Exercises for Algebraic Topology II

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Lev Pontryagin (1908 — 1988)

Exercise 3.1 (Maps from a Lie group)

Let G be a connected Lie group of dimension m and X a path-connected CW complex; let $D \subset G$ be a closed disk around the unit $1 \in G$.
(a) Show that we have a diagram

$$\begin{array}{ccc} C(G-D;X,x_0) & \xrightarrow{\gamma} & \mathrm{map}(G,1;\Sigma^m X) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle j} \\ C(G;X,x_0) & \xrightarrow{\gamma} & \mathrm{map}(G;\Sigma^m X) \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle \mathrm{eval}_1} \\ C(G,G-D;X,x_0) & \xrightarrow{\gamma} & \Sigma^m X \end{array}$$

with a quasifibration Q on the left, a fibration eval_1 (the evaluation at the unit) on the right, and the horizontal maps (weak) homotopy equivalences.

(b)* There are obvious translation actions by G on the two middle spaces. Is the approximation map γ in this case equivariant ? What about the conjugation action (assuming, that D is invariant under conjugation) ?

(c) For $G = \mathbb{S}^1 = SO(2)$, the fibration on the right is the evalutaion fibration of the free loop space of ΣX with fibre the based loop space of ΣX. What are the filtration quotients on the left ?

Exercise 3.2 (Pontryagin product of $J(X)$)
The reduced product space $J(X)$ of a space X is an H-space. We know from Exc. ..... its homology, at least for a sphere $X = \mathbb{S}^n$. Determine the Pontryagin product

$$H_i(J(X)) \otimes H_j(J(X)) \to H_{i+j}(J(X) \times J(X)) \to H_{i+j}(J(X)),$$

when $H_*(X)$ is given, where the first map is the homology cross product and the second is induced by the H-space multiplication. Here we assume, that the homology is free (or we work over a field).

Bust of L. Pontryagin in Moscow

Exercise 3.3 (Splitting of a reduced product space $J(X)$)
Let $J_n(X)$ denote the filtration of the reduced product $J(X)$ by word length. We know for the filtration quotients $D_n(X) = J_n(X)/J_{n-1}(X) \cong X^{(n)}$, the n-fold smash product.
For any word $w = x_1 x_2 \dots x_m$ we consider all subwords w_α of length n as an element in $D_n(X)$, order them lexicographically with respect to the indices, and concatenate them in this lexicographic order to a word $W_{(n)} := w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_r}$ of length $r = \binom{m}{n}$. Obviously, $W_{(n)}$ is an element in $J(D_n(X))$,

(1) The map $w \mapsto W_{(n)}$ is a well-defined and continous map $h_n \colon J(X) \to J(D_n(X))$.

(2) We concatenate all these maps h_n to a single map $h \colon J(X) \to J(\bigvee_{n \geq 1} D_n(X))$, identify $J(Y) \simeq \Omega\Sigma Y$ and denote its adjoint by
$$h' \colon \Sigma J(X) \to \Sigma(\bigvee_{n \geq 1} D_n(X)) \simeq \bigvee_{n \geq 1} \Sigma X^{(n)}.$$

(3) Study the restrictions $h'_k \colon \Sigma J_k(X) \to \Sigma(\bigvee_{n=1}^{n=k} D_n(X))$ of h' to the filtrations $J_k(X)$ and show that we have

homotopy commutative diagrams,

$$\begin{array}{ccc}
\Sigma J_k(X) & \xrightarrow{h'_k} & \Sigma \bigvee_{n=1}^{n=k} D_n(X) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
\Sigma J_{k+1}(X) & \xrightarrow{h'_{k+1}} & \Sigma \bigvee_{n=1}^{n=k+1} D_n(X) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
\Sigma D_{k+1}(X) & \xrightarrow{\mathrm{id}} & \Sigma D_{k+1}(X)
\end{array}$$

where q and p are the obvious quotient maps and i and j the obvious inclusions.

(4) Show by induction, that each h'_k and thus h' is a weak homotopy equivalence (for X connected).

Exercise 3.4 (Splitting of a product space)
Recall the splitting

$$\Sigma(X \times Y) \simeq \Sigma(X \vee Y \vee (X \wedge Y)),$$

its homological analogue. Find for the powers X^n a connection to labelled configuration spaces $C(M; X)$ and their splitting, when M is a compact manifold of dimension 0.