S4B1: GRADUATE SEMINAR IN ANALYSIS

GROMOV'S NON-SQUEEZING THEOREM

WINTERSEMESTER 2026/27

INSTRUCTORS: Herbert Koch und Jan Bohr (lastname@math.uni-bonn.de)
PRELIMINARY MEETINGS: 20 July (2:15 pm in SR N0.008) and 15 October (12:15 pm in SR 0.007)
TIME & ROOM: Thursdays, 12:15pm, SR 0.007 — Starting on 15 October
WEBSITE: https://www.math.uni-bonn.de/ag/ana/WiSe2627/S4B1/index.htmpl

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New masters students are welcome! — Just drop in on 15th October.

OVERVIEW

In *Hamiltonian mechanics* the time evolution of a particle $q(t) \in \mathbb{R}^n$ with momentum $p(t) \in \mathbb{R}^n$ is governed by the following system of ordinary differential equations:

$$\begin{cases} \dot{q}_i = \dfrac{\partial}{\partial p_i} H(q,p) \\ \dot{p}_i = -\dfrac{\partial}{\partial q_i} H(q,p) \end{cases}, \quad i = 1, \dots, n.$$

The function $H\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is called the *Hamiltonian* and typically models the energy of the underlying physical system. For example, a pendulum is described by

$$H(q,p) = \frac{1}{2}p^2 + 1 - \cos q.$$

Given a subset $E_0 \subset \mathbb{R}^{2n}$ of initial states, let us write

$$E_t = \Big\{ \big(q(t), p(t)\big) : (q(0), p(0)) \in E_0 \Big\} \subset \mathbb{R}^{2n}$$

for the set of states reached at time $t \in \mathbb{R}$, starting in E_0 and evolving according to Hamilton's equations (we assume for simplicity that solutions exist for all times). Then *Liouville's theorem* states that the Lebesgue measure of E_t is constant in time:

$$|E_t| = |E_0|, \quad t \in \mathbb{R}.$$

The *non-squeezing theorem* of Gromov (1985) can be seen as a striking refinement of this result: Namely, if E_0 contains a ball of radius $R > 0$, then for all $t \in \mathbb{R}$ and all $0 < r < R$ it holds that:

$$E_t \not\subset Z_r = \{(q,p) \in \mathbb{R}^{2n} : q_1^2 + p_1^2 < r\}.$$

That is, the set E_0 cannot be squeezed into the cylinder Z_r by the Hamiltonian flow. For $n = 1$ this is a direct consequence of Liouville's theorem, but for $n \geq 2$ the cylinder has infinite measure and it less apparent where the obstruction comes from.

Mathematically, the influence of Gromov's paper is hard to overstate: It marks the advent of modern symplectic geometry and the techniques used to prove the nonsqueezing theorem have evolved into the theory of J-holomorphic curves, which underpins much of today's research in symplectic topology.

The goal of the seminar is to follow the early developments of J-holomorphic disks to the point where we can prove the nonsqueezing theorem. The focus will lie on analytical aspects around the equation for J-holomorphic disks, which is a quasilinear PDE for maps

$$u\colon \Omega \to \mathbb{R}^{2n}, \quad \Omega \subset \mathbb{C}.$$

Given a map $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ with

$$J^2(x) = -I,$$

and writing $z = s + it \in \Omega$, this PDE is given by

$$\partial_s u(s,t) + J(u(s,t)) \cdot \partial_t u(s,t) = 0 \text{ in } \Omega.$$

The basic example is the constant matrix

$$J_0 \equiv \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

where solutions $u(z) = (u_1(z), \dots, u_n(z)) \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ are simply vectors of holomorphic functions. Proving the non-squeezing theorem then requires developing a good understanding of this PDE (behaviour of individual solutions, bubbling, regularity and existence for suitable constraints). This partially parallels the development of the general theory discussed in the Nonlinear PDE lectures, while at the same time offering a glimpse into more advanced aspects pertaining to J-holomorphic curves.

PREREQUISITES

Basics of functional analysis (Banach spaces, Sobolov spaces) and a bit of complex analysis in one variable will be necessary. Some prior exposure to differential geometry will occasionally be useful, but is not a requirement.

TALK & WRITE-UP

Each topic should be presentend in a **talk** of ca. 75 minutes and summarised both on a brief **handout** (1 to 2 pages, typed with LaTeX) *and* a more detailed **write-up** (8 to 10 pages, typed with LaTeX). Each participant should arrange a meeting with one of the organisers until 2 weeks before their talk.

#	Topic	Reference
1	**Hamiltonian mechanics and symplectomorphisms** Hamiltonian equations, differential forms, Cartan's formula, Liouville's theorem, symplectomorphisms	[GZ] I.1 + I.2 & [Si]
2	**Pseudoholomorphic disks and nonsqueezing theorem** Almost complex structures in $\mathbb{R}^{2n}$, ω-compatibility, J-holomorphic disks, symplectic energy and area, proof idea for non-sequeezing	[GZ] I.3 + I.4.2
3	**Monotonicity lemma** Maximum principles (recap), isoperimetric inequalities, proof of monotonicity lemma	I.4 + I.5 (minus I.4.2)
4	**Moduli space and geometric bounds** Automorphisms of the disk (recap), Lagrangians, main theorem on moduli space (only statement), proof of C^0-bounds	[GZ] I.8 + II.1
5	**Bubbling and gradient bounds** Schwarz reflections (recap), the bubbling-off phenomenon, asymptotic isoperimetric inequality, proof of C^1-bounds (mod bootstrap)	[GZ] II.2+II.3+II.4
6	**Calderón–Zygmund inequality and a priori estimates** Marcinkiewicz interpolation, CZ-estimate, *a priori* estimates for $\bar{\partial}$	[GZ] III.1 + [MS] Appendix B.2
7	**Elliptic bootstrap and higher order bounds** Boundary and interior localisation, bootstrap, properness of evaluation map	[GZ] III.2+III.3
8	**Elliptic regularity** The operator $\bar{\partial}_J$ on Sobolev spaces, bounds on shift operator, smoothness of $W^{1,p}$-solutions ($p > 2$)	[GZ] IV.3
9	**Banach manifolds** Inverse function theorem for Banach spaces (recap), submanifolds of Banach spaces, Fredholm maps, Sard–Smale Theorem	[GZ] V.1.1 + V.1.4 + V.2 + [Sa]
10	**Fredholm theory** Banach manifold of $W^{k,p}$-disks, differentiability of $\bar{\partial}_J$, computation of linearisation D_u and, Fredholm property and $\mathrm{ind}(D_u)$	[GZ] IV.1 + V.1 (minus V.1.1 + V.1.4)
11	**Carleman similiarity principle and injective points** Boundary behaviour of holomorphic disks, Carleman similarity, injective points and local normal form	[GZ] V.3+V.4
12	**Universal Moduli space** Floer norm, Floer space of almost complex structures, transversaility for universal moduli space	[GZ] V.5 + V.6
13	**Degree of the evaluation map** Moduli space of generic J, degree of evaluation map, complete non-squeezing proof	[V.7]

References

[GZ] Geiges and Zehmisch, *A course on holomorphic discs*, Birkhäuser Advanced Texts. Basler Lehrbücher
[MS] McDuff and Salamon, *J-holomorphic curves and symplectic topology*, Colloq. Publ., Am. Math. Soc.
[Si] Cannas da Silva, *Lectures on symplectic geometry*, Lect. Notes Math.
[Sm] Smale, *An infinite dimensional version of Sard's theorem*, Am. J. Math.