.4. *Let $f \in L^1_{\text{loc}}(\Omega)$ and $T_f \in \mathscr{D}'(\Omega)$ the corresponding distribution. $p \in \Omega$ is in $\operatorname{supp} T_f$ iff*

$$\forall_{\varepsilon>0} \int_{\|x-p\|\le\varepsilon} |f(x)|dx > 0.$$

In particular, for $f \in C(\Omega)$ we have

$$\operatorname{supp} T_f = \operatorname{supp} f.$$

Examples 5.5.

(1) $\operatorname{supp}\delta_a = \text{sing supp}\,\delta_a = \{a\}$.

(2)

$$\begin{aligned} f(x) &= \begin{cases} \log x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}, \\ \operatorname{supp} f &= [0,\infty), \\ \text{sing supp}\, f &= \{0\}. \end{aligned}$$

(3) Exercise: $T \in \mathscr{D}'(\Omega)$. Then $T \in \mathscr{E}'(\Omega) \Leftrightarrow \operatorname{supp} T$ compact. This justifies calling them distributions with compact support.

6. Sobolev spaces

6.1. Integer order Sobolev spaces.

Definition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $k \in \mathbb{Z}_+$. $H^k(\Omega)$ denotes the space of distributions $T \in \mathscr{D}'(\Omega)$ such that $D^\alpha T \in L^2(\Omega)$ for $0 \leq |\alpha| \leq k$. We equip $H^k(\Omega)$ with the scalar product*

$$(f,g)_{H^k} := \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2} .$$

Lemma 6.2. *$H^k(\Omega)$ is a Hilbert space.*

Proof. If (f_n) is a Cauchy sequence in H^k then, for each $|\alpha| \leq k$, $D^\alpha f_n$ is a Cauchy sequence in $L^2(\Omega)$. By the completeness of $L^2(\Omega)$ put $g^{(\alpha)} = D^\alpha g^{(0)}$ and $f_n \to g^{(0)}$ in $H^k(\Omega)$. □

Example 6.3. Let $I = (a,b) \subset \mathbb{R}$. The $H^k(I)$ with $k \geq 1$ consists of those $f \in C^{k-1}[a,b]$ for which $f^{(k-1)}$ is absolutely continuous with square-integrable derivative. Now let $\Omega = \mathbb{R}^n$ and $k \in \mathbb{Z}_+$, $f \in H^k(\mathbb{R}^n)$. Then $f \in \mathscr{S}'(\mathbb{R}^n)$ (since $f \in L^2(\mathbb{R}^n)$) and

$$\begin{aligned} \forall_{|\alpha|\leq k} D^g af \in L^2(\mathbb{R}^n) &\Leftrightarrow \\ \forall_{|\alpha|\leq k} \xi^\alpha \mathscr{F} f(\xi) \in L^2(\mathbb{R}^n) &\Leftrightarrow \\ (1+|\xi|^2)^{\frac{k}{2}} \mathscr{F} f(\xi) \in L^2(\mathbb{R}^n) &\Leftrightarrow \\ \mathscr{F} \in L^2(\mathbb{R}^n, (1+|\xi|^2)^k d\xi)&. \end{aligned}$$

Furthermore we obtain an equivalent scalar product on $H^k(\mathbb{R}^n)$ by putting

$$(f,g)_{H^k} := \int \overline{\hat{f}(\xi)} \hat{g}(\xi)(1+|\xi|^2)^k d\xi.$$

6.2. Real order Sobolev spaces.

Definition 6.4. *Let $H^s(\mathbb{R}^n)$ be the space of those $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\mathscr{F} f \in L^2(\mathbb{R}^n, (1+|\xi|^2)^s ds)$. Put*

$$(f,g)_{H^s}(\mathbb{R}^n) := \int \hat{f}(\xi)\hat{g}(\xi)(1+|\xi|^2)^s d\xi.$$

Lemma 6.5. *(Peetre's inequality) For $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$*

$$(1+|x|)^s(1+|y|)^{-s} \leq (1+|x-y|)^{|s|}.$$

Proof. The inequality for $s \leq 0$ follows from the one for $s \geq 0$ by interchanging x and y. Thus without loss of generality let $s \geq 0$. For $u, v \in \mathbb{R}^n$ we have

$$1+|u+v| \leq 1+|u|+|v| \leq (1+|u|)(1+|v|)$$

and hence

$$(1+|u+v|)^s \leq (1+|u|)^s (1+|v|)^s .$$

Now put $x = u+v$ and $y = v$. □

{prop:I.6.6}

Proposition 6.6. *Let* $H^s := H^s(\mathbb{R}^n)$

(1) $H^s(\mathbb{R}^n)$ *is a Hilbert space.*
(2) $\Lambda_t : f \mapsto \mathscr{F}^{-1}\left((1+|.|^2)^{\frac{t}{2}}\hat{f}\right)$ *is an isometry of* H^s *onto* H^{s-t}*.*
(3) $\mathscr{S}(\mathbb{R}^n)$ *is dense in* H^s*.*
(4) *For all* $s \in \mathbb{R}$ *and* $\alpha \in \mathbb{Z}_+^n$ *is* $D^\alpha : H^s \to H^{s-|\alpha|}$ *continuous.*
(5) *For* $f \in \mathscr{S}(\mathbb{R}^n)$ *multiplication by* f *is continuous* $H^s \to H^s$ *for all* $s \in \mathbb{R}$*.*
(6) *The scalar product on* $\mathscr{S}(\mathbb{R}^n)$ *extends to an anti dual pairing* $H^s \times H^{-s} \to \mathbb{C}$*.*

Proof.

This is obvious.

(1)(2) This is obvious.

(3) Λ_{-s} is an isometry of L^2 onto H^s. Hence Λ_{-s} maps the dense subspace $\mathscr{S} \subset L^2$ onto a dense subspace of H^s. But $\Lambda_{-s}(\mathscr{S}) \subset \mathscr{S}$ and thus $\mathscr{S}$ is dense in H^s. {lbl:I.6.6.3}

(4) For $u \in \mathscr{S}$ we have

$$|\widehat{(D^\alpha u)}(\xi)| = |\xi^\alpha \hat{u}(\xi)| \le (1+|\xi|^2)^{\frac{|\alpha|}{2}}|\hat{u}(\xi)|.$$

Together with (3) this implies the claim.

(5) We have $\widehat{fg} = (2\pi)^{-n}\hat{f} * \hat{g}$. For any $t > 0$ we have $f \in H^t$. Cauchy-Schwartz yields for $u \in H^s$ to

$$\begin{aligned} \left|\hat{f} * \hat{u}(x)\right|^2 &\le \left(\int |\hat{f}(y)\hat{u}(x-y)|dy\right)^2 \\ &\le \underbrace{\int \left|\hat{f}(y)\right|^2\left(1+|y|^2\right)^t dy}_{=cst\|f\|^2_{H^t}} \int |\hat{u}(x-y)|^2\left(1+|y|^2\right)^t dy, \end{aligned}$$

thus

$$\int \left|\hat{f} * \hat{u}(x)\right|^2\left(1+|x|^2\right)^s dx \le \|f\|^2_{H^t}\iint |\hat{u}(x-y)|^2\left(1+|x|^2\right)^s\left(1+|y|^2\right)^{-t} dxdy.$$

Peetre implies

$$\left(1+|x-y|^2\right)^{-s} \le c(s,n)\left(1+|x|^2\right)^{-s}\left(1+|y|^2\right)^{|s|},$$

hence

$$\le \|f\|^2_{H^t}\|u\|^2_{H^s}c(s,n)\int\left(1+|y|^2\right)^{-t+|s|}dy.$$

Now choose $t > \frac{n}{2}+|s|$. Then $\int\left(1+|y|^2\right)^{-t+|s|}dy < \infty$ and the claim is proved.

(6) This follows immediately from the fact that $\mathscr{S}$ is dense in H^s and from the identity for $f, g \in \mathscr{S}(\mathbb{R}^n)$

$$\begin{aligned}(f,g)_{L^2} &= \int \overline{\hat{f}(\xi)}\hat{g}(\xi)d\xi \\ &= \int \overline{\hat{f}(\xi)\left(1+|\xi|^2\right)^{\frac{s}{2}}}\left(1+|\xi|^2\right)^{-\frac{s}{2}}\hat{g}(\xi)d\xi \\ &\leq \|f\|_{H^s}\|g\|_{H^{-s}}.\end{aligned}$$

For the last inequality we used Cauchy-Schwartz.

□

Proposition 6.7. *(Trace theorem) Let $s > \frac{1}{2}$. Then the map $\mathscr{S}(\mathbb{R}^n) \ni f \mapsto f\big|_{\mathbb{R}^{n-1}} \in \mathscr{S}(\mathbb{R}^{n-1})$ extends to a bounded linear map $H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. Let $g := f(0, \bullet)$.

$$\begin{aligned}g(x') &= \int_{\mathbb{R}^n} e^{i\langle x',\xi'\rangle}\hat{f}(\xi_1,\xi')d\xi \\ &= \int_{\mathbb{R}^{n-1}} e^{i\langle x',\xi'\rangle}\underbrace{\int_{\mathbb{R}}\hat{f}(\xi_1,\xi')d\xi_1}_{=\hat{g}(\xi')}d\xi'.\end{aligned}$$

Furthermore ($\xi = (\xi_1, \xi')$),

$$\begin{aligned}|\hat{g}(\xi')|^2 &= |\int_{\mathbb{R}}\left(\hat{f}(\xi)\left(1+|\xi|^2\right)^{\frac{s}{2}}\right)\left(\left(1+|\xi|^2\right)^{-\frac{s}{2}}\right)d\xi_1|^2, \qquad |\xi|^2 = \xi_1^2 + |\xi'|^2 \\ &\overset{\text{C-S}}{\leq} \int_R |\hat{f}(\xi)|^2\left(1+|\xi|^2\right)^s d\xi_1 \underbrace{\int_{\mathbb{R}}\left(1+\xi_1^2+|\xi'|^2\right)^{-s}d\xi_1}_{\substack{<\infty \Leftrightarrow s>\frac{1}{2} \\ \leq c(n,s)\left(1+|\xi'|^2\right)^{-s+\frac{1}{2}}}}\end{aligned}$$

and integration over $\mathbb{R}^{n-1}$ yields

$$\underbrace{\int_{\mathbb{R}^{n-1}}|\hat{g}(\xi')|^2\left(1+|\xi'|^2\right)^{s-\frac{1}{2}}d\xi'}_{\|g\|^2_{H^{s-\frac{1}{2}}}} \leq c(n,s)\underbrace{\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\left(1+|\xi|^2\right)^s d\xi}_{\|f\|^2_{H^s}}.$$

□

Proposition 6.8. *(Sobolev embedding theorem) Let $k \in \mathbb{Z}_+$ and $s > k + \frac{n}{2}$. Then*

$$H^s(\mathbb{R}^n) \hookrightarrow C_0^k(\mathbb{R}^n)$$

is continuously embedded.

