Exercises for Algebraic Topology II

Prof. Dr. C.-F. Bödigheimer

Summer Term 2018

Blatt 11 due by: 9.07.2018

Jean Leray (1906 - 1998), invented spectral sequences, while he was prisoner-of-war between 1940 and 1945.
The foto was taken 1961 in Oberwolfach.

Exercise 11.1 (Spectral sequences in Linear Algebra)

Let $E = V \oplus W$ be a vector space over a field $\mathbb{F}$; we regard the decomposition as a coming from a filtration $F_0E = 0$, $F_1E = V$ and $F_2E = E$. Let $\partial\colon E \to E$ be a differential which preservs the filtration; written as a matrix in a basis of V and of W the differential has the form

$$\partial = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

and since $\partial \circ \partial = 0$ we must have $A^2 = 0$, $D^2 = 0$ and $AB + BD = 0$. We consider the associated spectral sequence as a kind of basic example. Our aim is the homology $H(E; \partial)$ of this differential object, or say the Betti number $b(E) := \dim H(E; \partial) = \dim(\ker(\partial)) - \dim(\mathrm{im}(\partial))$.
(Note that we have no grading on E; so we set the homological grading $q = 0$ for all terms.)

(1) The 0-page consists of $E^0_{0,1} = F_1E/F_0E = V$ and $E^0_{0,2} = F_2E/F_1E = W$. The differential d^0 is A on V and is D on W.

(2) Then on the 1-page we have $E^1_{0,1} = H(V; A)$ and $E^1_{0,2} = H(W; D)$. The differential $d^1 \colon E^1_{0,2} \to E^1_{0,1}$ is

$$d^1(y + \operatorname{im} D) := B(y) + \operatorname{im} A \quad \text{for } y \in W.$$

(3) If we regard $B \colon W \to V$ as a differential morphism ('chain map') — after sneaking in some sign — from the differential object (W, D) to (V, A), then the induced map $H(B) = B_* \colon H(W; D) \to H(V; A)$ is exactly d^1.

(4) The 2-page is $E^2_{0,1} = \operatorname{coker}(B_*)$ and $E^2_{0,2} = \ker(B_*)$.

(5) Find the isomorphism $H(E; \partial) \cong \operatorname{coker}(B_*) \oplus \ker(B_*)$.

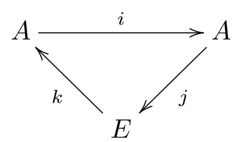
Exercise 11.2 (Bockstein spectral sequences)
Consider the extensions

$$(A) \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{\ p\ } \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

$$(B) \qquad 0 \longrightarrow \mathbb{Z}/p \xrightarrow{\ p\ } \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

for some (prime) number p.
Choose (A) or (B) and construct the exact (Bockstein) couple associated these extensions. Determine if the spectral sequence converges; and if so, to what does it converge ?

Exercise 11.3 (Exact couples)
Consider an exact couple (E, A)



in the category of R-modules. Assume, Φ is an exact covariant endo-functor.

- We get a new exact couple $(\Phi(E), \Phi(A))$.
- If the spectral sequence for (E, A) converges to M, does the spectral sequence for $(\Phi(E), \Phi(A))$ converge to $\Phi(M)$?

Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

Ravi Vakil, Lecture notes

Exercise 11.4 (Wang sequence)
Let ξ be a fibre bundle $F \to E \to B$ over $B = \mathbb{S}^1$ with structure group G. Recall that ξ is determined by the homotopy class of a (based) clutching function $c_\xi \colon \mathbb{S}^0 = \{\pm 1\} \to G$, so by a single element $c_\xi(+1) \in G$, in other words by some homeomorphism $\gamma \colon F \to F$. In fact, E is homeomorphic to the mapping torus of γ.
Invoking the Mayer-Vietoris sequence (for the obvious decomposition E_1 resp. E_2 the restrictions of E to the upper resp. lower hemisphere of $\mathbb{S}^1$) we saw — a long time ago — that the homology of E is isomorphic to (in the appropriate degrees) the kernel resp. cokernel of the endomorphism $\gamma_* - \mathrm{id} \colon H_*(F) \to H_*(F)$.
Relate this to Exercise 11.1.