Nodal domains and landscape functions

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1 A remark on gradients of harmonic functions

After W. Wang [Wa]

A summary written by Anna Skorobogatova

Abstract

Given any $C^{1,\alpha}$ domain D in Euclidean space, we show that there exists a non-trivial harmonic function that is C^1 up to the boundary, such that both the function and its gradient vanish on a set of positive measure in the boundary. This extends the result Bourgain and Wolff [BW] from the upper half space to general $C^{1,\alpha}$ domains.

1.1 Introduction

In [Wo], it was shown that it is possible to construct a non-zero $C^{1+\varepsilon}$ harmonic function on the upper half space $\mathbb{R}^d_+$, $d \geq 3$, whose gradient vanishes on a positive measure set of the boundary $\mathbb{R}^{d-1} \equiv \partial\mathbb{R}^d_+$. Subsequently, in [BW], this was sharpened to the existence of a harmonic function, C^1 up to the boundary, that vanishes along with its gradient on a positive measure set of the boundary $\mathbb{R}^{d-1}$.

We discuss the extension of this to general $C^{1,\alpha}$ domains in $\mathbb{R}^d$. More precisely:

Theorem 1 ([Wa])**.** *Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a $C^{1,\alpha}$ domain for some $\alpha > 0$. Then there exists a harmonic function $u \in C^1(\bar{D})$ such that*

$$\mathcal{H}^{d-1}(\partial D \cap \{u = 0,\ \nabla u = 0\}) > 0.$$

Remark 2.

1. *The aforementioned results fail in the plane, since for any planar harmonic function u, the function $\log |\nabla u|$ is subharmonic.*

2. *If instead one asks for a non-zero harmonic map u to vanish on an* ***open*** *subset of ∂D (say, C^1 domain), then Tolsa in [T] showed that one has*

$$\mathcal{H}^{d-1}(\partial D \cap \{u = 0,\ \nabla u = 0\}) = 0.$$

Under better regularity assumptions on D (namely, $C^{1,\mathrm{Dini}}$), it was shown that the boundary singular set $\partial D \cap \{u = 0, \nabla u = 0\}$ has Hausdorff dimension at most $d-2$ ([AE]) and has locally finite $(d-2)$-dimensional Minkowski content ([KZ]).

1.2 Notation

We henceforth fix a large constant $M > 0$ arbitrarily. Let $\mathscr{D}_M$ be the class of domains $\Omega \subset \mathbb{R}^d$ such that

$$\Omega = \{X = (x, x_d)\colon x_d > \varphi(x)\},$$

where $\varphi \in C^{1,\alpha}$ with $\varphi(0) = 0$, $\nabla\varphi(0) = 0$, and

$$\|\nabla\varphi\|_\infty + [\nabla\varphi]_{C^{0,s}} \le e^{-M}.$$

$\mathbf{B}_R(X)$ will denote a d-dimensional ball of radius R centered st X in $\mathbb{R}^d$, while $B_R(x)$ will denote a $(d-1)$-dimensional ball centered at x in $\mathbb{R}^{d-1}$. $Q(0,\ell)$ denotes a $(d-1)$-dimensional cube in $\mathbb{R}^{d-1}$. We will denote by $\ell(Q)$ its side length when it is not specified.

ν will denote the (outward) normal vector (taken either at $x \in \mathbb{R}^{d-1}$ or $X \in \partial\Omega$). $T_X\partial\Omega$ will denote the tangent space to $\partial\Omega$ at the point X.

We will use the notation $\lesssim_\square$ to demonstrate that the left-hand side of the given inequality is controlled by the right-hand side, up to a constant dependent on the quantities $\square$.

1.3 Preliminary lemmas

Lemma 3. *If $p > 0$ is sufficiently small and $M > 0$ is sufficiently large, then for every $\Omega \in \mathscr{D}_M$, there exists $\eta = \eta(d) > 0$ such that for every $\varepsilon > 0$ and for each cube $Q = Q(0,\ell) \subset \mathbb{R}^{d-1}$, $\ell(Q) < 1$, there exists a harmonic function $h_\varepsilon^Q \in C^1(\bar\Omega)$ with $\operatorname{supp} h_\varepsilon^Q\big|_{\partial\Omega} \subset \varphi(B_{\varepsilon\ell}(0))$ with the following properties:*

(a)

$$\frac{1}{|\varphi(Q)|}\int_{\varphi(Q)} \left(|1 + \partial_\nu h_\varepsilon^Q|^p - 1\right) d\mathcal{H}^{d-1} \lesssim_{d,p} -\eta e^{-(d-1)M},$$

(b)

$$|\nabla h_\varepsilon^Q(X)| \lesssim_d \min\left\{\varepsilon^{-d}, e^{(d-\frac{1}{2})M}\left|\frac{X}{\ell}\right|^{-d}\right\}.$$