Proof. We show for $f \in \mathscr{S}(\mathbb{R}^n)$ is $\|f\|_{\infty,k} \le c\|f\|_{H^s}$. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in H^s and since certainly $\mathscr{S} \subset C_0^k$ the claim then follows. So let $f \in \mathscr{S}(\mathbb{R}^n)$ and $|\alpha| \le k$.

$$\begin{aligned}
|(\widehat{D^\alpha f})(\xi)| &= |\xi^\alpha \hat{f}(\xi)| \\
&= \left(1+|\xi|^2\right)^{\frac{s-k}{2}} |\xi^\alpha||\hat{f}(\xi)| \left(1+|\xi|^2\right)^{-\frac{s-k}{2}} \\
&\le c\left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{f}(\xi)| \left(1+|\xi|^2\right)^{-\frac{s-k}{2}},
\end{aligned}$$

hence

$$\begin{aligned}
\|D^\alpha f\|_\infty &\le \|\widehat{D^\alpha f}\|_{L^1} \\
&\le c\int \left(\left(1+|\xi|^2\right)^{\frac{s}{2}} |\hat{f}(\xi)|\right) \left(1+|\xi|^2\right)^{-\frac{s-t}{2}} d\xi \\
&\overset{\text{C-S}}{\le} \underbrace{c\left(\int \left(1+|\xi|^2\right)^{-(s-k)}\right)^{\frac{1}{2}}}_{<\infty \text{ since } s-k<\frac{n}{2}} \|f\|_{H^s}.
\end{aligned}$$

□

6.3. Localization. Let $\Omega \in \mathbb{R}^n$ be open. Any compactly supported function on Ω (e.g. in $C_c^\infty(\Omega)$) has a natural extension by σ to $\mathbb{R}^n$. More generally, there is a continuous embedding $i : \mathscr{E}'(\Omega) \to \mathscr{E}'(\mathbb{R}^n)$ by putting

$$\langle iT, \phi\rangle := \langle T, \phi\big|_\Omega\rangle, \quad \phi \in \mathscr{E}(\mathbb{R}^n).$$

We won't write i all the time and consider $\mathscr{E}'(\Omega) \subset \mathscr{E}'(\mathbb{R}^n)$.

Definition 6.9. *Let $s \in \mathbb{R}$. $H^s_{\text{comp}}(\Omega)$ consists of those $T \in H^s(\mathbb{R}^n)$ with* $\operatorname{supp} T$ *compact and subset of Ω. $H^s_{\text{loc}}(\Omega)$ consists of those $T \in \mathscr{D}'(\Omega)$ such that $\phi T \in H^s_{\text{comp}}(\Omega)$ for all $\phi \in \mathscr{D}(\Omega)$.*

Remark 6.10. It is routine to check that, analogously to $L^p_{\text{loc}}(\Omega)$ and $L^p_{\text{comp}}(\Omega)$, $H^s_{\text{comp}}(\Omega)$ has a natural LF structure and $H^s_{\text{loc}}(\Omega)$ has a natural Fréchet space structure. Moreover, the scalar product extends to a (anti) dual pairing

$$H^s_{\text{loc}}(\Omega) \times H^{-s}_{\text{comp}}(\Omega) \to \mathbb{C}.$$

More concretely, for $f \in H^{-s}_{\text{comp}}(\Omega)$, $g \in H^s_{\text{loc}}(\Omega)$ choose $\phi \in \mathscr{D}(\Omega)$ with $\phi f = f$. Then

$$(g, f) = (\overline{\phi} g, f) = \int_{\mathbb{R}^n} \overline{(\widehat{\overline{\phi} g})(\xi)}\hat{f}(\xi) d\xi.$$

It follows from Proposition 6.6, (5), that the definition of $H^s_{\text{comp}}(\Omega)$ and $H^s_{\text{loc}}(\Omega)$ is consistent with the definition of $H^s(\Omega)$. I.e. $H^s_{\text{comp}}(\Omega) \subset H^s(\Omega) \subset H^s_{\text{loc}}(\Omega)$.

6.4. Compactness.

Proposition 6.11. *(Rellich lemma) Let* $f_j \in H^s(\mathbb{R}^n)$ *with support in a fixed compact set* K*. Suppose that* $\|f_j\|_{H^s} \leq C$ *for all* j*. Then there is a subsequence* (f_{j_l}) *which converges in* $H^t(\mathbb{R}^n)$ *for every* $t < s$*.*

PROOF. □

7. The Schwartz Kernel Theorem

The Kernel Theorem is a far reaching generalization of the principle that linear maps between finite-dimensional vector spaces are given by matrices.

Examples 7.1.

(1) Let $(k(i,j))_{1\leq i,j\leq n}$ be a Matrix, $\#$ the counting measure on $\{1,\dots n\}$. The vectors are functions $\{1,\dots n\} \to \mathbb{C}$

$$Kf(i) = \sum_{j=1}^{n} k(i,j)f(j) = \int_{\{1,\dots n\}} k(x,y)f(y)d\#(y).$$

(2) Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathrm{C}^\infty(\Omega\times\Omega)$. $K : \mathscr{D}(\Omega) \to \mathscr{E}(\Omega)$

$$Kf(x) := \int_\Omega k(x,y)f(y)dy.$$

(3) $k \in \mathscr{D}'(\Omega)$. Then k is a generalized function. Kf is now in $\mathscr{D}'(\Omega)$

$$\begin{aligned}\langle Kf,\phi\rangle &= \langle \text{``}\int_\Omega k(.,y)f(y)dy\text{''},\phi\rangle\\ &= \text{``}\iint_{\Omega\times\Omega} k(x,y)\phi(x)f(y)dxdy\text{''}\\ &= \langle K, \phi\otimes f\rangle.\end{aligned}$$

Skipping the two intermediate steps define

$$\begin{aligned}K &: \mathscr{D}(\Omega) \to \mathscr{D}'(\Omega),\\ \langle Kf,\phi\rangle &:= \langle K,\phi\otimes f\rangle.\end{aligned}$$

We will show that this describes all continuous linear maps $\mathscr{D}(\Omega) \to \mathscr{D}'(\Omega)$. Before we need some preparations which are of interest in themselves.

{prop:I.7.2}

Proposition 7.2. *Let* $s > \frac{p}{2}$*. Then for* $a \in \mathbb{R}^p$ *we have* $\delta_a \in H^{-s}(\mathbb{R}^p)$*. Moreover* $\mathbb{R}^p \ni a \mapsto \delta_a \in H^{-s}(\mathbb{R}^p)$ *is continuous and for* $f \in \mathscr{S}(\mathbb{R}^p)$ *we have* $f = \int_{\mathbb{R}^p} f(a)\delta_a da$ *ad a Bochner integral in* $H^{-s}(\mathbb{R}^p)$*.*

PROOF.

(1) $(\mathscr{F}\delta_a)(\xi) = e^{-i\langle a,\xi\rangle}$, hence $\int_{\mathbb{R}^p} |(\mathscr{F}\delta_a)(\xi)|^2 \left(1+|\xi|^2\right)^{-s} d < \infty$ since $2s > p$.

(2) Since $a \mapsto \delta_a$ is a group homomorphism it suffices to prove continuity at $a = 0$.
$$\|\delta_a - \delta_0\|_{H^{-s}}^2 = \int_{\mathbb{R}^p} \left|e^{-i\langle a,\xi\rangle} - 1\right|^2 \left(1 + |\xi|^2\right)^{-s} d\xi$$
If $a_n \to 0$ then the corresponding integrands converge pointwise to 0. Moreover they are uniformly bounded by $C\left(1 + |\xi|^2\right)^{-s}$ and dominated convergence implies $\|\delta_a - \delta_0\|_{H^{-s}} \to 0$.

(3) For $f \in \mathscr{S}(\mathbb{R}^n)$ the map $\mathbb{R}^p \ni a \mapsto f(a)\delta_a \in H^{-s}(\mathbb{R}^p)$ is now continuous by (1) and (2). Moreover
$$\int_{\mathbb{R}^p} \|f(a)\delta_a\|_{H^{-s}} da \leq \|\delta_0\|_{H^{-s}} \int |f|$$
since by (1) $\|\delta_a\|_{H^-s} = \|\delta_0\|_{H^{-s}}$. Thus the integral exists. Futhermore, for $\phi \in \mathscr{S}(\mathbb{R}^p)$.
$$\begin{aligned}\langle \int_{\mathbb{R}^p} f(a)\delta_a, \phi\rangle &= \int_{\mathbb{R}^p} f(a)\langle\delta_a, \phi\rangle da \\ &= \int_{\mathbb{R}^p} f(a)\phi(a) da \\ &= \langle f, \phi\rangle,\end{aligned}$$
where $\int f(a)\delta_a da = f$.

□

{rem:I.7.3}

Remark 7.3. (Exercise) If $s > \frac{p}{2} + k$ then $a \mapsto \delta_a \in H^{-s}$ is k-times continuously differentiable.

