Aufgaben zur Topologie

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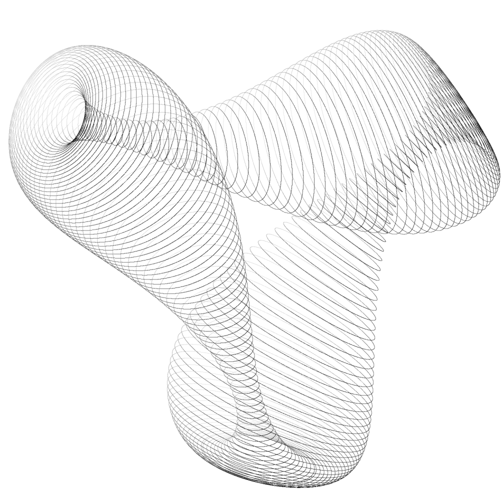
Wintersemester 2016/17

Week 2 — Covering spaces to be handed in on 09.11.2016 (before the lecture)



A 3-fold covering of the Klein bottle.

Exercise 2.1 (The higher spheres are simply-connected: $\pi_1(\mathbb{S}^n, x_0) = 1$ for $n \geq 2$.)

We prove this by showing in several steps, that any closed curve $\alpha\colon [0,1] \to \mathbb{S}^n$ with $\alpha(0) = \alpha(1) = x_0$ is contractible relative to $\{0, 1\}$. In most steps we use a homeomorphism $(\mathbb{S}^n - \{P\}, x_0) \to (\mathbb{R}^n, 0)$ of pointed spaces.

(1) If α does not cover the entire sphere, then α is contractible relative to $\{0, 1\}$. (N.B. There are curves (e.g. the Peano curves), which cover an entire sphere, even for $n > 1$.)
(2) There are finitely many $0 = t_0 < t_1 < \ldots < t_m = 1$, such that, for each curve $\alpha([t_k, t_{k+1}])$, there is (at least) one of the 2(n+1) open half-spheres $U_i^{\pm} := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \pm x_i > 0\}$, that entirely contains it, $k = 0, \ldots, m-1$, $i = 1, \ldots, n+1$. (This is an application of the Lemma of Lebesgue.)
(3) Any path $\beta\colon [a, b] \to \mathbb{S}^n$ that is contained in some half-sphere $U_i^{\pm}$ is homotopic, in $U_i^{\pm}$ and relative to $\{a, b\}$, to a path running along a section of the great circle from $\beta(a)$ to $\beta(b)$.
(4) Given (2), and using (3), there is a homotopy relative to $\{0, 1\}$ between α and a closed path γ, which runs piecewise along sections of finitely many great circles.
(5) Since a path γ as in (4) satisfies the condition of (1), we are done. Where did we use $n > 1$?

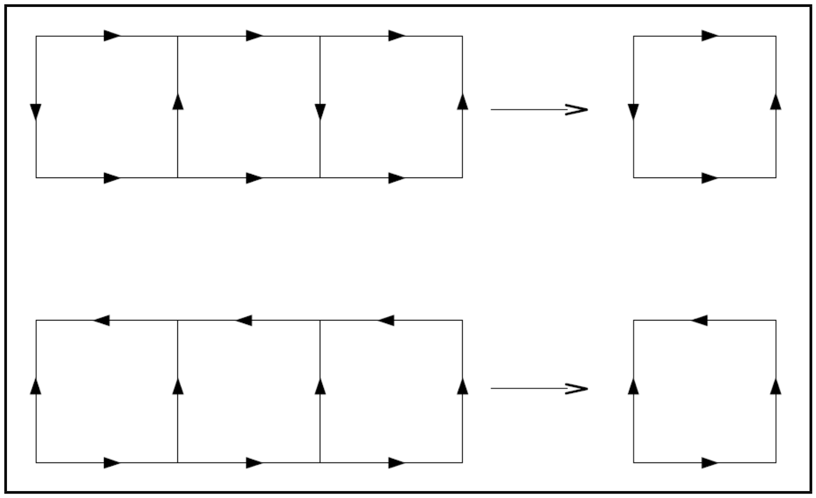
Exercise 2.2 (Coverings of the figure-eight space)

Find all 2-fold and 3-fold coverings of the figure-eight space $X = \mathbb{S}^1 \vee \mathbb{S}^1$: first classify all coverings, connected or disconnected, by giving two permutations; then sort by the number of connected components.

Exercise 2.3 (Sums, products and compositions of coverings)

(1) The sum $\tilde{X} \sqcup \tilde{Y}$ of two coverings $\tilde{X} \to X$ and $\tilde{Y} \to Y$ is a covering of the sum $X \sqcup Y$.

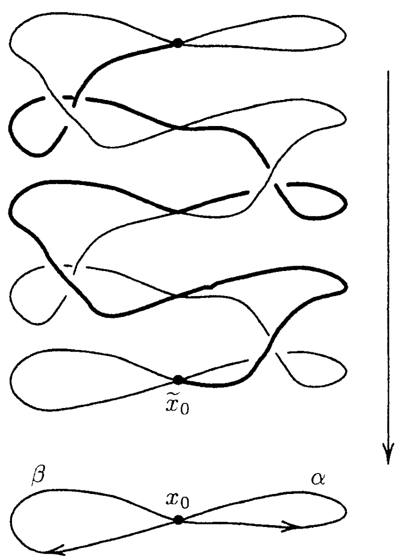
(2) The product $\tilde{X} \times \tilde{Y}$ of two coverings $\tilde{X} \to X$ and $\tilde{Y} \to Y$ is a covering of the product $X \times Y$.



