Exercises for Algebraic Topology II

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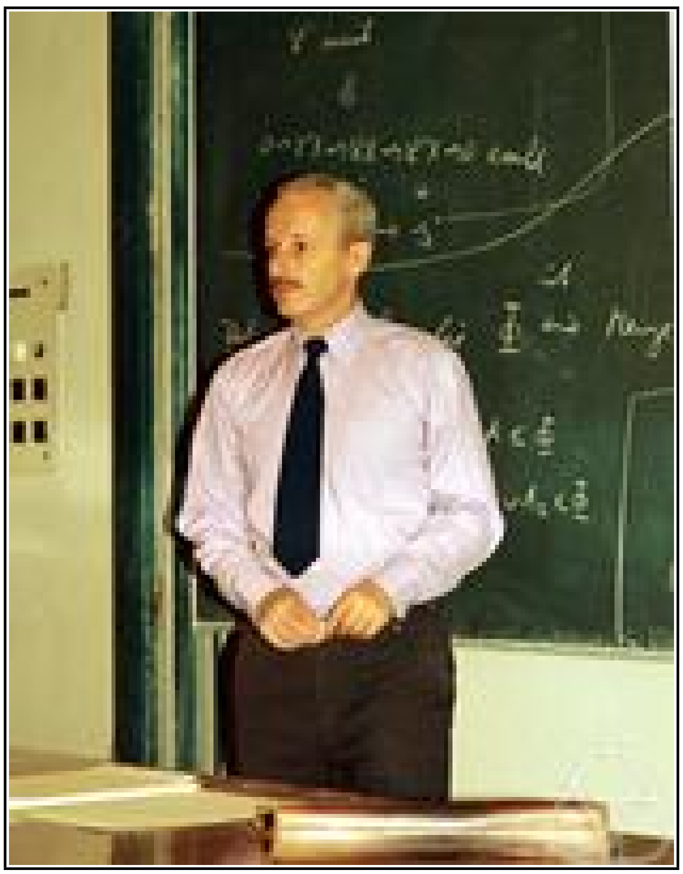
Blatt 6 due by: 4.06.2018



Albrecht Dold (1928 - 2011)

Exercise 6.1 (Fundamental Theorem of Algebra, topological version)

Regard an element of SP_n as a (non-negative) divisor and associate to it the (projective class of the) coefficients of any polynomial with zeroes and their multiplicities given by the divisor. This is a homeomorphism $\Phi\colon \mathrm{SP}_n(\mathbb{S}^2) \to \mathbb{C}P^n$. Give the details and conclude:

- $\mathrm{SP}_n(\mathbb{S}^2) \cong \mathbb{C}P^n$
- $\mathrm{SP}(\mathbb{S}^2), \infty \cong \mathbb{C}P^\infty$
- $\mathrm{SP}_n(\mathbb{R}P^2) \cong \mathbb{R}P^{2n}$

- $\mathrm{SP}(\mathbb{R}P^2, \infty) \cong \mathbb{R}P^\infty$

Exercise 6.2 (An nfinite loop space)

Show that $\mathrm{SP}(X, x_0)$ is an infinite loop space, if X is connected.

Exercise 6.3 (Eilenberg MacLane spaces)

(a) Show: For any abelian group G and $n \geq 1$ there is a Moore space $M(G, n)$.

[Hint: Take a presentation of G by generators and relations, thus a short exact sequence (or resolution of G): $R = \bigoplus_j \mathbb{Z} \to F = \bigoplus_i \mathbb{Z} \to G$, realize F and R by bouquets of n-spheres $X_F = \bigvee_i \mathbb{S}^n$ resp. $X_R = \bigvee_j \mathbb{S}^n$, realize the homomorphism $\phi\colon R \to F$ (which is a matrix $(\lambda_{i,j})$ by a map $f\colon X_R \to X_F$ from one bouquet to the other (such that the degree on the j-th sphere to the i-th sphere is $\lambda_{i,j}$), and study the mapping cone of f.]