Proposition 7.4. *Let $s > \frac{p}{2} + k$, $t > \frac{q}{2} + k$ and $K : H^{-s}(\mathbb{R}^p) \to H^t(\mathbb{R}^q)$ a bounded linear operator. Then there is a unique C^k-function $k \in C^k(\mathbb{R}^q \times \mathbb{R}^p)$ such that for $f \in \mathscr{S}(\mathbb{R}^p)$ we have*
$$Kf(x) = \int_{\mathbb{R}^p} k(x,y)f(y)dy.$$

PROOF.
$$\begin{aligned}\mathbb{R}^q \times \mathbb{R}^p &\to H^{-t}(\mathbb{R}^q) \times H^{-s}(\mathbb{R}^p) \\ (x,y) &\mapsto (\delta_x, \delta_y)\end{aligned}$$
is C^k by Remark 7.3 and Proposition 7.2. By assumption
$$\begin{aligned}H^{-t} \times H^{-s} &\to \mathbb{C} \\ (f,g) &\mapsto \langle Kg, f\rangle_{H^t, H^{-t}}\end{aligned}$$
is bilinear and continuous Hence the composition
$$(x,y) \mapsto \langle K\delta_y, \delta_x\rangle =: k(x,y)$$

is a C^k-function. Furthermore, for $f \in \mathscr{S}(\mathbb{R}^p)$ and $g \in \mathscr{S}(\mathbb{R}^q)$ we find again using Proposition 7.2

$$\begin{aligned}\int_{\mathbb{R}^q}(Kf)(x)g(x)dx &= \langle Kf, g\rangle \\ &= \int_{\mathbb{R}^q}\langle Kf, \delta_x\rangle g(x)dx \\ &= \int_{\mathbb{R}^q\times\mathbb{R}^p}\langle K\delta_y, \delta_x\rangle f(y)g(x)dydx \\ &= \int_{\mathbb{R}^q}\left(\int_{\mathbb{R}^p} k(x,y)f(y)dy\right)g(x)dx.\end{aligned}$$

□

Theorem 7.5. *(Kernel Theorem of Schwartz) Let $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$ be open.*

(1) *If $k \in \mathscr{D}'(Y \times X)$ then $K : \mathscr{D}(X) \to \mathscr{D}'(Y)$*

$$\langle K\varphi, \psi\rangle = \langle k, \psi \otimes \varphi\rangle, \quad \varphi \in \mathscr{D}(X), \quad \psi \in \mathscr{D}(Y) \tag{32}$$ {eqn:I.7.6.1}

is a continuous linear map.

(2) *Conversely for each continuous linear map $K : \mathscr{D}(X) \to \mathscr{D}'(Y)$ there exits a unique $k \in \mathscr{D}'(Y \times X)$ such that (32) holds.*

k is called the Schwartz Kernel of K. (1) is relatively easy. The difficult part (2) will rarely be used since in practice it will always be clear what k is.

PROOF. □

CHAPTER 3

Symbol spaces and oszillatory integrals

1. Symbol spaces

1.0. Motivation. Recall that for a differential operator $D : \mathscr{D}(Y) \to \mathscr{D}(X) \subset \mathscr{D}'(X)$ we have

$$\begin{aligned}\langle Du, v\rangle &= \langle \int e^{i\langle .,\xi\rangle}\sigma(.,\xi)\hat{u}(xi)d\xi, v\rangle \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(x,\xi)\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}u(y)dyd\xi v(x)dx \\ &= \text{``}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\underbrace{\int_{\mathbb{R}^n} e^{\langle x-y,\xi\rangle}\sigma(x,\xi)d\xi}_{k_D(x,y)}\left(v\otimes u\right)(x,y)dx\text{''}.\end{aligned}$$

- Integral for $k_D(x,y)$ does not converge.
- Make such integrals rigorous to understand Schwartz kernels of pseudodifferential operators.

1.1. Spaces of symbols.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $N \in \mathbb{Z}_+$ (often $n = N$ or $n = 2n$) and $m \in \mathbb{R}(\mathbb{C})$. The symbol functions of Hörmander type $(1,0)$, $S^m(\Omega\times\mathbb{R}^N) \subset \mathrm{C}^\infty(\Omega\times\mathbb{R}^N)$, consists of those $a : \Omega\times\mathbb{R}^N \to \mathbb{C}$ such that*

$$\forall_{K\subset\Omega \text{ compact}}\forall_{\alpha,\beta}\exists_{C=C(\alpha,\beta,K)}\forall_{(x,\xi)\in K\times\mathbb{R}^N} : |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \le C(1+|\xi|)^{m-|\beta|}. \qquad (33)$$ `{eqn:II.1.2.1}`

$a \in S^m(\Omega\times\mathbb{R}^N)$ is a classical symbol, if in addition there exists $a_{m-j} \in \mathrm{C}^\infty(\Omega\times\mathbb{R}^N)$ such that

- *$a_{m-j}(x,\lambda\xi) = \lambda^{m-j}a_{m-j}(x,\xi)$, for $\lambda \ge 1$, $|\xi| \ge 1$,*
- *$\forall_N : a - \sum_{j=0}^N \in S^{m-N-1}(\Omega\times\mathbb{R}^N)$.*

For the second one we write for short

$$a \sim \sum_{j\ge 0} a_{m-j}.$$

We write $CS^m(\Omega\times\mathbb{R}^N) \subset S^m(\Omega\times\mathbb{R}^N)$ for the set of classical symbols.

Remark 1.2. (1) The best constants in (33) are seminorms on S^m for which S^m becomes a F-space.
(2) From $m \le m'$ follows $S^m \subset S^{m'}$

  - Exercise: $CS^m \subset CS^{m'}$ only if $m' - m \in \mathbb{Z}_+$. $CS^{-\infty} = S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m$
  - Exercise: Note that the countable intersection of F-spaces is again naturally a F-space.

(3) Let $p(x,\xi) = \sum_{|\sigma|\in k} a_\alpha(x)\xi^\alpha$ with $a_\alpha \in \mathrm{C}^\infty(\Omega)$. Then $p \in CS^k(\Omega\times\mathbb{R}^N)$

(4) Let $a \in \mathrm{C}^\infty(\Omega\times\mathbb{R}^N)$ positively homogeneous, i.e.

$$a(x,\lambda\xi) = \lambda^m a(x,\xi), \text{ for } \lambda \geq 1, |\xi| \geq 1$$

Then $a \in CS^m(\Omega\times\mathbb{R}^N)$.

(5) The definition can be extended to cones $\Gamma \subset \mathbb{R}^N$ instead of $\mathbb{R}^N$ (later).

Next we analyze the topology on S^m.

{thm:II.1.3}

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be open and $m < m'$. On bounded subsets of $S^m(\Omega\times\mathbb{R}^N)$ the topology of $S^{m'}$ is the topology of pointwise convergence! Concretely: let $(a_j)_j \subset S^m(\Omega\times\mathbb{R}^N)$ be a bounded sequence which converges pointwise to a function $a : \Omega\times\mathbb{R}^N \to \mathbb{C}$. Then $a \subset S^m(\Omega\times\mathbb{R}^N)$ (in particular smooth!) and $a_j \xrightarrow{S^{m'}} a$.*

Proof. $\square$

{thm:II.1.4}

Theorem 1.4. *Let $m < m'$. In the topology of $S^{m'}$, the space $S^{-\infty}(\Omega\times\mathbb{R}^N)$ is dense in $S^m(\Omega\times\mathbb{R}^N)$.*

{prop:II.1.5}

Proposition 1.5. *(Asymptotic summation lemma) Let $m_1 > m_2 > \dots$ be real numbers with $\lim_{j\to\infty} = -\infty$ and suppose given $a_j \in S^{m_j}(\Omega\times\mathbb{R}^N)$. Then there exists $a \in S^{m_1}(\Omega\times\mathbb{R}^N)$ such that*

$$a \sim \sum_{j=1}^\infty a_j.$$

Recall: $\sim$ means that $\forall_N : a - \sum_{j=1}^N a_j \in S^{m_{N+1}}$. a is unique modulo $S^{-\infty}$.

Proof. $\square$

Proposition 1.6. *Under the prerequisites of Proposition 1.5 let $b \in \mathrm{C}^\infty(\Omega\times\mathbb{R}^N)$ such that*

(1) *For $K \in \Omega$ compact, multiindices α and β exists $C(\alpha,\beta,K)$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leq C(\alpha,\beta,K)(1+|\xi|)^{M(\alpha,\beta,K)}, \quad \text{for } (x,\xi) \in K\times\mathbb{R}^N.$$

(2) *There exists $m'_j \in \mathbb{R}$, $m'_j \to -\infty$ such that for $K \subset \Omega$ compact*

$$|\partial_x^\alpha\partial_\xi^g ba(x,\xi)| \leq C(k,K)(1+|\xi|)^{m'_k}, \quad \text{for } (x,\xi) \in K\times\mathbb{R}^N.$$

Then $b \in S_1^m$ and $b \sim \sum_{j=1}^\infty a_j$.

Proof. $\square$

Lemma 1.7. *For* $f \in C^2[-\varepsilon, \varepsilon]$

$$|f'(0)|^2 \leq 4\|f\|_\infty \left(\frac{1}{\varepsilon^2}\|f\|_\infty + \|f''\|_\infty \right).$$

PROOF. Exercise. □

2. Phase functions and oscillatory integrals

Definition 2.1. $\varphi \in \mathrm{C}^\infty(\Omega \times \dot{\mathbb{R}}^N)$ *(*$\dot{\mathbb{R}}^N := \mathbb{R}^N \setminus \{0\}$*) is called a phase function, if*

- $\Im\varphi \geq 0$,
- $\varphi(x, \lambda\xi) = \lambda\varphi(x,\xi)$, *for* $\lambda > 0$, $\xi \in \dot{\mathbb{R}}^N$,
- $d\varphi(x,\xi) \neq 0$, *for* $(x,\xi) \in \Omega \times \dot{\mathbb{R}}^N$.

