KAM AND NASH–MOSER

GRADUATE SEMINAR ON ANALYSIS (S4B1)
WINTER TERM 2024/25

Organisers: Jan Bohr (`bohr@math.uni-bonn.de`) and Herbert Koch
Preliminary Meeting: Wednesday 24th July, 4:15 pm in SR 0.011.
Time & place: Wednesdays 10:15 am, starting at 16th October, SR 1.007
Website: `https://www.math.uni-bonn.de/ag/ana/WiSe2425/S4B1`

Main references:

- [1] *Alinhac-Gérard:* Pseudodifferential Operators and the Nash-Moser theorem (in particular Chapter III)
- [4] *Pöschel*: A lecture on the classical KAM theorem

Additional background reading:

- [3] *Hubbard:* The KAM Theorem
- [2] *Cannas da Silva:* Lecture notes on symplectic geometry

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Overview

Inverse Function Theorems. *Inverse function theorems (IFT)* concern the solvability of the equation

$$f(x) = y$$

near a point x_0 where the derivative $f'(x_0)$ is surjective. For C^1-maps $f: \mathbb{R}^d \supset U \to \mathbb{R}^d$ between finite dimensional spaces the IFT is proved in undergraduate analysis, but the same result holds true also for C^1-maps

$$f\colon B_1 \supset U \to B_2,$$

where B_1 and B_2 are (infinite dimensional) Banach spaces and $f'(x_0)$ has a bounded right inverse. A typical application of this occurs in elliptic non-linear PDE, where we might have:

$$\textit{Sobolev spaces: } B_1 = H^{s+2}(\Omega),\ B_2 = H^s(\Omega), \quad \textit{Laplacian: } f'(0)h = \Delta h.$$

Showing that the linearisation $f'(0)\colon B_1 \to B_2$ has a right inverse $f'(0)^{-1}$ thus amounts to solving a Laplace equation, together with a bound of the form

$$||f'(0)^{-1}u||_{H^{s+2}} \lesssim ||u||_{H^s}. \tag{1}$$

Date: October 7, 2024.

The applicability of a Banach space IFT is warranted by the right inverse $f'(0)^{-1}$ gaining as many derivatives as an application of f costs—one refers to this by saying that there is no '*loss of derivatives*'.

Nash-Moser theory. The *Nash–Moser theorem* deals with situations where there *is* a loss of derivatives. Indeed, given a non-linear map

$$f\colon C^\infty(\mathbb{T}^d) \supset U \to C^\infty(\mathbb{T}^d)$$

that is C^1 (in a suitable sense), the linearisation $f'(x_0)\colon C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ might be surjective, but instead of (1) one might only have bounds of the form

$$||f'(x_0)^{-1}u||_{H^{s+m}} \lesssim ||u||_{H^{s+r}}, \tag{2}$$

where, morally speaking, m is the number of derivatives an application of f 'costs' and $r \geq 0$ is the number of 'lost derivatives'. In a situation like this one might not be able to fix a pair of Banach spaces, while the Nash–Moser theorem might still be applicable.

The phenomenon of a *loss of derivatives* occurs in several problems across analysis and differential geometry, a prominent example being Nash's *isometric embedding theorem*. As it turns out, this particular problem can be solved *without* the Nash–Moser theorem (and this is not the only example where a theorem, originally proved using Nash-Moser, has been reproved using Banach space methods). There are however situations where the Nash–Moser technique seems indispensable.

KAM theory. An important example of this occurs in *KAM Theory* (KAM = Kolmogorov–Arnold–Moser), which is concerned with the perturbation theory of integrable Hamiltonian systems.

Much of the early interest in this stems from the question of stability of our solar system: assuming that the n planets are not mutually attracted to each other, they orbit around the sun with constant frequencies $\omega_1, \ldots, \omega_n$, which is to say that the orbit of the dynamical system is constrained to an n-dimensional torus. A more realistic model is obtained by allowing for a small perturbation of the system, taking into account a small (in comparison to the sun) mutual attractive force. One then asks whether the invariant torus survives the perturbation and thus the motion is qualitatively comparable to the zero-mass situation or not, allowing for chaotic behaviour.

The classical KAM theorem has an answer to the question, albeit one that is extremely sensitive to the frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ and should rather be interpreted as a probabilistic statement: in the perturbed system, a randomly chosen orbit lies on an invariant torus with high probability.

List of topics

Each topic should be covered in one 60-75 minute talk. Talks marked with a [†] can be left out if there are too few participants. In case of additional demand, further topics can be provided (14 talks maximum).

#	**Topic**	**Reference**
1	Hamiltonian mechanics, integrable systems and action-/angle variables	[2, Chapter 18]
2	Classical KAM theorem: statement and reduction to Thm. A+B	[4, Chapters 1-2]
3	Classical KAM theorem: overview of proof	[4, Chapter 3]
4	Classical KAM theorem: KAM-step and end of proof	[4, Chapter 4-5]
5	Littlewood Payley theory and regularisation operators	[1, Chapter II.A.1]
6[†]	Hölder space estimates of products and compositions	[1, Chapter II.A.2]
7	Banach space IFT and applications	[1, Chapter III.A]
8[†]	Fixed point method 1 (symmetric hyperbolic systems)	[1, Chapter III.B.1]
9[†]	Fixed point method 2 (isometric embeddings)	[1, Chapter III.B.2]
10	Nash-Moser: overview, examples, tame estimates	[1, Chapter III.C.1-3]
11	Nash-Moser theorem – Part 1	[1, Chapter III.C.4]
12	Nash-Moser theorem – Part 2	[1, Chapter III.C.4]

References

[1] Serge Alinhac and Patrick Gérard. *Pseudo-differential operators and the Nash-Moser theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson.

[2] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[3] John H. Hubbard. The KAM theorem. In *Kolmogorov's heritage in mathematics*, pages 215–238. Springer, Berlin, 2007.

[4] Jürgen Pöschel. A lecture on the classical KAM theorem. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 707–732. Amer. Math. Soc., Providence, RI, 2001.