(b) Show: $\mathrm{SP}(M(G, n))$ is an Eilenberg-MacLane space $K(G, n)$.

(c) Show: $M(G_1, n) \vee M(G_2, n)$ is a Moore space, and $K(G_1, n) \times K(G_2, n)$ is an Eilenberg-MacLane space.

Rene Thom (1923 - 2002)

Exercise 6.4 (Yoneda Lemma)

Let $\mathcal{C}$ be any category. Any object K in $\mathcal{C}$ defines a contravariant functor

$$\Phi_K \colon \mathcal{C} \longrightarrow \text{Set}, \quad \Phi_K(X) := \text{mor}_{\mathcal{C}}(X, K), \quad \Phi_K(\gamma)(f) = f \circ \gamma$$

for $\gamma \colon Y \to X$ and $f \in \text{mor}_{\mathcal{C}}(X, K)$. Let $F \colon \mathcal{C} \to \text{Set}$ be any contravariant functor.
Then there is a one-to-one correspondence between all natural transformations $\theta \colon \Phi_K \to F$ and the elements $\omega \in F(K)$.

Exercise 6.5* (Configuration spaces with labels, again)

For a cofibration sequence $A \to X \to (X, A)$ the functor 'infinite symmetric product' gives a quasifibration sequence $\text{SP}(A) \to \text{SP}(X) \to \text{SP}(X, A)$. Inspired by this fact we ask: Does (for a fixed manifold pair (M, M_0)) the functor 'configuration space of particles in (M, M_0) with labels in ..... ' give a quasifibration sequence $F = C(M; A) \to E = C(M; X) \to B = C(M; X, A)$?

The decisive moment in the proof is the decomposition of a configutation $\zeta = [z_1, z_2, \ldots; x_1, x_2, \ldots]$ into the sub-configuration ζ_A resp. ζ_{X-A} of all particles with labels in A resp. in $X - A$. This decomposition $\zeta \mapsto (\zeta_A, \zeta_{X-A})$ is a continuous map $D \colon E_n - E_{n-1} \to F \times (B_n - B_{n-1})$ on each stratum of E. Here we refer to the usual filtration of B by the number of particles, and the induced filtration on E; note that $B_n - B_{n-1} = C_n(M - M_0; X - A)$.

The problem: the inverse map should be the 'addition' $C(M, M_0; A) \times C_n(M - M_0; X - A) \longrightarrow C(M, M_0; X)$, $(\zeta_A, \zeta_{X-A}) \mapsto \zeta_A + \zeta_{X-A}$, where the $+$ stands for juxtaposition of two configurations. But such an addition is not possible, since the particles of ζ_A and ζ_{X-A} may not be disjoint.

So what can we do ? Perhaps we thicken $M, M_0)$ by an intervall I to $(M \times I, M_0 \times I)$ and put ζ_A on the 0-level and ζ_{X-A} on the 1-level. Thus we have $\mathcal{A} \colon C(M, M_0; A) \times C_n(M - M_0; X - A) \longrightarrow C(M \times I, M_0 \times I; X)$. The composition $\mathcal{A} \circ D$ is of course not the identity, but homotopic to the map, which puts the particles all on level 0. And running to the limit $C(M \times I^\infty, M_0 \times I^\infty; X)$ turns this inclusion into an identity.

This is the rough sketch to prove:

- The functor $\mathcal{C}(Y) := C(M \times I^\infty, M_0 \times I^\infty; Y)$ turns a cofibration sequence $A \to X \to (X, A)$ of connected spaces into a quasifibration sequence $\mathcal{C}(A) \to \mathcal{C}(X) \to \mathcal{C}(X, A)$, for each pair (M, M_0).
- Thus $h_q(Y) := \pi_q(\mathcal{C}(Y))$ is a homology theory of connected spaces, for each pair (M, M_0).
- For M a point and $M_0 =$ we have $\mathcal{C}(Y) = \Omega^\infty \Sigma^\infty Y$. Which homology theory is $h_*(Y) := \pi_*(\Omega^\infty \Sigma^\infty Y)$?