Example 2.2. For $\Omega = \mathbb{R}^n$ and $N = n$, $\varphi(x,\xi) = \langle x, \xi\rangle$ is a phase function. Similarly for $\Omega = \mathbb{R}^n \times \mathbb{R}^n$ and $N = n$, is $\varphi(x,y,\xi) = \langle x - y, \xi\rangle$ a phase function.

Given a phase function φ and $a \in S^m(\Omega \times \mathbb{R}^N)$ with $m < -N$. Then we obtain a smooth function by putting

$$I(a,\varphi)(x) := \int_{\mathbb{R}^N} e^{i\varphi(x,\xi)} a(x,\xi) d\xi, \quad x \in \Omega.$$

Namely, for $K \subset \Omega$ compact and $(x,\xi) \subset K \times \mathbb{R}^N$ we have

$$|\partial_x^\alpha a(x,\xi)| \leq c(K,\alpha)(1 + |\xi|)^m.$$

So the integral exists and can be differentiated under the integral. We wish to relax the requirement $m < -N$. As a distribution we have for $\pm(a,\varphi)$

$$\langle I(a,\varphi), u\rangle = \int_{\mathbb{R}^N} \int_\Omega e^{i\varphi(x,\xi)} a(x,\xi) u(x) dx d\xi, \quad u \in \mathscr{D}(\Omega).$$

Note the order of integration!

{lem:II.2.4}

Lemma 2.3. *There exist* $a_j \in S^0(\Omega \times \mathbb{R}^N)$ *and* $b_j, c \in S^{-1}(\Omega \times \mathbb{R}^N)$ *such that with*

$$L = \sum_{j=1}^N a_j(x,\xi)\frac{\partial}{\partial \xi_j} + \sum_{j=1}^n b_j(x,\xi)\frac{\partial}{\partial x_j} + c(x,\xi)$$

we have $L^t e^{i\varphi} = e^{i\varphi}$*. Here is* L^t *the adjoint with regard to the* bilinear *pairing* $\int fg$*. Furthermore,* $L(S^m(\Omega \times \mathbb{R}^N)) \subset S^{m-1}(\Omega \times \mathbb{R}^N)$*.*

PROOF. □

{thm:II.2.3}

Theorem 2.4. *Let* $\varphi \in \mathrm{C}^\infty(()\Omega \times \dot{\mathbb{R}}^N)$ *be a phase function. Then* $a \mapsto I(a,\varphi)$ *has a unique continuous extension*

$$S^\infty(\Omega \times \mathbb{R}^N) := \bigcup_{m \in \mathbb{R}} S^m(\Omega \times \mathbb{R}^N) \to \mathscr{D}'(\Omega).$$

PROOF. □

Remark 2.5. The Theorem says that $\langle I(a,\varphi),u\rangle = \lim\langle I(a_j,\varphi),u\rangle$ whenever $a_j \xrightarrow{S^{m'}} a$. E.g. if $a \in S^m$ then the proof of Theorem 1.4 shows that $a_j(x,\xi) = \varrho\left(\frac{\xi}{j}\right) a(x,\xi)$ is in $S^{-\infty}$ and $a_j \xrightarrow{S^{m'}} a$. Thus we have

$$\langle I(a,\varphi),u\rangle = \lim_{j\to\infty}\int_{\mathbb{R}^N}\int_{\Omega} e^{i\varphi(x,\xi)}\varrho\left(\frac{\xi}{j}\right) a(x,\xi)u(x)dxd\xi.$$

This can also be used to give an alternative existence proof.

Example 2.6. Let $p \in S^m(\{0\}\times\mathbb{R}^n)$. Then p is naturally a tempered distribution. The oscillatory integral

$$I(p,-\langle x,\xi\rangle) = \int e^{i\langle x,\xi\rangle}p(\xi)d\xi$$

is nothing but the Fourier transform of p.

Next we want to investigate the singular support of $I(a,\varphi)$.

Definition 2.7. *For a phase function $\varphi : \Omega\times\dot{\mathbb{R}}^N \to \mathbb{C}$ put*

$$d_\xi\varphi := \sum_{j=1}^N \frac{\partial\varphi}{\partial\xi_j}d\xi_j$$

and

$$C_\varphi = \left\{(x,\xi) \;\middle|\; d_\xi\varphi\big|_{(x,\xi)} = 0\right\} \subset \Omega\times\dot{\mathbb{R}}^N.$$

Remark 2.8. (1) $\varphi(x,y,\xi) = \langle x-y,\xi\rangle$

$$C_\varphi = \left\{(x,x,\xi) \;\middle|\; x\in\varphi, \xi\in\dot{\mathbb{R}}^N\right\} = (\text{diagonal of }\Omega)\times\dot{\mathbb{R}}^N.$$

(2) C_φ is *conic* in sense of that for $(x,\xi)\in C_\varphi$ and $\lambda>0$ follows $(x,\lambda\xi)\in C_\varphi$.

{prop:II.2.9}

Proposition 2.9. *If $a\in S^m(\Omega\times\mathbb{R}^N)$ vanishes in a conic neighborhood U of C_φ then $I(a,\varphi)\in \mathrm{C}^\infty(\Omega)$. For arbitrary $a\in S^m(\Omega\times\mathbb{R}^N)$ we have*

$$\operatorname{sing\,supp} I(a,\varphi) \subset \pi_\Omega(C_\varphi) = \left\{x\in\Omega \;\middle|\; \exists_{\xi\in\dot{\mathbb{R}}^N} : (x,\xi)\in C_\varphi\right\}.$$

3. Operator phase functions

Let $X\subset\mathbb{R}^p$, $Y\subset\mathbb{R}^q$, $\Omega = X\times Y$ open, $\varphi:\Omega\times\dot{\mathbb{R}}^N\to\mathbb{C}$ a phase function and $a\in S^m(\Omega\times\mathbb{R}^N)$. By the Schwartz Kernel Theorem $I(a,\varphi)$ gives rise to a continuous linear operator

$$A = A(a,\varphi) : \mathscr{D}(Y)\to\mathscr{D}'(X).$$

{thm:II.3.1}

Theorem 3.1. *The following statements are equivalent.*

1. *If for all $x\in X$ already $\varphi(x,\bullet,\bullet)$ is a phase function (i.e. $d_{(y,\xi)}\varphi(x,y,\xi)\neq 0$) the A maps $\mathscr{D}(Y)$ continuously into $\mathscr{E}(X)$.*
2. *If for all $y\in X$ already $\varphi(\bullet,y,\bullet)$ is a phase function (i.e. $d_{(x,\xi)}\varphi(x,y,\xi)\neq 0$) then A (can uniquely be extended) maps $\mathscr{E}'(Y)$ continuously into $\mathscr{D}'(X)$.*

Definition 3.2. *A phase function on* $X \times Y \times \dot{\mathbb{R}}^N$ *is called* operator phase function, *if it fulfills condition (1) and (2) of the Theorem. Then*

$$\begin{aligned} A : \mathscr{D}(Y) &\to \mathscr{E}(X), \\ \mathscr{E}'(Y) &\to \mathscr{D}'(X), \\ A^t : \mathscr{D}(X) &\to \mathscr{E}(Y), \\ \mathscr{E}'(X) &\to \mathscr{D}'(Y). \end{aligned}$$

Example 3.3. Let $X = Y$ and $n = \dim X$. Then $\varphi(x, y, \xi) = \langle x - y, \xi \rangle$ is an operator phase function.

Definition 3.4. *The corresponding operator to the phase function of the example is called a* pseudodifferential *operator of order* m.

4. Conormal distributions

We elaborate a bit more on oscillatory integrals with *linear* phase function. First, let $T \subset \mathscr{S}'(\mathbb{R}^n)$ and $f \in \mathscr{S}(\mathbb{R}^n)$. Suppose $\mathscr{F}T \in S^m(\{0\} \times \mathbb{R}^n)$[1]. Then

$$\begin{aligned} \langle T, u \rangle &= \langle \mathscr{F}T, \mathscr{F}^{-1} u \rangle \\ &= \int_{\mathbb{R}^n} (\mathscr{F}T)(\xi) \mathscr{F}^{-1} u(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} (\mathscr{F}T)(\xi) u(\xi) dx d\xi \\ &=: \langle I(\mathscr{F}T\varphi), u \rangle, \end{aligned}$$

where $\varphi(x, \xi) = \langle x, \xi \rangle$. From now on use this φ. In other words a tempered distribution T with $\mathscr{F}T \in S^m(\{0\} \times \mathbb{R}^n)$ is given by an oscillatory integral.

{prop:II.4.1}

Proposition 4.1. *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $a \in S^m(\Omega \times \mathbb{R}^n)$, *Then for* $u \in \mathscr{D}(\Omega)$

$$\langle I(a, \varphi), u \rangle = \int_{\mathbb{R}^n} \left(\int_\Omega e^{i\langle x, \xi \rangle} a(x, \xi) u(x) dx \right) d\xi,$$

where the iterated (TODO: was heisst das?) integral converges. If $\operatorname{supp} a \subset B \times \mathbb{R}^n$ *with* $B \subset \Omega$ *compact. Then for* $O \notin B$ *(TODO: was ist* O*, oder soll das eine* 0 *seien?) follows* $I(a, \varphi) \in \mathrm{C}_c^\infty(\Omega)$, *otherwise define* $\sigma := I(\hat{a}, \varphi) \in S^m(\{0\} \times \mathbb{R}^n)$ *and we have*

$$\langle I(a, \varphi), u \rangle = \int_{\mathbb{R}^n} \int_\Omega e^{i\langle x, \xi \rangle} \sigma(\xi) u(x) dx d\xi$$

and

$$\sigma(\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha a(x, \xi) \bigg|_{x=0}.$$

Proof. □

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[1] $\mathscr{F}u(\xi) = \mathscr{F}u(-\xi) = \int e^{i\langle x, \xi \rangle} u(x) dx$

This Proposition will be a key tool for establishing the calculus of pseudodifferential operators. It has an obvious extension to the following parametric situation:

Let $a \in S^m(X \times Y \times \mathbb{R}^q)$, $X \subset \mathbb{R}^p$ and $Y \subset \mathbb{R}^a$ such that for each $x \in X$ there is a compact neighborhood $L \ni x$ and a compact set $L' \subset Y$ such that for $a(x, y, \xi) \neq 0$ and $x \in L$ follows $y \in L'$. Consider $I(a, \varphi) \in \mathscr{D}'(X \times Y \times \mathbb{R}^q)$ which is given by the iterated integral

$$\langle I(\sigma, \varphi), u\rangle = \int_{\mathbb{R}^n} \int_{X \times Y} e^{i\langle y, \xi\rangle} \sigma(x, \xi) u(x, y) dx dy d\xi$$

and

$$\sigma(x, \xi) \sim \sum_{\alpha} \frac{i^{|\sigma|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha a(x, y, \xi)\Big|_{y=0}$$

(hence is unique $\mod S^{-\infty}$). σ is obtained as partial Fourier transform of $I(a, \varphi)$ in the y variables.

