Exercises for Algebraic Topology II

Prof. Dr. C.-F. Bödigheimer

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Blatt 10 due by: 2.7.2018

Exercise 10.1 (The first Stiefel-Whitney class)
A real vector bundle ξ is orientable if and only if its first Stiefel-Whitney class $w_1(\xi)$ vanishes.

Exercise 10.2 (Classifying spaces of categories)
If two categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent, show that there classifying spaces $B\mathcal{C}$ and $B\mathcal{C}'$ are homotopy-equivalent.

Norman Steenrod (1910 — 1971)

Exercise 10.3 (Milnor construction)
Let G be a topological group. We denote by $EG := \lim G \star \ldots \star G$ the Milnor construction of G, where $G \star \ldots \star G$ denotes the $(n+1)$-fold join of G. We denote elements in EG by the equivalence class $[\underline{t}, \underline{g}]$, where $\underline{t} = (t_i)$ is a sequence of barycentric coordinates and $\underline{g} = (g_i)$ is a sequence of group elements.

- If $\phi\colon G \to G'$ is a continuous homomorphism of groups, there is an induced map $E\phi\colon EG \to EG'$.
- $E\phi$ is equivariant in the sense $E\phi(\gamma(\underline{t}, \underline{g})) = \phi(\gamma)E\phi(\underline{t}, \underline{g})$.
- If $\phi_0 \simeq \phi_1$ are homotopic through a homotopy of homomorphisms ϕ_t, then there is a G-equivariant homotopy $E\phi_0 \simeq E\phi_1$.

- If $G = G'$ and $\phi\colon G \to G$ is an automorphism, then $E\phi$ is a G-equivariant homeomorphism.

Exercise 10.4 (Inner automorphisms: conjugation in a group)

Let G be a topological group and consider for an arbitrary element $\gamma \in G$ the inner automorphism $\kappa_\gamma\colon G \to G, \quad g \mapsto \gamma\, g\, \gamma^{-1}$. It induces on BG a map homotopic to the identity.
(Hint: the self-map $[\underline{t},\, \underline{g}] \mapsto [\underline{t},\, \underline{g}\gamma^{-1}]$ of EG is G equivariant and thus (even G-equivariant) homotopic to the identity. The self-map $[\underline{t},\, \underline{g}] \mapsto [\underline{t},\, \gamma\underline{g}\gamma^{-1}]$ induces on the quotient $BG = EG/G$ the same map (where we act on EG on the left).)

Exercise 10.5* (Clutching construction)

Let $\xi\colon E \to B$ be an (F, G)-bundle over a suspension $B = \Sigma X$. We assume that the G action on the fibre F is faithful.

(a) We decompose the base space in to the closed upper and lower hemispheres $B^+ = \Sigma^+ X$ resp. $B^- = \Sigma^- X$, we identify their intersection as the equator X, and choose trivializations $h_+\colon E^+ = \xi^{-1}(B^+) \to B^+ \times F$ resp. $h_+\colon E^- = \xi^{-1}(B^+) \to B^- \times F$. Their restrictions $h^\pm E^0 := \xi^{-1}(X) \to X \times F$ give the map $h_- \circ h_+^{-1}\colon X \times F \to X \times F$ of the form $(x, y) \to (x, H(x, y))$ for some map $H : X \times F \to F$. Since H must be G-equivariant, the adjoint is a map

$$\mathrm{cl}_\xi = \mathrm{cl}\colon X \to G, \qquad \text{with} \qquad H(x, y) = \mathrm{cl}(x)\, y,$$

which is called a *clutching function* for ξ.

(b) Show: The homotopy class of cl_ξ does not depend on the choice of the trivialisations over $\Sigma^\pm X$.

(c) Show: If $\xi \cong \xi'$, then $\mathrm{cl}_\xi \simeq \mathrm{cl}'_\xi$.

(d) Vice versa, show that any function $c\colon X \to G$ determines a bundle $\xi = \xi_c$ over ΣX with total space $E := (\Sigma^+ X \times F) \sqcup (\Sigma^- X \times F)/(0, x, y) \sim (0, x, c(x)y)$. And c is obviously a clutching function for this ξ_c.

(e) Show: If $c \simeq c'$, then $\xi_c \cong \xi'_c$.

Thus altogether we have an isomorphism

$$\mathrm{Bun}_G^F(\,\Sigma X\,) \xrightarrow{\cong} [X, G],$$

and in particular for principal G bundles $\mathrm{Prin}_G(\Sigma X) \cong [X, G]$. What is the relation to the classification theorem $\mathrm{Bun}_G^F(B) \cong [X, BG]$ for arbitrary base spaces ?