NODAL DOMAINS AND LANDSCAPE FUNCTIONS

1. Unique continuation

An analytic function vanishing to infinite order at a point is identically zero, a generic smooth function not. A part of such behavior of analytic functions persists when passing to solutions to divergence form elliptic PDE

$$-\operatorname{div}(A\nabla u) = 0.$$

This imposes constraints on the structure of zero sets of harmonic functions, and further of Laplace eigenfunctions. The first two papers discuss the question how much a harmonic function can vanish at the boundary of a domain without being identically zero. The following three focus on what can happen in the interior.

(1) W.S. Wang, *A remark on gradients of harmonic functions*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 227–245.
This paper extends a theorem of Bourgain and Wolff on failure of boundary unique continuation on upper half-space to domains. There exists a regular non-zero harmonic function which together with its gradient vanishes in a set of positive surface measure on the boundary. Discuss the main result and its proof in detail. **(Skorobogatova)**

(2) X. Tolsa, *Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constant*, arXiv:2004.10721.
This paper shows that if vanishing as above takes place in an open set, then the harmonic function must be identically zero. Discuss the main theorem and its proof. **(Covi)**

(3) N. Garofalo and F.-H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), no. 2, 245–268.
This paper uses the frequency function in the study of unique continuation for elliptic PDE. This notion is important in later talks. Put priority in presenting Theorems 1.2 and 1.3 well, but time permitting try to include as much as you can discuss clearly and understandably. **(Ciccone)**

(4) G. Alessandrini, L. Rondi, E. Rosset and S. Vessella, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems **25** (2009), no. 12, 123004, 47 pp.
This paper can be viewed as a quantitative version of boundary unique continuation. Focus on proving Theorem 1.7, which will be needed later. **(de Albuquerque)**

(5) A. Logunov and E. Malinnikova, *Quantitative propagation of smallness for solutions of elliptic equations*, arXiv:1711.10076.
This paper is a quantitative version of unique continuation, propagation of smallness. Prove and motivate the main theorem (Theorem 2.1). References [17,18,19] are presented in other talks, so those results, in particular, can be quoted. **(Neelakantan Memana)**

2. Yau's conjecture on Laplace eigenfunctions

The zero set of a Laplace eigenfunction

$$-\Delta u = \lambda u$$

divides the ambient domain into connected components, the nodal domains. Courant's nodal domain theorem gives an upper bound on their number and is a starting point for the questions concerning the Laplace eigenfunctions. As every Laplace eigenfunction $u(x)$ admits a harmonic extension $e^{-t\sqrt{\lambda}}u(x)$ that shares the zero set Z_λ with its restriction to the plane $\{t = 0\}$, the harmonic polynomials let one (Yau's conjecture) guess

$$\mathcal{H}^{n-1}(Z_\lambda) \sim \lambda^{1/2}.$$

The first talk proves Yau's conjecture on very smooth manifolds. The following three discuss it on more general manifolds. The last one verifies it in a large class of Euclidean domains. Each of the papers below has one or two clear main results, which are intended to be the contents of the talks. If it however remains unclear where to focus, you can ask the organizers.

(6) H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), no. 1, 161–183.
(Miranda)

(7) A. Logunov and E. Malinnikova, *Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimensions two and three*, 50 years with Hardy spaces, 333–344, Oper. Theory Adv. Appl., 261, Birkhäuser/Springer, Cham, 2018.
(Lin)

(8) A. Logunov, *Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure*, Ann. of Math. (2) **187** (2018), no. 1, 221–239.
(Gallegos)

(9) A. Logunov, *Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture*, Ann. of Math. (2) **187** (2018), no. 1, 241–262.
(Becker)

(10) A. Logunov, E. Malinnikova, N. Nadirashvili and F. Nazarov, *The sharp upper bound for the area of the nodal sets of Dirichlet Laplace eigenfunctions*, Geom. Funct. Anal. **31** (2021), no. 5, 1219–1244.
(Stipčić)

3. Other aspects: p-Laplace eigenfunctions and heat flows

One geometric interpretation of the Laplace eigenfunctions is as extremizers of Poincaré's inequality. Inequalities for different Sobolev spaces lead to p-Laplace eigenfunctions

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u$$

with $p \neq 2$. In both cases, there is an associated heat flow whose large time values can be used to study the eigenfunctions. The first two papers discuss p-Laplace eigenfunctions when $p \neq 2$. The third talk is on a heat flow/probabilistic approach to 2-Laplace eigenfunctions. The last talk of the section is about a (not unique) heat flow for the p-Laplace setting.

