S4B2 – Topics in Incompressible Fluid Dynamics

Dimitri Cobb – Winter Semester 2024-2025

In this seminar, we will study a wide range of properties related to the incompressible Euler and Navier-Stokes equations of fluid dynamics [9, 8]. These equations are the main models used for describing the motion of respectively non-viscous and viscous fluids, and, although they have been intensively studied, many deep questions remain unanswered. The goal of this seminar is to introduce some of the different types of problems that have been explored by mathematicians on these equations. We will also cover related models, such as incompressible magnetohydrodynamics [6, 10].

We will start by studying general aspects and questions of the Euler and Navier-Stokes equations, mainly the existence and uniqueness of solutions. Topics involved will include Yudovich solutions (for the 2D Euler system) [9], as well as weak energy solutions and strong solutions of the Navier-Stokes equations (with $H^{d/2-1}$ or L^3 initial data) [1, 7]. As we cover these topics, we will see how tools from harmonic analysis, such as Singular Integral Operators, are crucially involved.

However, hydrodynamics is not limited proving the existence and uniqueness of solutions: we will also examine the qualitative properties of solutions. For example, we may wonder what their asymptotic behavior is like at large scales and whether they display recognizable geometric structure [2]. Likewise, an interesting question it to determine the behavior of different norms of the solution, say $\|u(t)\|_{C^{1,\alpha}}$, with respect to the time variable $t \geq 0$. This question in the Euler equations [5] is closely related to the mixing properties of the fluid flow. Another topic which we will cover is the study of solutions with particular geometry, point vortices, which represent solutions with very intense swirls localized around certain points [3, 7]. We will also examine the distribution of kinetic energy in the Navier-Stokes flow [4].

Prerequisites: basic knowledge of Fourier series and transform, functional analysis, Sobolev spaces and ODEs. No prior experience in physics is required.

Preliminary meeting: Thursday July the 18th at 16h. For students who cannot attend the meeting, another one will be organized at the beginning of the term.

References

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[4] T. Gallay: *Infinite energy solutions of the two-dimensional Navier-Stokes equations.* (Lecture notes), Annales de la Faculté des Sciences de Toulouse **26** (2017), pp. 979–1027.

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[8] P. G. Lemarié-Rieusset: *"Recent developments in the Navier-Stokes problem"*. Chapman & Hall/CRC Research Notes in Mathematics, n. 431. Chapman & Hall/CRC, Boca Raton, FL, 2002.

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