Summer school on
Analysis of multiple ergodic averages

Kopp, September 24- September 29, 2023

I. Convergence of ergodic averages

1. **Tao, T. (2008). Norm convergence of multiple ergodic averages for commuting transformations. *Ergodic Theory and Dynamical Systems, 28(2)*, 657-688. doi:10.1017/S0143385708000011**
   ArXiv link: https://arxiv.org/abs/0707.1117
   Remarks: Prove Theorems 1.1 and 1.6 in the case $l = 2$ (for *two* commuting transforms).
   James Leng, UCLA

2. **Austin, T. (2010). On the norm convergence of non-conventional ergodic averages. *Ergodic Theory and Dynamical Systems, 30(2)*, 321-338**
   Arxiv link: https://arxiv.org/abs/0805.0320
   Remarks: Prove Theorem 1.1 in the special case of $\mathbb{Z}$-actions ($r = 1$), with the standard sequence of averaging sets $I_N + a_N = \{-N, \ldots, N\}$ rather than Følner sequences, and if need be, for two commuting actions ($d = 2$).
   Noa Bihlmaier, Tübingen

3. **Walsh, M. N. (2012). Norm convergence of nilpotent ergodic averages. *Annals of Mathematics*, 1667-1688.**
   Arxiv link: https://arxiv.org/abs/1109.2922
   Remarks: Prove Theorem 1.1 (generalization of the above results to nilpotent groups).
   Chiara Paulsen and Lars Niedorf, Kiel

II. Szemeredi's and Roth Thoerems

4. **Furstenberg, H., Katznelson, Y., & Ornstein, D. (1982). The ergodic theoretical proof of Szemerédi's theorem. *Bulletin of***

***the American Mathematical Society, 7(3)*, 527-552.**
Access Link: https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-7/issue-3/The-ergodic-theoretical-proof-of-Szemerédis-theorem/bams/1183549768.full
Remarks: State and prove Theorem II (briefly state but don't spend time on the proof of Theorem I). The proof is contained in §6-§10. If there is time, briefly discuss some special cases (in §2).
Patrick Hermle, Wuppertal

5. **Ruzsa, I. Z. (1994). Generalized arithmetical progressions and sumsets. *Acta Mathematica Hungarica, 65(4)*, 379-388.**
   and
   Gowers, W. T. (1998). A new proof of Szemerédi's theorem for arithmetic progressions of length four. *Geometric & Functional Analysis GAFA, 8(3)*, 529-551.
   Access links: https://link.springer.com/article/10.1007/BF01876039
   and
   https://link.springer.com/article/10.1007/s000390050065
   Remarks: Prove Theorem 1.1 (Freiman's theorem) from the first reference. Using this, prove the results in Sections 3 and 4 of the second reference. These will be used in the following talk.
   Dimas de Albuquerque and Gautam Neelakantan Memana, UW Madison

6. **Gowers, W. T. (1998). A new proof of Szemerédi's theorem for arithmetic progressions of length four. *Geometric & Functional Analysis GAFA, 8(3)*, 529-551.**
   Access link: https://link.springer.com/article/10.1007/s000390050065
   Remarks: Prove Theorem 20, assuming the results from Sections 3 and 4 (in particular, Proposition 9 and Corollary 14), presented in the preceding talk.
   Kornélia Héra, Budapest/Bonn

7. **Green, B. (2006). Montreal Lecture Notes on Quadratic Fourier Analysis. *Proceedings of the CRM-Clay Conference on Additive Combinatorics, Montreal 2006.***
   Arxiv link: https://arxiv.org/abs/math/0604089
   Remarks: Start with Definition 1.10, prove Proposition 2.2, Propositions 1.13 and 1.14 are covered in the preceding two talks. Apply Proposition 2.2 to find arithmetic progressions of length 4 in subsets of $\mathbb{F}_5^n$.
   Bora Çalim and Nihan Tanısalı, Istanbul

8. **Prendiville, S. (2017). Quantitative bounds in the polynomial Szemerédi theorem: the homogeneous case. *Discrete Analysis,***

***2017*(5).**
ArXiv link: https://arxiv.org/pdf/1409.8234.pdf
Remarks: Give the proof of Corollary 5.3 (Local von Neumann theorem) in the integer setting as discussed in §3-5. Briefly discuss how these arguments can be used to deduce Proposition 2.2 in the following paper (in the finite field setting).
Borys Kuca, Crete