CHAPTER 4

Local Theory of pseudodifferential operators

1. Basic definitions and proper support

Definition 1.1. *Let $X \subset \mathbb{R}^n$ be open, $a \in S^m(X \times X \times \mathbb{R}^n)$. Put for $u \in \mathscr{D}(X)$*

$$Au(x) := \int_{\mathbb{R}^n} \int_X e^{i\langle x-y,\xi\rangle} a(x,y,\xi) u(y) dy d\xi.$$

A is called a pseudodifferential operator with amplitude a.

Obviously, $\varphi(x,y,\xi) = \langle x-y,\xi\rangle$ is an operator phase function in the sense of Section II.3. The iterated integral exists by Proposition 4.1. It is straightforward to see that one can differentiate by x under the integral. By Theorem 3.1 A maps $\mathscr{D}(X) \to \mathscr{E}(X)$ $\mathscr{E}'(X) \to \mathscr{D}'(X)$ continuously.

{lem:III.1.2}

Lemma 1.2. *Let $X \subset \mathbb{R}^p$, $Y \subset \mathbb{R}^q$ be open.*

(1) *Let $K : \mathscr{D}(Y) \to \mathscr{D}'(X)$ with Schwartz Kernel k. Then for $u \in \mathscr{D}(Y)$*

$$\operatorname{supp}(Ku) \subset \operatorname{supp} k \circ \operatorname{supp} u \tag{34}$$

{eqn:III.1.1}

(2) *If $K : \mathscr{D}(Y) \to \mathscr{E}(X)$ has a continuously extension $\mathscr{E}'(Y) \to \mathscr{D}'(X)$, then for $u \in \mathscr{E}'(Y)$*

$$\operatorname{sing\,supp}(Ku) \subset \operatorname{sing\,supp} k \circ \operatorname{sing\,supp} u$$

and equation (34) persists to hold for $u \in \mathscr{D}'(Y)$

{rem:III.1.3}

Remark 1.3. For $R \subset Y \times X$ and $L \subset Y$

$$R \circ L := \left\{ x \in X \;\middle|\; \exists_{y \in Y} (x,y) \in R \wedge y \in L \right\}$$

We leave it as an exercise to show that if L is compact, then $R \circ L$ is closed. Furthermore, if in addition R is *proper* then $R \circ L$ is compact. R proper means

$$Y \xleftarrow{\pi_Y} R \xrightarrow{\pi_X} X$$

where π_X and π_Y are proper maps on R.

Proof. $\square$

Proposition 1.4. *Let A be a pseudodifferential operator on X. Thern $\operatorname{sing\,supp} K_A \subset \Delta = \left\{(x,x) \;\middle|\; x \in X\right\}$ and $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$ with $u \in \mathscr{E}'(X)$.*

Proof. This follows from Proposition 2.9 and Lemma 1.2. $\square$

{def:III.1.5}

Definition 1.5. *A pseudodifferential operator A on X is called* properly supported *if K_A is properly supported in $X \times X$.*

Notation:

$$L^m(X) = \big\{A \bigm| A \text{ pseudodifferential operator of order } m \text{ which is properly supported}\big\}$$
$$CL^m(X) = \big\{A \in L^m(x) \bigm| A = \mathrm{Op}(a) \text{ with a classical amplitude } a \in CS^m\big\}$$

{prop:III.1.5}

Proposition 1.6. *For $A \in L^m(X)$ there is an amplitude $a \in S^m(X \times X \times \mathbb{R}^n)$ which is properly supported in the sense that*

$$\overline{\big\{(x,y) \in X \times X \bigm| \exists_\xi : a(x,y,\xi) \neq 0\big\}}$$

is proper subset of $X \times X$.

Proof. No proof. □

It is an immediate consequence of the definition and Remark 1.3 that $A \in L^m(X)$ maps $\mathscr{D}(X)$, $\mathscr{E}(X)$, $\mathscr{E}'(X)$ and $\mathscr{D}'(X)$ (continuously) in itself respectively. We leave the continuity as an exercise.

1.1. Properly supported smoothing operators.

Let $k \in C^\infty_{\mathrm{proper}}(X \times X)$ and consider the smoothing operator

$$Ku(x) = \int_X k(x,y)u(y)dy.$$

We will show that $K \in L^{-\infty}(X)$ and we will construct three *different* amplitudes representing K.

Definition 1.7. *Fix $X \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} X = 1$. Then for $u \in \mathscr{E}(X)$ and $x \in B \subset\subset X$, $L := \pi_2(\pi^-1_1 B \cap \operatorname{supp} k)$ (L compact since* $\operatorname{supp} k$ *is proper).*

$$\begin{aligned} Ku(x) &= \int_L k(x,y) \int_{\mathbb{R}^n} X(\xi) e^{i\langle x-y,\xi\rangle} e^{-i\langle x-y,\xi\rangle} d\xi u(y) dy \\ &= \int_{\mathbb{R}^n}\int_L e^{i\langle x-y,\xi\rangle} \underbrace{\left\{k(x,y)e^{-i\langle x-y,\xi\rangle}X(\xi)\right\}}_{a(x,y,\xi)} u(y)dyd\xi. \end{aligned}$$

$a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$ is a properly supported amplitude *for K.*

An amplitude depending only on Y
Let now $u \in \mathscr{D}(X)$, then $B = (\operatorname{supp} k) \circ \operatorname{supp} u$ compact. Put for any $y \in X$

$$\begin{aligned} \widetilde{a}(y,\xi) &:= \left(\mathscr{F}_{x \rightsquigarrow \xi} k(\bullet, y)\right)(\xi) \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} k(x,y) dx. \end{aligned}$$

Again since k is properly supported it follows that $\widetilde{a} \in S^{-\infty}(X \times \mathbb{R}^n)$. Moreover

$$\begin{aligned} Ku(x) &= \int_{\operatorname{supp} u} \underbrace{\int_{\mathbb{R}^n} a(y,\xi) e^{i\langle x,\xi\rangle} d\xi}_{=k(x,y)} u(y)dy \\ &= \int_{\mathbb{R}^n} \int_X e^{i\langle x-y,\xi\rangle} \underbrace{\left\{\widetilde{a}(y,\xi) e^{i\langle y,\xi\rangle}\right\}}_{a(y,\xi)} u(y)dyd\xi \end{aligned}$$

for $a \in S^{-\infty}(X \times \mathbb{R}^n)$. Note that a priori the iterated integral exists only for $u \in \mathscr{D}(X)$.

An amplitude depending only on X
Again for $u \in \mathscr{D}(X)$ we compute[1]

$$\begin{aligned} Ku(x) &= \int_X k(x,z) \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle} \hat{u}(\xi) d\xi dz \\ &= \int_{\mathbb{R}^n} \underbrace{\int_X k(x,z) e^{i\langle z,\xi\rangle} dz}_{=:e^{i\langle x,\xi\rangle}\sigma(x,xi), \quad \sigma\in S^{-\infty}(X\times\mathbb{R}^n)} \hat{u}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \int_X e^{i\langle x-y,\xi\rangle} \sigma(x,\xi) u(y) dyd\xi \end{aligned}$$

2. The (complete) symbol of a properly supported pseudodifferential operator

{dfn:III.2.1}

Definition 2.1. *Let $A \in L^m(X)$. Put*

$$\sigma_A(x,\xi) := e^{-i\langle x,\xi\rangle} \left(A e^{i\langle \bullet,\xi\rangle} \right)(x).$$

σ_A is called the (complete) symbol *of A. Certainly $\sigma_A \in \mathrm{C}^\infty(X \times \mathbb{R}^n)$.*

{thm:III.2.2}

Theorem 2.2. *σ_A is an amplitude for A and $\sigma_A \in S^m(X \in \mathbb{R}^n)$. For any amplitude a of A*

$$\sigma_A \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\Big|_{y=x}.$$

Proof. ☐

---

[1] Again it is essential that k is properly supported to justify the performed manipulations of integrals.