Two 3-fold coverings of the Klein bottle.

(3) If $\tilde{X} \to X$ and $\tilde{\tilde{X}} \to \tilde{X}$ are two finite coverings, then the composition $\tilde{\tilde{X}} \to X$ is a covering.

Exercise 2.4 (Klein bottle covering itself.)
Show that the two maps in the figure above are 3-fold coverings of the Klein bottle K.

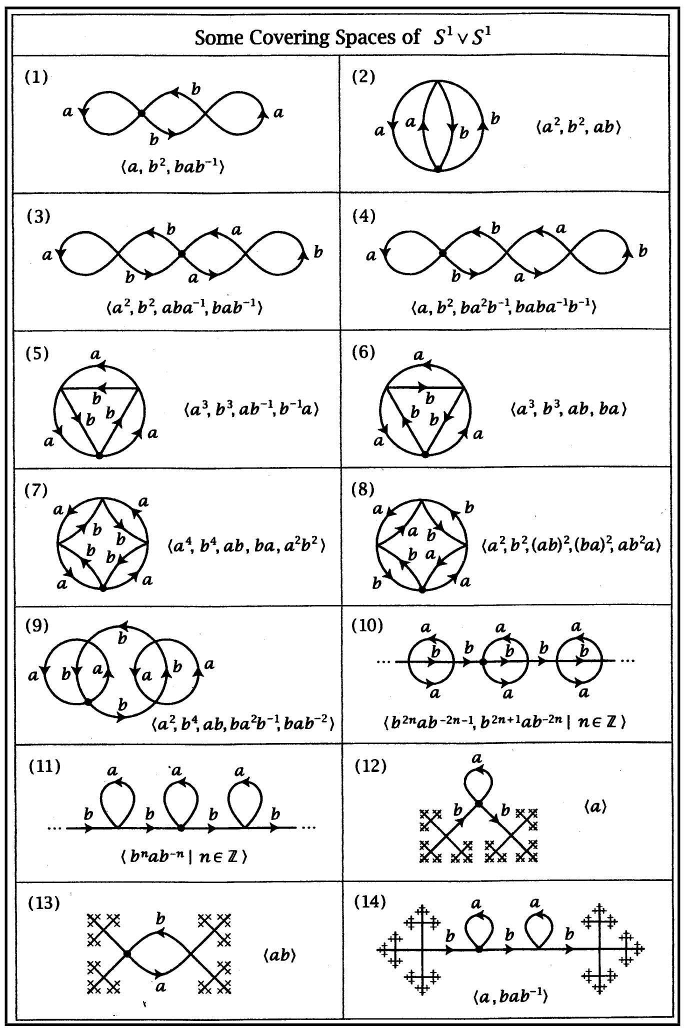


A 5-fold covering of the figure-eight space.
From A.Fomenko, D.Fuchs: *Homotopical Topology*, p.70.

Exercise 2.5 (A non-commutative fundamental group.)
The fundamental group $\pi_1(X, x_0)$ of the figure-eight space $X = \mathbb{S}^1 \vee \mathbb{S}^1$ is non-abelian. Assume it were commutative; consider the commutator $[\gamma] := [\alpha]\,[\beta]\,[\alpha]^{-1}\,[\beta]^{-1}$, where α resp. β is the closed curve running counter-clockwise along the right resp. clockwise along the left leaf of the bouquet $\mathbb{S}^1 \vee \mathbb{S}^1$, as in the figure above. If $[\gamma]$ were the trivial element, the lift $\widetilde{\gamma}$ of $\gamma = \alpha * \beta * \bar{\alpha} * \bar{\beta}$ in any covering $\tilde{X} \to X$ would be a closed curve.

Exercise 2.6 (Some ∞-fold coverings.)

As in the previous exercise, let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ be the figure-eight space and let a and b be the closed curves described above. For each of the following subgroups G of $\pi_1(X, x_0)$, draw a covering $\tilde{X} \to X$ with the property that the image of the induced map of fundamental groups $\pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is G.
(a) $G =$ the normal subgroup generated by $[a]$.
(b) $G =$ the normal subgroup generated by the element $[c]$ defined in the previous exercise.
(c) $G =$ the subgroup generated by $[c]$.
(d) Now let w be any finite sequence of elements of the set $\{a, b, \bar{a}, \bar{b}\}$ and take $G =$ the subgroup generated by $[w]$.
In each case, once you have constructed a covering $\tilde{X} \to X$ which potentially corresponds to the correct subgroup G, what you need to check is that a based loop in X lifts to a *loop* (not just a path) in $\tilde{X}$ if and only if it represents an element of $\pi_1(X, x_0)$ that lies in G.



Some coverings of the figure-eight space.
From A.Hatcher: *Algebraic Topology*, Cambridge Univ. Press 2002, p. 58.