(11) M. Degiovanni and M. Marzocchi, *On the dependence on p of the variational eigenvalues of the p-Laplace operator*, Potential Anal. **43** (2015), no. 4, 593–609.
The goal of the talk is to discuss how the geometry of the domain relates to the continuity of the Poincaré constants (nonlinear variational eigenvalues). Motivate the notion of non-linear eigenvalues; show the continuity in p from the right; and discuss the counterexample for continuity from the left. Discuss the characterizations for continuity from the left. You will need

P. Lindqvist, *On nonlinear Rayleigh quotients*, Potential Anal. **2** (1993), no. 3, 199–218.

in addition to the first reference. **(Gietl)**

(12) M. Cuesta, D. de Figueiredo and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), no. 1, 212–238.
The goal of this talk is to give a nonlinear result in the spirit of Courant's nodal domain theorem: the nodal set of the first non-zero nonlinear eigenfunction divides the domain in two connected components. Focus on presenting the result in

M. Cuesta, D. de Figueiredo and J.-P. Gossez, *A nodal domain property for the p-Laplacian*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 8, 669–673.

and take only what is necessary from the JDE paper above. **(Niedorf)**

(13) B. Georgiev and M. Mukherjee, *Nodal geometry, heat diffusion and Brownian motion*, Anal. PDE **11** (2018), no. 1, 133–148.
The main goal is to give an overview of the method of heat flow and probabilistic tools. Discuss the preliminary material carefully and prove at least one of the main results (any of the statements labeled Theorem). **(Denson)**

(14) R. Hynd and E. Lindgren, *Large time behavior of solutions of Trudinger's equation*, J. Differential Equations **274** (2021), 188–230.
Theorem 1.1 is the main result (large time behaviour of the Trudinger equations), and this is the first priority to present. Do not attempt to include anything about fractional smoothness PDEs. **(Lee)**

4. Landscape functions

The Schrödinger eigenfunctions with a potential

$$(-\Delta + V)u = Eu$$

form a third class of functions of interest. The landscape function of Filoche and Mayboroda describes their localization into subdomains when the energy is small. The following talks discuss the method of landscape function and some of its applications.

(15) M. Filoche and S. Mayboroda, *Universal mechanism for Anderson and weak localization*, Proc. Natl. Acad. Sci. USA **109** (2012), no. 37, 14761–14766
This short paper comes with an appendix of proofs. Present the idea of the landscape function and the proof from the appendix. **(Hou)**

(16) S. Steinerberger, *Localization of quantum states and landscape functions*, Proc. Amer. Math. Soc. **145** (2017), no. 7, 2895–2907.
Do everything. **(Poggi)**

(17) D.N. Arnold, G. David, M. Filoche, D. Jerison and S. Mayboroda, *Localization of eigenfunctions via an effective potential*, Comm. Partial Differential Equations **44** (2019), no. 11, 1186–1216.
Try to discuss all the main results, Theorem 1.1 and 1.2. **(de Dios)**

(18) Z. Shen, *On fundamental solutions of generalized Schrödinger operators*, J. Funct. Anal. **167** (1999) 521–564.
Prove the two-sided bounds for the fundamental solution in terms of the Fefferman-Phong maximal functions. **(Chica)**

(19) G. David, M. Filoche and S. Mayboroda, *The landscape law for the integrated density of states*, Adv. Math. **390** (2021), Paper No. 107946, 34 pp.
You might not be able to include everything, but discuss very well at least the first two theorems, the later ones subject to time constraints. **(Bergmann)**

(20) B. Poggi, *Applications of the landscape function for Schrödinger operators with singular potentials and irregular magnetic fields*, arXiv:2107.14103.
Focus on constructing the landscape function and connecting it to the Fefferman–Phong–Shen maximal function. **(Seye)**

5. Quasiconformal methods

Schrödinger equation can be rewritten as a divergence form elliptic equation for an auxiliary function. While conformal mappings respect harmonicity, quasiconformal mappings behave well with divergence form elliptic equations of n-Laplacian type making them an important tool. In addition, the planar quasiconformal mappings are tightly connected to another equation, the Beltrami system on the plane

$$\partial_z u = \mu \partial_{\overline{z}} u.$$

The first two talks of this section discuss fundamental results on quasiconformal mappings. The remaining two are results where these methods can be used.

(21) Use Sections 3 and 13 from K. Astala, T. Iwaniec, M. Gaven, *Elliptic partial differential equations and quasiconformal mappings in the plane* (PMS-48), Princeton University Press, 2009, to give an understandable exposition of necessary Definitions, Mori's theorem (in the form of Thm 3.10.2) and area distortion (Thm 13.1.5). **(Guillen)**
(22) Use Section 5 of the book above to discuss Stoilow factorization (Thm 5.5.1). **(Brocchi)**
(23) F. Nazarov, L. Polterovich, M. Sodin, *Sign and area in nodal geometry of Laplace eigenfunctions*, Amer. J. Math., 127, 2005, 879–910. **(Dosidis)**
(24) A. Logunov, E. Malinnikova, N. Nadirashvili and F. Nazarov, *The Landis conjecture on exponential decay*, arXiv:2007.07034. **(Hsu)**