3. The calculus of pseudo differential operators

{thm:III.3.1}

Theorem 3.1. *Let $A \in L^m(X)$. The adjoint, A^*, with regard to the L^2-scalar product $(f,g) = \int_X \overline{f} g$ is in $L^m(X)$ and*

$$\sigma_{A^*}(x,\xi) \sim \sum \frac{i^{-\alpha}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{\alpha_A(x,\xi)}.$$

Furthermore, $\overline{\sigma_{A^*}(y,\xi)}$ is an x-independent amplitude for A and hence for $u \in \mathscr{D}(X)$

$$\widehat{(Au)}(\xi) = \int e^{-i\langle y,\xi\rangle} \overline{\sigma_{A^*}(y,\xi)} u(y) dy.$$

Proof. ☐

{thm:III.3.2}

Theorem 3.2. *Let $A \in L^m(X)$ and $B \in L^{m'}(X)$ be properly supported pseudodifferential operators. Then $A \circ B \in L^{m+m'}(X)$ and*

$$\sigma_{A \circ B}(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma_A(x,\xi)\right) \partial_x^\alpha \sigma_B(x,\xi).$$

Proof. ☐

3.1. Change of coordinates.

Let $X, Y \subset \mathbb{R}^n$ be open and $\kappa : X \to Y$ a diffeomorphism. Furthermore, let $A \in L^m(Y)$. TODO: Bild. Let $a \in S^m(Y \times Y \times \mathbb{R}^n)$ be a properly supported amplitude A. Then

$$\begin{aligned}
(\kappa^*)u(x) &= \{Au \circ \kappa^{-1}\}(\kappa(x)) \\
&= \int_{\mathbb{R}^n} \int_Y e^{i\langle \kappa(x) - \widetilde{y}, \xi\rangle} a(\kappa(x), \widetilde{y}, \xi) u(\kappa^{-1}\widetilde{y}) d\widetilde{y} d\xi, \qquad \widetilde{y} := \kappa(y) \\
&= \int \mathbb{R}^n \int_X e^{i\langle \kappa(x) - \kappa(y), \xi\rangle} \underbrace{a(\kappa(x), \kappa(y), \xi) |\det D\kappa(y)|}_{=: b(x,y,\xi)} u(y) dy d\xi.
\end{aligned}$$

Thus $\kappa^* A$ is a Fourier Integral Operator with operator phase function

$$\varphi(x,y,\xi) = \langle \kappa(x) - \kappa(y), \xi\rangle.$$

We are going to prove in principle a result about *equivalence of phase functions.* Choose a neighborhood U of the diagonal $\Delta \subset X \times X$ sucht that there exists a smooth $F : U \to \mathrm{GL}(n, \mathbb{R})$ such that $\kappa(x) - \kappa(y) = F(x,y)(x-y)$ for $x, y \in U$. The following calculation (modulo details) shows that this is possible

$$\kappa(x) - \kappa(y) = \int_0^1 \frac{d}{dt} \kappa(y + t(x-y)) dt \underset{\text{chain rule}}{=:} F(x,y)(x,y)$$

with $F(x,y) = D\kappa(x)$. It follows

$$(\kappa^* A)u(x) = \int_{\mathbb{R}^n}\int_X e^{\langle x-y,F(x,y)^t\xi\rangle} b(x,y,\xi)dyd\xi, \qquad \xi := \underbrace{(F(x,y)^t)^{-1}}_{=:G(x,y)}\theta$$

$$= \int_{\mathbb{R}^n}\int_X e^{i\langle x-y,\theta\rangle}\underbrace{b(x,y,G(x,y)\theta)|\det G(x,y)|}_{c(x,y,\theta}u(y)d\theta.$$

The chain rule indeed shows $c \in S^m(X\times X\times\mathbb{R}^n)$. Moreover, *modulo* S^{m-1} we find

$$\begin{aligned}
\sigma_{\kappa^*A}(x,\xi) &\equiv c(x,x,\xi) && \bmod S^{m-1}\\
&\equiv b(x,x,G(x,x)\xi)|\det G(x,x)| && \bmod S^{m-1}\\
&\equiv a(\kappa(x),\kappa(x),\{D\kappa(x)^t\}^{-1}\xi)\underbrace{|\det D\kappa(x)||\det G(x,x)|}_{=1} && \bmod S^{m-1}\\
&\equiv \sigma_A(\kappa(x),\kappa(x),(D\kappa(x)^t)^{-1}\xi) && \bmod S^{m-1}.
\end{aligned}$$

{thm:III.3.3}

Theorem 3.3. *Let $X, Y \subset \mathbb{R}^n$ be open, $A \in L^m(Y)$, $\kappa : X \to Y$ diffeomorphism. Then*

$$\kappa^* A := \kappa^* \circ A \circ \kappa_* \in L^m(X)$$

and

$$\sigma_{\kappa^*A}(x,\xi) \equiv \sigma_A\left(\kappa(x), \left(D\kappa(x)^t\right)^{-1}\xi\right) \qquad \bmod S^{m-1}.$$

4. Elliptic regularity

{def:III.4.1}

Definition 4.1. *Let $X \subset \mathbb{R}^n$ open. $a \in S^m(X\times\mathbb{R}^n)$ is called* elliptic, *if for each compact subset $K \subset\subset X$ there exists a $\xi_0 \in \mathbb{R}^n$, such that $a(x,\xi)$ is invertible for $x \in K$, $|\xi| \geq |\xi_0|$ and*

$$|a(x,\xi)^{-1}| \leq c(K)(1+|\xi|)^{-m}.$$

$A \in L^m(X)$ is called elliptic *if σ_A is elliptic.*

Let $a \in S^m(X\times\mathbb{R}^n)$. Using a partition of unity we find $b \in \mathrm{C}^\infty(()X\times\mathbb{R}^n)$ such that $b(x,\xi) = a(x,\xi)^{-1}$ for $|\xi| \geq |\xi_0|$, $\xi_0 = \xi_0(K)$, $x \in K$.

{lem:III.4.2}

Lemma 4.2. $b \in S^{-m}(X\times\mathbb{R}^n)$

Proof. Exercise: Use the Leibnitz rule. □

Remark 4.3. Exercise: For a classical pseudodifferential operator ellipticity is equivalent to the invertibility of the homogeneous principal symbol $\sigma_A^m(x,\xi)$ for *all* $(x,\xi) \in X\times(\mathbb{R}^n\setminus\{0\})$.

{thm:III.4.4}

Theorem 4.4. *Let $A \in L^m(X)$ be elliptic and properly supported. Then there exists $B \in L^{-m}(X)$ such that $AB - I, BA - I \in L^{-\infty}(X)$. B is called a* parametrix *for A.*

Proof. □

`{cor:III.4.5}`

Corollary 4.5. *(Weak form of the regularity theorem) Let $A \in L^m(X)$ be properly supported and elliptic. Then for $u \in \mathscr{D}'(X)$*

$$\operatorname{sing\,supp} Au = \operatorname{sing\,supp} u.$$

In particular $Au = 0$ implies $u \in \mathrm{C}^\infty(()X)$. For $A = \Delta = -\sum_{j=1}^n \frac{pl^2}{\partial x_j^2}$ this is the so-called Weyl Lemma.

Proof. We already know $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$. Let $BA - I \in L^{-\infty}$, then

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} BAu \subset \operatorname{sing\,supp} Au.$$

□

5. L^2-continuity

In this section we want to investigate continuity properties of pseudodifferential operators between Sobolev spaces. This requires norm estimates for integral operators.

`{prop:III.5.1}`

Proposition 5.1. *(Version of Schur's test) Let (X_j, μ_j) be measure spaces and let $k : X_1 \times X_2 \to \mathbb{C}$ be a measurable kernel. Suppose that*

$$C_1 = \sup_{x \in X_1} \int_{X_2} |k(x,y)| d\mu_2(y) < \infty \qquad \text{and}$$

$$C_2 = \sup_{y \in X_2} \int_{X_1} |k(x,y)| d\mu_1(x) < \infty.$$

Then

$$Kf(x) = \int_{X_2} k(x,y) f(y) d\mu_2(y)$$

defines a bounded operator $L^2(X_2, \mu_2) \to L^2(X_1, \mu_1)$ with $\|K\| \le \sqrt{C_1 C_2}$.

`{rem:III.5.2}`

Remark 5.2. For $X_1 = X_2 = \mathbb{R}^n$, $\mu_1 = \mu_2$ Lebesgue measures we note the following simple sufficient criterion for applying Proposition 5.1. Suppose

$$k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$$

with

$$|k(x,y)| \le C_p (1 + \|x - y\|)^{-p}$$

for some $p > n$ then we have

$$C_1, C_2 \le C_p \int_{\mathbb{R}^n} (1 + \|x\|)^{-p} dx$$

and thus

$$\|K\| \le C_p \int_{\mathbb{R}^n} (1 + \|x\|)^{-p} dx.$$

Definition 5.3. *For* $a \in S^m_{\mathrm{comp}}(U \times \mathbb{R}^n)$ *put*

$$q_a(\zeta, \xi) := \int_U e^{i\langle x, \zeta\rangle} a(x, \xi) dx.$$

Thus $\mathscr{F} \operatorname{Op}(a) \mathscr{F}^{-1}$ is an integral operator

$$(\mathscr{F} \operatorname{Op}(a) \mathscr{F}^{-1}) u(\tau) = \int_{\mathbb{R}^n} q_a(\xi - \tau, \xi) u(\xi) d\xi$$

in $\mathbb{R}^n$ with kernel $k_a(\tau, \xi) = q_a(\xi - \tau, \xi)$. To estimate the operator norm TODO: Bild

{dia:III.5.1}

of $\operatorname{Op}(a)$ we can equivalently estimate the norm of $\widetilde{K_a}$ with kernel

$$(1 + |\tau|)^{s-m} k_a(\tau, \xi)(1 + |\xi|)^{-1} = (1 + |\tau|)^{s-m} q_a(\xi - \tau, \xi)(1 + |\xi|)^{-s}$$

on $L^2(\mathbb{R}^n)$. First we estimate $q_a(\xi, \xi)$; note that $\operatorname{supp} a \subset K \times \mathbb{R}^n$. For any $\alpha \in \mathbb{Z}^n_+$ we have

$$|\zeta^\alpha q_a(\zeta, \xi)| = |\int_K e^{i\langle x, \xi\rangle} \partial^\alpha_x a(x, \xi) dx| \le \operatorname{vol}(K) c_{\alpha, K}(a)(1 + |\xi|)^m.$$

Since α was arbitrary, we conclude that for any $N \in \mathbb{N}$

$$|q_a(\zeta, \xi)| \le c_N (1 + |\zeta|)^{-N} (1 + |\xi|)^m.$$

Thus we find for the kernel $\widetilde{k_a}$ of $\widetilde{K_a}$

$$|\widetilde{k_a}(\tau, \xi)| \le c_N (1 + |\tau|)^{s-m} (1 + |\xi - \tau|)^{-N} (1 + |\xi|)^{m-s}.$$

Peetre's inequality gives

$$(1 + |\tau|)^{s-m} (1 + |\xi|)^{m-s} \le (1 + |\xi - \tau|)^{|m-s|},$$

thus

$$|\widetilde{k_a}(\tau, \xi)| \le c_N (1 + |\xi - \tau|)^{|m-s|-N}.$$

Since N was arbitrary, we conclude from Remark 5.2 that $\widetilde{K_a}$ is bounded in $L^2(\mathbb{R}^n)$, which in view of the diagram on page 87 proves the boundedness of $\operatorname{Op}(a)$.