9. **Peluse, S. (2019). On the polynomial Szemerédi theorem in finite fields. *Duke Mathematical Journal, 168(5)*, 749-774.** (Part 1)
ArXiv link: https://arxiv.org/abs/1802.02200
Remarks: Cover sections 1-3. State Theorem 1.1, reduce it to Theorem 2.1 and prove this theorem in the case $m_1 = 1$ (the base case of the induction), assuming Proposition 2.2. Illustrate the inductive step using the simpified example in §4.1.
Lars Becker, Bonn

10. **Peluse, S. (2019). On the polynomial Szemerédi theorem in finite fields. *Duke Mathematical Journal, 168(5)*, 749-774.** (Part 2)
ArXiv link: https://arxiv.org/abs/1802.02200
Remarks: Cover sections 4-6 (except §4.1). State Theorem 2.1 in the general case by induction on m_1. Reduce it to Lemma 4.1 and then prove it.
Jonathan Chapman, Bristol

11. **Peluse, S., & Prendiville, S. (2019). Quantitative bounds in the nonlinear Roth theorem. *arXiv preprint arXiv:1903.02592.*** (Part 1)
ArXiv link: https://arxiv.org/abs/1903.02592
Remarks: Cover §2 and §6-§8. Take what is needed from §1 (Introduction). Would be good to cover the outline of the proof in §1 (coordinate with the person giving the next talk).
Alternative reference:
Prendiville, S. (2020). The inverse theorem for the nonlinear Roth configuration: an exposition. *arXiv preprint arXiv:2003.04121.*
Guo-Dong Hong, Caltech

12. **Peluse, S., & Prendiville, S. (2019). Quantitative bounds in the nonlinear Roth theorem. *arXiv preprint arXiv:1903.02592.*** (Part 2)
ArXiv link: https://arxiv.org/abs/1903.02592

Remarks: Cover §3-5 and the relevant parts from §1 (coordinate with the person giving the preceding talk).
Alternative reference:
Prendiville, S. (2020). The inverse theorem for the nonlinear Roth configuration: an exposition. *arXiv preprint arXiv:2003.04121.*
Seljon Akhmedli, Northwestern

III. Applications of Peluse-Prendiville theory

13. **Frantzikinakis, N. (2023). Joint ergodicity of sequences. *Advances in Mathematics, 417*, 108918.**
ArXiv link: https://arxiv.org/abs/2102.09967
Remarks: The goal is to prove Theorem 1.1 and, if time permits, Corollaries 1.3, 1.4. This covers §1-4. §2 gives the ideas of the proof in the special, simpler case of the Furstenberg-Weiss theorem (Theorem 2.1). First prove this, and then go on to the proof of Theorem 1.1.
Andreas Mountakis, Warwick

14. **Durcik, P., & Roos, J. (2022). A new proof of an inequality of Bourgain. *arXiv preprint arXiv:2210.01326.***
ArXiv link: https://arxiv.org/abs/2210.01326
Remarks: Prove Theorem 1.
Leonidas Daskalakis, Rutgers

IV. Wiener–Wintner, return time's theorem, and pointwise convergence on nilmanifolds

15. **Leibman, A. (2005). Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold. *Ergodic Theory and Dynamical Systems, 25(1)*, 201-213.**
Access link: https://people.math.osu.edu/leibman.1/preprints/pen.pdf
Remarks: Work out the proofs of Theorems A, B and C only for to Følner sequence $[-N, N]$ on $\mathbb{Z}$.
Konstantinos Tsinas, Crete

16. **Rudolph, D. J. (1994). A joinings proof of Bourgain's return time theorem. *Ergodic Theory and Dynamical Systems, 14(1)*, 197-203.**
Remarks: Prove Theorem 1.
Zi Li Lim, UCLA

17. **Assani, I., Duncan, D., & Moore, R. (2016). Pointwise characteristic factors for Wiener–Wintner double recurrence theorem. *Ergodic Theory and Dynamical Systems, 36(4)*, 1037-1066.**
ArXiv link: https://arxiv.org/abs/1402.7094
Remarks: Focus on proving (1) of Theorem 2.3. This is divided into Theorems 4.1 and 5.1. Try to cover the proof of both (§1-5). This paper uses results from the previous two papers.
Leon Duensing, Tübingen

