Exercises for Algebraic Topology II

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Blatt 2 due by: Monday, 30.04.2018

The missing picture will come soon !

Exercise 2.1 (Configuration spaces of the circle)

Show that $C^n(\mathbb{S}^1)$ is homeomorphic to $(\mathbb{S}^1 \times \Delta^{n-1})/\mathbb{Z}_n$, where Δ^n denotes the open n-simplex and the action of a generator $T \in \mathbb{Z}_n$ is given by $T \cdot (\zeta, t_0, t_1, t_2, \ldots, t_{n-1}) = (\zeta \, \exp(2\pi i \, t_0), t_1, t_2, \ldots, t_{n-1}, t_0)$.

Exercise 2.2 (Quasifibrations I: fibre and homotopy fibre)

If a map $p\colon E \to B$ is a quasifibration, than the inclusion of the fibre $F_b = p^{-1}(b)$ over $b \in B$ into the homotopy fibre $\mathrm{hFib}(p, b)$ over b is a weak homotpy equivalence.

Exercise 2.3 (Quasifibrations II: gluing property)

Let $p\colon E \to B$ be a map of connected spaces and assume $B = B_1 \cup B_2$ is the union of two open, path-connected spaces with $B_0 := B_1 \cap B_2$ path-connected. If each $p_i\colon E_i := p^{-1}(B_i) \to B_i$ is a quasifibration ($i = 0, 1, 2$), then $p\colon E \to B$ is a quasifibration.

Exercise 2.4* (Complexes of spaces)

We have seen diagrams of spaces in the following way. Let Γ be a directed graph and assume that to each vertex $v \in \Gamma$ we have associated a space X_v and to each edge $v \to w$ we have associated a map $f_{w,v}\colon X_v \to X_w$. If we denote this collection of data by $\mathcal{X}$, we can build a space $\lim \mathcal{X}$ by

$$\lim \mathcal{X} := (\bigsqcup_v X_v)/\sim$$

where $x \in X_v$ is identified with $f_{w,v}(x) \in X_w$.

We have seen examples: (1) pushouts (or gluing) of two spaces over a third, where the diagram is $X_1 \leftarrow X_0 \rightarrow X_2$; (2) the direct limit, where the diagram is $X_0 \to X_1 \to X_2 \to \ldots$; (3) a quotient X/G of a group action by G on a space X, where the diagram is given by one vertex v with $X_v = X$ and an edge for each group element.

We have also seen how useful the homotopy versions of these constrctions are: homotopy push-outs (i.e. double mapping cylinders), homotopy colimits (i.e., telescopes) and homotopy quotients (i.e. Borel constructions $EG \times_G X$) are. So the general construction is a *complex of spaces* $\mathcal{X}$ given by a simplicial complex (or polyhedron) B whose 1-skeleton we denote by Γ; to each vertex v we again have associated a space X_v, to each edge $v \to w$ a map $f_{w,v}\colon X_v \to X_w$ such that for each n-simplex $\sigma \subset B$ with the vertices $v_0, v_1, \ldots v_n$ the maps

$$X_{v_0} \xrightarrow{f_1} X_{v_1} \xrightarrow{f_2} X_{v_2} \ldots \xrightarrow{f_n} X_{v_n}$$

and their composites form a commutative diagram. We then build iterated mapping cylinders $M(f_1, \ldots, f_n)$ to be the mapping cylinder of the composition $M(f_1, \ldots, f_{n-1}) \xrightarrow{\text{proj}} X_{n-1} \xrightarrow{f_{n-1}} X_n$, where the first map is the projection of the mapping cylinder onto the target space. There is a projection $M(f_1, \ldots, f_n) \to \Delta^n$ to the geometric realization $|\sigma| = \Delta^n$ of each simplex.

It is obvious how to glue all iterated mapping cylinders over the simplices together to a space $\mathrm{holim}\, \mathcal{X}$ and that we have a projection $p\colon \mathrm{hocolim}\, \mathcal{X} \to B$.

Finally, finally here is the claim: If all maps $f_{w,v}$ are weak homotopy equivalences, than $p\colon \operatorname{hocolim} \mathcal{X} \to B$ is a quasifibration.