{thm:III.5.4}

Theorem 5.4. *Let* $A \in L^m(\mathbb{R}^n)$ *with supp* K_A *compact. Then for all* $s \in \mathbb{R}$ *the operator* A *maps* $H^S(\mathbb{R}^n)$ *continuously into* $H^{s-m}(\mathbb{R}^n)$ *(in fact* $H^s_{\mathrm{loc}} \to H^{s-m}_{\mathrm{comp}}$*).*

CHAPTER 5

Global Theory of pseudodifferential operators

1. (Density) bundles and distributional sections on manifolds

In the sequel M denotes a smooth manifold. I will say a words about distributions on manifolds. However, I will avoid a too high level of abstraction. Therefor, I will adopt an informal style here.

1.1. Vector bundles. Recall that a vector bundle over M consists of the following data

- a smooth manifold E together with a submersion $\pi : E \to M$.
- a vector space structure on the fibres $E_p = \pi^{-1}(\{e\})$, $p \in M$.
- local tiviality
  TODO: Bild
  where h is a diffeomorphism which is fiberwise a *linear* isomorphism.

1.2. Density bundles. Linear Algebra: V n-dimensional $\mathbb{R}$ vector space. $|V|^\alpha$ denotes the space of maps $f : \Lambda^n y^* \to \mathbb{R}$ such that $f(\lambda v) = |\lambda|^\alpha v$, $\lambda \in \mathbb{R}$. $|V|^\alpha$ is a 1-dimensional $\mathbb{R}$ vector space. For a manifold M let Ω^α denote the bundle $|\Lambda^n T^* M|^\alpha$, Ω^α is called the bundle of α-densities over M. If g is a Riemannian metric on M then $|\mathrm{vol}_g|^\alpha = \sqrt{\det(g_{ij})}^\alpha |dx_1 \wedge \cdots \wedge dx_n|$ is a nowhere vanishing α-density. Hence Ω^α is a trivializable (but not canonically trivial) line bundle. In terms of cocycles, Ω^α is the bundle with transition maps given by

$$|\det D\varphi|^\alpha,$$

where φ is a change of the coordinate map. Therefor, there is a canonical bilinear map

$$\Omega^\alpha \times \Omega^\beta \to \Omega^{\alpha+\beta}.$$

Furthermore, sections of Ω^1 can be integrated. Hence there is a canonical bilinear pairing

$$\begin{aligned} \Gamma_c^\infty(M, \Omega^\alpha) \times \Gamma^\infty(M, \Omega^{1-\alpha}) &\to \mathbb{C} \\ (f, g) &\mapsto \int_M fg, \end{aligned}$$

hence a canonical embedding

$$\Gamma^\infty(M, \Omega^\alpha) \hookrightarrow \mathscr{D}(M, \Omega^{1-\alpha})'.$$

For any bundle E, $\mathscr{D}(M,E)$ denotes the space $\Gamma^\infty_{\mathrm{c}}(M,E)$ equipped with the LF space topologie analogous to $\mathscr{D}(X)$ with $X \subset \mathbb{R}^n$ open.

So functions on M have an interpretation as elements of $\mathscr{D}(M,\Omega)'$. There is no canonical way of embedding $C^\infty(M)$ into $\mathscr{D}(M)'$. Although the latter is possible non-canonical by *choosing* a trivialization of Ω^1.

Anyway, the natural space of distributions on M is the dual space $\mathscr{D}(M,\Omega^1)'$. Thus put

$$\mathscr{D}'(M) := \{\mathscr{D}(M,\Omega^1)\}'.$$

More generally, the distributional sections of a vector bundle E are defined as

$$\mathscr{D}'(M,E) := \{\mathscr{D}(M,E^* \otimes \Omega^1\}.$$

Then there is a natural embedding

$$\begin{aligned} \Gamma^\infty(M,E) &\hookrightarrow \mathscr{D}'(M,E) \\ f &\mapsto T_f \\ T_f(s) &:= \int_M \langle f, s\rangle, \end{aligned}$$

where $\langle \bullet, \bullet\rangle$ denotes the contraction map $E \times E^* \otimes \Omega^1 \to \Omega^1$.

2. (Elliptic) pseudodifferential operators on manifolds

{def:IV.2.1}

Definition 2.1. *Let M be a smooth manifold and E, F be vector bundles over M. A linear map*

$$P : \Gamma^\infty_{\mathrm{c}}(M,E) \to \Gamma^i nfty(M,E)$$

is called a pseudodifferential operator (classical, properly supported of order m,…) if for each chart $\kappa : M \supset U \to V \subset \mathbb{R}^n$ and bundle charts

$$\begin{tikzcd} E\big|_U \arrow[r, "\phi^E"] \arrow[d, "\pi"] & V \times \mathbb{C}^r \arrow[d] \\ U \arrow[r, "\kappa"] & V \end{tikzcd} ,$$

$$\begin{tikzcd} F\big|_U \arrow[r, "\phi^F"] \arrow[d, "\pi"] & V \times \mathbb{C}^s \arrow[d] \\ U \arrow[r, "\kappa"] & V \end{tikzcd}$$

the operator $\kappa_* P$ defined by

$$\begin{tikzcd} \Gamma^\infty_{\mathrm{c}}(E\big|_U) \arrow[r, "P"] \arrow[d, "\phi^E_*"] & \Gamma^\infty(F\big|_U) \arrow[d, "\phi^F_*"] \\ \mathrm{C}^\infty_c(V,\mathbb{C}^r) \arrow[r, "\kappa^P_*"] & \mathrm{C}^\infty(V,\mathbb{C}^s) \end{tikzcd}$$

is a pseudodifferential operator (classical,…) on V *with* $\mathrm{M}(s \times r, \mathbb{C})$*-valued symbol. We write* $\mathrm{L}^m(M; E, F)$ *for the set of properly supported pseudodifferential operators of order* m *acting between sections of* E *and* F *and we write* $CL^m(M; E, F)$ *for the set of classical operators in* $\mathrm{L}^m(M; E, F)$*.* P *is called* elliptic *if* $\kappa_* P$ *is elliptic.*

Warning: In the Definition 2.1 the charts may be non-connected. Otherwise

$$\begin{aligned} C^\infty(R \setminus \{0\}) &\to C^\infty(\mathbb{R} \setminus \{0\}) \\ f &\mapsto f(-\bullet) \end{aligned}$$

would qualify as a pseudodifferential operator (which it is *not*).

{thm:IV.2.2}

Theorem 2.2. *Let* M *be a manifold and* E*,* F *be vector bundles. Let* $P \in \mathrm{L}^m(M; E, F)$ *(or* CL^m*) be elliptic. The there exists* $Q \in \mathrm{L}^{-m}(M; E, F)$ *such that* $PQ - I, QP - I \in \mathrm{CL}^{-\infty}(M; E/F)$*.*

APPENDIX A

Tools from functional analysis

Status:

- 2012: Just typed, not checked
- 01.11.13: checked against handwritte notes, slightly edited.

Standing Assumptions. During this Chapter $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ stands for either the real or complex numbers.

1. Seminorms, LCTVS

Definition 1.1. *Let E be a $\mathbb{K}$ vector space. A seminorm on E is a map $p : E \to \mathbb{R}_+$ such that for $\lambda \in \mathbb{K}$ and $x, y \in E$*

(1) $p(\lambda x) = |\lambda| p(x)$,
(2) $p(x + y) \leq p(x) + p(y)$.

{A1.2}

Definition and Proposition 1.2. *Let $\mathscr{P} = (p_i)_{i \in I}$ be a family of seminorms on E. Call a subset $U \subset E$ open if for each $x \in U$ there exists an $\varepsilon > 0$ and* finitely *many indices $i_1, \dots, i_r \in I$ such that the subset*

$$x + \{ y \in E \mid p_{i_j} < \varepsilon, j = 1, \dots, r \} \subset U.$$

In this way E becomes a topological vector space. .

This is not the most general way of constructing a TVS, but the previous construction leads to the fairly general class of LCTVS; we will not bother to look at more general TVS.

Definition and Proposition 1.3. *The family of seminorms $\mathscr{P}$ is called seperating if*

$$\forall_{x \in E \setminus \{0\}} \exists_{i \in I} : \quad p_i(x) \neq 0.$$

$\mathscr{P}$ is separating iff the topology is separated. A locally convex topological vector space is a pair $(E, \mathscr{P})$ consisting of a $\mathbb{K}$ vector space E and a separating family of seminorms such that the topology on E is induced by $\mathscr{P}$ in the sense of Definition and Proposition 1.2.

LCTVShave a natural uniform structure: a net (sequence) $(x_\alpha)_{\alpha \in \mathscr{A}}$ is called a Cauchy net if

$$\forall_{p \in \mathscr{P}} \forall_{\varepsilon > 0} \exists_{\alpha_0 \in \mathscr{A}} \forall_{\beta, \alpha \geq \alpha_0} : \quad p(x_\beta - x_\alpha) \leq \varepsilon.$$

Convergence is defined accordingly and $(E, \mathscr{P})$ is said *complete* if every Cauchy net converges.

1.4. Important classes of LCTVS

1. Normed spaces. Here $\mathscr{P}$ may be chosen to consist of one element, $\mathscr{P} = \{\|\cdot\|\}$. The norm axioms imply that $\mathscr{P}$ is separating. A complete normed space is called *Banach space.*

2. Fréchet spaces. A complete LCTVS$(E, \mathscr{P})$ is called a *Fréchet space* if $\mathscr{P}$ is countable, i.e. $\mathscr{P} = (p_j)_{j\in\mathbb{N}}$. Then E is metrizeable: to see this put

$$d(x, y) := \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x, y)}{1 + p_j(, xy)}.$$

Then the topology of E equals the one induced by d.