V. Harmonic analysis of ergodic averages

18. **Durcik, P. (2015). An L^4 estimate for a singular entangled quadrilinear form. *Mathematical Research Letters, 22(5)*, 1317-1332.**
ArXiv link: https://arxiv.org/abs/1412.2384
Remarks: Prove Theorem 1.
Jacob Denson and Jacob Fiedler, UW Madison

19. **Durcik, P., Kovač, V., Škreb, K. A., & Thiele, C. (2019). Norm variation of ergodic averages with respect to two commuting transformations. *Ergodic Theory and Dynamical Systems, 39(3)*, 658-688.**
ArXiv link: https://arxiv.org/abs/1603.00631
Remarks: Prove only Theorem 2.
Fred Lin, Bonn and Martin Hsu, Purdue

20. **Tao, T. (2016). Cancellation for the multilinear Hilbert transform. *Collectanea Mathematica, (67)*, 191–206**
ArXiv link: https://arxiv.org/abs/1505.06479
Remarks: Prove Theorem 1.2 (use Theorem 3.9 as blackbox).
Wojciech Słomian, Wrocław

21. **Zorin-Kranich, P. (2017). Cancellation for the simplex Hilbert transform. *Mathematical Research Letters, 24(2)*, 581-592. ArXiv link**: https://arxiv.org/abs/1507.02436
Remarks: Prove Theorem 1.3.
Jianghao Zhang, Bonn

22. **Durcik, P., Kovač, V., & Thiele, C. (2019). Power-type cancellation for the simplex Hilbert transform. *Journal d'Analyse Mathématique, 139(1)*, 67-82.**
ArXiv link: https://arxiv.org/abs/1608.00156

Remarks: Prove Theorem 1 (cover §2-§4). Mention Corollary 2.
Jaume de Dios, UCLA/ETH

23. **Durcik, P., & Thiele, C. (2020). Singular Brascamp-Lieb inequalities with cubical structure. *Bulletin of the London Mathematical Society, 52(2)*, 283-298.**
ArXiv link: https://arxiv.org/abs/1809.08688
Remarks: Prove Theorem 1.
Michel Alexis, McMaster/Bonn

24. **Durcik, P., & Kovač, V. (2021, November). Boxes, extended boxes and sets of positive upper density in the Euclidean space. *In Mathematical Proceedings of the Cambridge Philosophical Society* (Vol. 171, No. 3, pp. 481-501). Cambridge University Press.**
ArXiv link: https://arxiv.org/pdf/1809.08692.pdf
Remarks: Prove Theorems 1 and 2, assuming the main result of the preceding paper.
Ethan Ackelsberg, IAS/EPFL

Additional topics

25. **Karagulyan G., Lacey M., Martirosyan, V. (2022) On the convergence of multiple ergodic means. *New York J. Math. 28 , 1448–1462.***
ArXiv link: https://arxiv.org/pdf/2208.00215.pdf
Gevorg Mnatsakanyan, Bonn

26. **Jones R., Seeger A., Wright, J.(2008) Strong variational and jump inequalities in harmonic analysis. *Trans. Amer. Math. Soc. 360 , no. 12, 6711–6742.*** On pages 2-3, the paper sketches an alternative proof of Birkhoff's ergodic theorem via a stronger variational bound using Lépingle. Collect together the ingredients for this sketch from the various ressources quoted here and present this proof in detail.
Joe Trate, Bonn

27. **Host, B., & Kra, B. (2005). Nonconventional ergodic averages and nilmanifolds. *Annals of Mathematics*, 397-488.**
Access link: https://annals.math.princeton.edu/wp-content/uploads/annals-v161-n1-p08.pdf **Remarks:** Workout everything in the Conze–Lesigne case $k = 2$ (see Section 8), and until and including Lemma 8.8.
Henrik Kreidler, Wuppertal

28. **Furstenberg, H., & Weiss, B. (1996). A Mean Ergodic Theorem for $\frac{1}{N}\Sigma_{n=1}^{N} f(T^n x) g(T^{n^2} x)$. In *Convergence in ergodic theory and probability* (pp. 193-228). de Gruyter.**
Yoav Cohn, HUJI