Exercises for Algebraic Topology II

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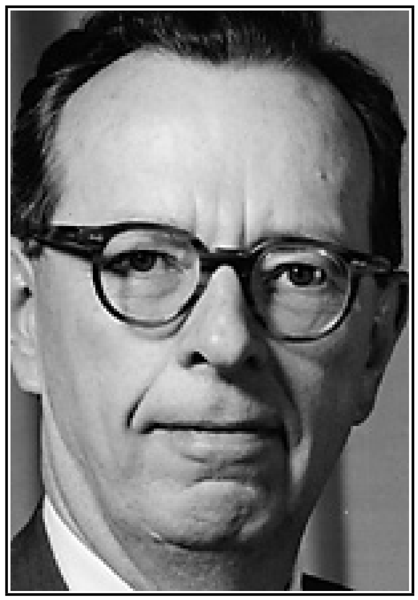
Blatt 4 due by: 14.05.2018



George Whitehead (1918 — 2004)

Exercise 4.1 (Whitehead product I)

Let X be a path-connected space with basepoint x_0. For $p, q \geq 0$ various we want to define a product

$$\pi_{p+1}(X, x_0) \times \pi_{q+1}(X, x_0) \longrightarrow \pi_{p+q+1}(X, x_0)\,,$$

inspired by the commutator product in the fundamental group, i.e., the case $p = q = 0$ below. For $\alpha = [a] \in \pi_{p+1}(X)$ and $\beta = [b] \in \pi_{q+1}(X)$ we define

$$[\alpha, \beta] := [(a \vee b) \circ W_{p,q}]\,,$$

where $W_{p,q} \colon \mathbb{S}^{p+q+1} \to \mathbb{S}^{p+1} \vee \mathbb{S}^{q+1}$ is a specific map, often called *Whitehead map*, namely: regard $a \colon (\mathbb{D}^{p+1}, \mathbb{S}^p) \to (X, x_0)$ and $b \colon (\mathbb{D}^{q+1}, \mathbb{S}^q) \to (X, x_0)$ as maps of pairs, consider $\mathbb{S}^{p+q+1}$ as the boundary of a cube, $\partial\mathbb{D}^{p+q+2} = \partial(\mathbb{D}^{p+1} \times \mathbb{D}^{q+1}) = (\mathbb{S}^p \times \mathbb{D}^{q+1}) \cup (\mathbb{D}^{q+1} \times \mathbb{S}^q)$ and $W_{p,q}$ is on the first part $\mathbb{S}^p \times \mathbb{D}^{q+1} \to \mathbb{D}^{q+1} \to \mathbb{D}^{q+1}/\mathbb{S}^q = \mathbb{S}^{q+1} \to \mathbb{S}^{p+1} \vee \mathbb{S}^{q+1}$ the composition of the projection, the quotient map and inclusion into the second wedge summand, and likewise on the second part, but with inclusion into the first wedge summand. And $a \vee b$ is the map given by a on the first and by b on the secand wedge summand.
Note that $W_{p,q}$ is the attaching map of the top $(p + q + 2)$-cell in the product $\mathbb{S}^{p+1} \times \mathbb{S}^{q+1}$ to the lower cells $\mathbb{S}^{p+1} \vee \mathbb{S}^{q+1}$.

- Make a drawing.

- Show that the Whitehead product is well-defined.
- For $p = q = 0$ the Whitehead product is the commutator product in $\pi_1(X, x_0)$: $[\alpha, \beta] = \alpha\,\beta\,\alpha^{-1}\,\beta^{-1}$.

  This explains the notation of the Whitehead product as a commutator, although for $p, q \geq 1$ the groups involved are abelian.
- For $p = 0 < q$ the Whitehead product is related to the action $\tau_\alpha \colon \pi_{q+1}(X, x_0) \to \pi_{q+1}(X, x_0)$ of the fundamental group: $[\alpha, \beta] = \tau_\alpha(\beta) - \beta$.

Exercise 4.2 (Whitehead product II)
Prove some or all of the following formulae:

(1) *Naturality*: $f_*([\alpha, \beta]) = [f_*(\alpha), f_*(\beta)]$ for a based map $f \colon (X, x_0) \to (Y, y_0)$.

(2) *Bilinearity*: $[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]$ and $[\alpha, \beta_1 + \beta_1] = [\alpha, \beta_1] + [\alpha, \beta_2]$ for $p, q \geq 1$.

(3) *Graded Commutativity*: $[\beta, \alpha] = (-1)^{(p+1)(q+1)}\,[\alpha, \beta]$ for $p + 1 = |\alpha|, q + 1 = |\beta|$.

(4) *Jacobi-Identity*: $(-1)^{(p+1)(r+1)}[[\alpha, \beta], \gamma] + (-1)^{(q+1)(p+1)}[[\beta, \gamma], \alpha] + (-1)^{(r+1)(q+1)}[[\gamma, \alpha], \beta] = 0$ for $p + 1 = |\alpha|, q + 1 = |\beta|, r + 1 = |\gamma|$.

Exercise 0.3 (Whitehead product III)
Conclude from Exercise 3.4 form last week that the suspension of a Whitehead product is always trivial: $\Sigma[\alpha, \beta] = 0$ in $\pi_{p+q+2}(X, x_0)$ for $\alpha \in \pi_{p+1}(X, x_0)$ and $\beta \in \pi_{q+1}(X, x_0)$.

Exercise 4.4* (EHP - sequence)
Let $n \geq 1$ and consider the first the case $X = \mathbb{S}^n$ and its reduced product space $J(\mathbb{S}^n)$.

- $(J(\mathbb{S}^n, \mathbb{S}^n)$ is $(2n-1)$-connected.
- Thus $\pi_i(J(\mathbb{S}^n), \mathbb{S}^n) \cong \pi_i(J(\mathbb{S}^n)/\mathbb{S}^n)$ for $i \leq 3n - 2$.
- The latter group is isomorphic to $\pi_i(\mathbb{S}^{2n})$, since the $(3n-1)$-skeleton of the quotient $J(\mathbb{S}^n)/\mathbb{S}^n$ is a $\mathbb{S}^{2n}$.
- Conclude that a (finite) portion of the relative homotopy sequence of the pair $(J(\mathbb{S}^n), \mathbb{S}^n)$ can be written as:

$$\pi_{3n-2}(\mathbb{S}^n) \xrightarrow{E} \pi_{3n-1}(\mathbb{S}^{n+1}) \xrightarrow{H} \pi_{3n-2}(\mathbb{S}^{2n}) \xrightarrow{P} \pi_{3n-3}(\mathbb{S}^n) \xrightarrow{E} \pi_{3n-2}(\mathbb{S}^{n+1}) \to \ldots$$

  This is called the EHP-sequence, since the first map E is the suspension (German Einhängung*), the second map H is a kind of higher Hopf invariant, and the third map P is related to the Whitehead product.*
- Study the sequence for $n = 2$, using the Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$.
- Generalize the above to an arbitrary $(n-1)$-connected space X, using $J_k(X) = X^{(k)}$ to investigate the $3n - 1$-skeleton.

A short biography of George Whitehead by Haynes Miller can be found here:
`http://www.nasonline.org/publications/biographical-memoirs/memoir-pdfs/whitehead-george.pdf`