3. LF*-spaces.* Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of Fréchet spaces sucht that $E_n \subset E_{n+1}$ (topological inclusion!) for each n. Then $E := \bigcup_{n=1}^{\infty} E_n$ is naturally a vector space. Call a seminorm $p : E \to \mathbb{R}_+$ admissible if $\forall_n p\big|_{E_n}$ is continuous. Let $\mathscr{P}$ be the set of admissible seminorms. Then $(E, \mathscr{P})$ is a LCTVS, a so called LF*-space.*

{thm:A.1.5}

Theorem 1.5. *Let E be an* LF*-space with defining sequence $(E_n)_n$.*

1. *A sequence (x_n) converges in E iff there is an index N such that $x_n \in E_N$ for all n and (x_n) converges in E_N to $x \in E_N$. There is an obvious analogous statement for Cauchy sequences.*
2. *An LF space is complete.*

The proof is beyond the scope of this course. Cf. [Trè06, Lemma 13.1 and Sec. I.7]. Crucial for the proof is the following result:

{prop:A.1.6}

Proposition 1.6. *Let E be a* LCTVS*, $E_0 \subset E$ a closed subspace, p a continuous seminorm on E_0 and $x_0 \in E \setminus E_0$. Then for $r \in \mathbb{R}_+$ there is a continuous seminorm $\widetilde{p}$ on E such that $\widetilde{p}\big|_{E_0} = p$ and $\widetilde{p}(x_0) \geq r$.*

{lem:A.1.7}

Lemma 1.7 (Equivalence of seminorms)**.** *Let $(E, (p_i)_{i\in I})$ be a* LCTVS*. A seminorm $p : E \to \mathbb{R}_+$ is continuous iff*

$$\exists_{i_1,\ldots,i_r} \exists_C \forall_{x\in E} \quad p(x) \leq C \max_{j=1}^{r} p_{i_j}(x).$$

It is clear that we can replace $(p_i)_{i\in I}$ by the set if all continuous seminorms without changing the topology on E. Furthermore, it is easy to see that families $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ of seminorms induce the same locally convex topology on E if p_i is continuous on $(E, (q_j)_{j\in J})$ for all i and q_j is continuous on $(E, (p_i)_{i\in I})$ for all j.

2. Linear Operators

The natural morphisms between LCTVS are *continuous linear* maps.

{prop:A.2.1}

Proposition 2.1. *Let $(E, (p_i)_{i\in I})$ and $(F, (q_j)_{j\in J})$ be* LCTVS*. For a linear operator $T : E \to F$ the following statements are equivalent:*

(1) *T is continuous in 0.*
(2) *T is continuous.*
(3) $\forall_{j\in J}\exists_{i_1,\dots,i_r\in I}\exists_{C>0}\forall_{x\in E}:\quad q_j(Tx)\le C\cdot\max_{d=1}^{r} p_{i_l}(x)$.

Proof. □

{prop:A.2.2}

Proposition 2.2. *Let E be an* LF*-space with defining sequence (E_n) of Frèchet spaces. Let F be a* LCTVS *and $T:E\to F$ a linear map. Then the following statements are equivalent:*

(1) *T is continuous.*
(2) *T is sequentially continuous.*
(3) *For each n the restriction $T\big|_{E_n}$ is continuous.*

Proof. □

2.1. The dual space. Let E be a LCTVS. Put

$$E' = \mathscr{L}(E,\mathbb{K}) = \{\text{continuous linear functionals } E\to\mathbb{C}\}\,.$$

The *Hahn-Banach Theorem* implies that E' is separating on E, that is for $x\in E\backslash\{0\}$ there exists $\phi\in E'$ with $\phi(x)\ne 0$. Each $x\in E$ gives rise to a seminorm $|\langle x,\cdot\rangle|$ on E'.

$E'_\sigma = (E',\{|\langle x,\cdot\rangle|\}_{x\in E})$ is called the weak-* dual of E. We will mostly consider this topology on dual spaces. Conversely, each $x'\in E'$ gives rise to a seminorm $|\langle\cdot,x'\rangle|$ on E. The corresponding locally convex topology on E is called the weak topology. It is always weaker (in generally strictly weaker) than the original topology on E.

If E, F are LCTVS and $T\in\mathscr{L}(E,F)$ the we define the transposed map $T^t: F'\to E'$ by

$$\langle e,T^t f'\rangle := \langle Te,f'\rangle,\quad \text{for } f'\in F', e'\in E'.$$

Since the seminorms on E'_σ are of the form $|\langle e,\cdot\rangle|$, $e\in E$, we conclude that $|\langle e,T^t f'\rangle|$ is dominated by $|\langle Te,f'\rangle|$. Hence $T^t: F'_\sigma\to E'_\sigma$ continuous.

2.2. Closed Graph and Open Mapping Theorem.

{prop:A.2.3}

Proposition 2.3. *Let E, F be Fréchet spaces and T a* surjective *linear map. Then T is an open map, i.e. T maps open subsets $U\subset E$ onto subsets $T(U)\subset F$. In particular, if T is bijective then T^{-1} is continuous, too (inverse operator Theorem).*

The proof relys on the Baire Theorem. Hence it is important that Fréchet spaces are complete metrizable.

Proof. E.g. [Rud91, 2.11]. □

{prop:A.2.4}

Proposition 2.4. *Let E, F be Fréchet spaces. For a linear map $T:E\to F$ the following statements are equivalent:*

(1) *T is continuous.*
(2) *The graph $\Gamma(T)=\big\{(x,y)\in E\times F \;\big|\; y=Tx\big\}$ is closed.*

Proof. □

3. The Banach-Steinhaus Theorem

{thm:A.3.1}

Theorem 3.1. *Let E be a Fréchet space and F a* LCTVS. *Suppose that $\{T_i\}_{i\in I}$ is a family of continuous linear maps $E \to F$ which is* pointwise uniformly *bounded, that is*

$$\forall_{x\in E}\forall_{p \textit{ continuous seminorm on } F}\exists_C\forall_{i\in I}: \quad p(Ti_x) \le C. \tag{35}$$

Then the family is equicontinuous. *That is*

$$\forall_{p \text{ is continuous seminorm on } F}\exists_{q \text{ continuous seminorm on } E}\forall_{i\in I}\forall_{x\in E}: \quad p(T_i x) \le q(x).$$

Proof. □

We mention the following application:

{prop:A.3.2}

Proposition 3.2. *Let E, F be Fréchet spaces and let $\beta : E \times F \to \mathbb{C}$ be a bilinear separately continuous map. Then β is continuous. Hence there exist continuous seminorms p on E, q on F such that*

$$|\beta(x, y)| \le p(x)q(y), \qquad \textit{for } x \in E, y \in F.$$

Separately continuous means that for $x \in E$ the map $\beta(x,\cdot) : F \to \mathbb{C}$ and for $y \in F$ the map $\beta(\cdot, y) : E \to \mathbb{C}$ are continuous.

Proof. □

4. Hilbert spaces

Recall from linear algebra that a unitary vector space H is a $\mathbb{C}$-vector space with a scalar product

$$(\cdot,\cdot) : H \times H \to \mathbb{C}.$$

I adopt the convention that scalar products are antilinear in the *first* variable:

$$(\lambda x, y) = \overline{\lambda} \cdot \langle x, y\rangle.$$

The scalar product induces a natural *norm* which is given by

$$\|x\| := \sqrt{(x, x)}.$$

{def:A.4.1}

Definition 4.1. *A unitary vector space is called a* Hilbert space *if it is complete with regard to $\|\cdot\|$.*

Examples 4.2. $\ell^2(\mathbb{Z}_+)$ and $L^2(\Omega, \varrho dx)$ with $\Omega \subset \mathbb{R}^n$.

4.3. Important facts

1. Cauchy-Schwarz inequality.[1]

$$|(x, y)| \le \|x\|\|y\|,$$

in particular $\|x\| = \sup_{\|y\|\le 1} |(x, y)|$. Equality iff x and y are linear dependent.

---

[1] After Hermann Amandus Schwarz, not Laurent Schwar**t**z.

2. Parallelogram law.

$$\forall_{x,y\in H} : \|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The most important feature which distinguishes Hilbert spaces from general Banach spaces is the Projection Theorem and its consequences:

Theorem 4.4 (Projection Theorem). *Let H be a Hilbert space and $L \subset H$ a closed subspace. For $x \in H$ there is a unique $P_L(x) \in L$ such that*

$$\|x - P_L(x)\| = \min_{y\in L} \|x-y\|.$$

P_L is a self-adjoint linear operator with $P_L^2 = P_L$, i.e. it is the orthogonal projection *onto L.*

Proof. ☐

An immediate consequence of the Projection Theorem is that $H = L \oplus L^{\perp}$ and $L^{\perp\perp} = L$.

{cor:A.4.4}

Corollary 4.5. 1. *If $\varphi \in H'$ there is a unique vector $x_\varphi \in H$ such that $\varphi = (x_\varphi, \cdot)$. In other words a Hilbert space is complex antilinearly isometrically isomorphic to its dual space. It particular, Hilbert spaces are reflexive.*

2. *For any $T \in \mathscr{L}(H) = \{$*bounded linear operators $H \to H$*$\}$ there exists a unique adjoint $T^* \in \mathscr{L}(H)$. One has $\|T\| = \|T^*\|$.*

4.1. Orthonormal bases.

{def:A.4.5}

Definition 4.6. *For an orthonormal system $(e_i)_{i\in I}$, $((e_i, e_j) = \delta_{ij})$ is called a* Hilbert basis *if $\overline{\operatorname{span}(e_i)_{i\in I}} = H$. Gram-Schmidt shows that Hilbert bases exist; if H is not separable then also Zorn's Lemma is needed.*

{thm:A.4.6}

Theorem 4.7. *Let $(e_i)_{i\in I}$ be an orthonormal basis. Then for each $x \in H$ we have a convergent series representation*

$$x = \sum_{i\in I} (e_i, x)\, e_i, \quad \|x\|^2 = \sum |(e_i, x)|^2.$$

Proof. ☐

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