Exercises for Algebraic Topology II

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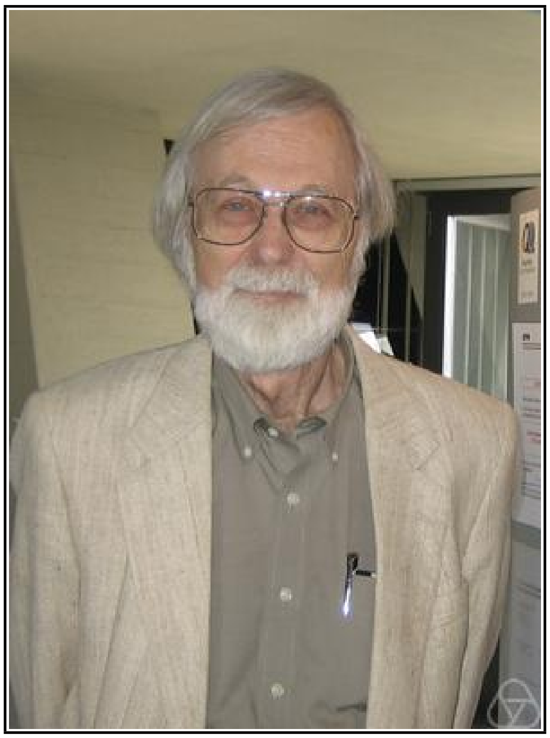
Blatt 7 due by: 11.06.2018



John Milnor, born 1931

Exercise 7.1 (Maps between Eilenberg-MacLane spaces)

Let $n \geq 1$ and G and G' be two groups, both abelian if $n \neq 1$, and assume we are given a homomorphism $\phi\colon G \to G'$. Then there is a map $f\colon K(G,n) \to K(G',n)$ which induces ϕ on the n-th homotopy group.
(Hint: We know how to construct $K(G,n)$ as $\mathrm{SP}(M(G,n))$, if $M(G,n)$ is the corresponding Moore space. So it will be enough to construct a map $g\colon M(G,n) \to M(G',n)$ which induces ϕ on the n-th homology group. But this can be done in the spirit of Excercise 6.3.)

Exercise 7.2 (Co-H-spaces)

Recall the definition of an H-space. Define the dual concept of a co-H-space and the properties h-unit, h-co-inverse, h-co-associativity and h-co-commutativity. Then show:

(a) A suspension ΣX is a h-coassociative co-H-space with h-co-unit. Is there a h-co-inverse ?

(b) A 2-fold suspension $\Sigma^2 X$ is in addition h-co-commutative.

(c) If X is a co-H-space, than the set $[Y, K]$ of homotopy classes of maps $X \to K$ is a (discrete) monoid. And if X has a h-co-unit, has a h-co-inverse, is h-associiative resp. h-co-commutative, then this monoid has a unit, is a group, is associative resp. is commutative.

Exercise 7.4 (Brown Representability, example)

Consider for a connected space X with base point x_0 the set $F(X)$ of based isomorphism classes of 2-fold coverings of X. This is a contravariant functor $\Phi\colon CW_0^{\mathrm{conn}} \longrightarrow \mathrm{Set}$ from the category of connected, based CW complexes to sets. Show:

(1) Φ is homotopy invariant.

(2) Φ satisfies the Mayer-Vietoris Axiom.

(3) Φ satisfies the Wedge Axiom.

By Brown's Representability Theorem, there is a representing space K. — Any guess ?
And furthermore, there must be universal element $\omega \in \Phi(K)$, i.e., a non-trivial 2-fold covering $E \to K$, such that any 2-fold covering ζ over any X is the pull-back $\zeta = f^*(\omega)$ of ω under a suitable map $f\colon X \to K$. Show

(4) E is (weakly) contractible.

Putting this together we derive from the antipodal action on the contractible space $\mathbb{S}^\infty$: (I) the representing space of Φ is the quotient $K = \mathbb{R}P^\infty$ of the antipodal action, and (II) the universal element of Φ is the antipodal covering $\omega\colon \mathbb{S}^\infty \to \mathbb{R}P^\infty$.

Exercise 7.4 (Brown Representability and classifying spaces)

Now we consider more generally bundles over X with fiber F and any topological group G as structure group, possibly not discrete. Denote the set of isomorphism classes of such bundles $\Phi(X) := \mathrm{Bun}_G^F(X)$.

Is there any reasonable doubt, that the statements corresponding to those in Exercise 7.3 are also true in this general setting ?

The resulting representing space is called the *classifying space* of G, denoted by BG; if G is discrete and $n = 1$, then $BG = K(G, 1)$ is an Eilenberg-MacLane space. The universal element is called the universal F-bundle with structure group G, namely $\omega\colon EG \times_G F \to EG/G = BG$. Here EG is any (nice) contractible space with a (nice) free G-action; for example, the Milnor construction gives one such space EG for any G.

Exercise 7.5* ('Dold-Thom Splitting' - the homology of an infinite symmetric product splits)

Let X be a connected space withbase point x_0 and consider on its infinite symmetric product $\mathrm{SP}(X, x_0)$ the filtration by the finite symmetric products $\mathrm{SP}_n(X, x_0)$. The basepoint in $\mathrm{SP}(X, x_0)$ be denote suggestively by 0. (*To simplify notation, we drop the base points from now on everywhere.*) We denote the filtration quotients by $D_n(X) := \mathrm{SP}_n(X)/\mathrm{SP}_{n-1}(X)$, its finite bouquets by $V_n(X) := \bigvee_{k=1}^n D_k(X)$ and the infinite bouquet by $V(X) := \bigvee_{k\geq 1} D_k(X)$. Note that $D_n(X) = X^{(n)}/\mathfrak{S}_n$, the n-th symmetric smash product. (Compare this setting with the Snaith splitting.)

Prove the following statement: *There is a weak homotopy equivalence*

$$\Psi\colon \mathrm{SP}(\mathrm{SP}(X)) \longrightarrow \mathrm{SP}(\bigvee_{k\geq 1} \mathrm{SP}_n(X)/\mathrm{SP}_{n-1}(X)).$$

Thus

$$H_*(\mathrm{SP}(X)) \cong \bigoplus H_*(\mathrm{SP}_n(X)/\mathrm{SP}_{n-1}(X)).$$

Hint: This is much easier than the Snaith splitting. For a $\zeta = [x_1, \dots x_n] \in \mathrm{SP}_n(X)$ consider all subdivisors $\zeta_\alpha \in \mathrm{SP}_k(X)$, where α is a subset of the index set $\{1, \dots, n\}$; there $\binom{n}{k}$ many of length k. Denote their image under $\mathrm{SP}_k(X) \to D_k(X)$ by $\bar{\zeta}_\alpha$. Summing all these gives a map $\Psi_n'\colon \mathrm{SP}_n(X) \to \mathrm{SP}(V_n(X))$. All Ψ_n' fit together to give a map $\Psi'\colon \mathrm{SP}(X) \to \mathrm{SP}(V(X))$, which extends to the desired map Ψ. This Ψ now restricts to maps $\Psi_n\colon \mathrm{SP}(\mathrm{SP}_n(X)) \to \mathrm{SP}(V_n(X))$. Show by induction, that each Ψ_n, and thus Ψ, is a weak homotopy equivalence.