Proof. For $X = (x, x_d) \in \Omega$ and $a > 0$, one can consider the function

$$F_\varepsilon^a(X) := -a\frac{\epsilon + \frac{x_d}{a}}{\left|\frac{X}{a} + \epsilon e_d\right|^d},$$

which in [AK] is shown to satisfy

$$\int_{\mathbb{R}^{d-1}} (|1+\partial_\nu F_\varepsilon^a(x)|^p - 1)d\mathcal{H}^{d-1} \leq -2\eta a^{d-1}.$$

Letting $\rho = 1 - \sum_j \psi_j$ for a Littlewood-Paley partition of unity $\{\psi_j\}_{j\geq 0}$ supported on the annuli $\{2^{j-1}\ell \leq |x| \leq 2^{j+1}\ell\} \subset \mathbb{R}^{d-1}$, define $h_\varepsilon^Q := \rho F_\varepsilon^a\big|_{\partial\Omega}$, with $a = e^{-M}\ell$. One can check, using standard Littlewood-Paley estimates, that this function satisfies the desired conclusion. We refer to [BW] or [Wa] for the details. □

Corollary 4. *Let p, M and ε be as in Lemma 3 and let $\Omega \in \mathscr{D}_{2M}$. Let $X_Q = (x_Q, \varphi(x_Q)) \in \partial\Omega$, let $s_0 > 0$ and let $Q \subset B_{s_0}(x_Q) \subset \mathbb{R}^{d-1}$. There exists $\beta = \beta(d, M) > 0$ such that the following holds.*

Suppose that $J : \Omega \to \mathbb{R}$ is a function with supp$J \subset \varphi(\Omega)$*, $J(X_Q) \neq 0$ and*

$$|J(X) - J(X_Q)| \leq \frac{\eta}{2} e^{-(d-1)M} |J(X_Q)|. \tag{1}$$

Then the map $\tilde{h}_\varepsilon^Q := h_\varepsilon^{Q(0,\ell)} \circ \tau_{-x_Q}$, for $h_\varepsilon^{Q(0,\ell)}$ from Lemma 3 and τ_{-x_Q} the translation by $-x_Q$ in $\mathbb{R}^{d-1}$, satisfies

(a)

$$\left(\frac{1}{|\varphi(Q)|}\int_{\varphi(Q)} |J(X) + J(X_Q)\partial_\nu h_\varepsilon^Q(X)|^p d\mathcal{H}^{d-1}(X)\right)^{\frac{1}{p}} \leq e^{-2\beta}|J(X_Q)|,$$

(b)

$$|\nabla h_\varepsilon^Q(X)| \lesssim_d \min\left\{\varepsilon^{-d}, e^{-(d-\frac{1}{2})M}\left|\frac{X - X_Q}{\ell}\right|^{-d}\right\}.$$

Proof. Let $\tilde{\Omega}$ be the translation of Ω by $-X_Q$ (so that X_Q is the new origin), followed by a rotation so that $T_{X_Q}\partial\tilde{\Omega} = \mathbb{R}^{d-1}$. Then $\tilde{\Omega} \in \mathscr{D}_M$ and so we can apply Lemma 3. Combining this with the fact that one can replace 1 by $\tilde{J} := \frac{J}{J(X_Q)}$ up to a negative exponential error term and letting $\beta > 0$ be such that

$$e^{-2\beta} = 1 - \frac{\eta}{2} e^{-(d-1)M},$$

the result follows. □

1.4 Proof of localized version of Theorem 1

In order to prove Theorem 1, we will first prove the following weaker, localized version:

Theorem 5. *Suppose that M is as in Lemma 3 and let $\Omega \in \mathscr{D}_{2M}$. Then for every $s_0 > 0$ sufficiently small, there exists a harmonic map $u \in C^1(\bar{\Omega})$ with*

$$\mathcal{H}^{d-1}\left(\{X \in \partial\Omega \cap \mathbf{B}_{s_0}(0) \colon u(X),\ \nabla u(X) = 0\}\right) > 0. \tag{2}$$

Sketch proof. We will proceed to construct u via a recursive procedure, starting with a function $\hat{u}_0$ that vanishes on an open subset of $\partial\Omega$ and correcting it in such a way that

- we decrease the normal derivative on a large measure subset of $\partial\Omega$ with each step;
- we keep $\hat{u}_0$ unchanged on a large measure subset.

If this is done carefully enough, in a quantitative manner, clearly it will yield the desired result in the limit as we increase the number of iterations.

Let $Q_0 = Q_0(0, s_0) \subset \mathbb{R}^{d-1}$. Let $\delta_n = 2^{-k_n} s_0$, $k_n \in \mathbb{Z}$, $k_n \nearrow +\infty$ to be determined later. Let $K_n \to +\infty$, $\varepsilon_n \to 0$, both to be determined later also. Let

$$\mathcal{F}_n \coloneqq \{\text{disjoint cubes } Q \subset Q_0 \text{ with } \ell(Q) = \delta_n \text{ and vertices in } \delta_n \mathbb{Z}^{d-1}\}.$$

Note that $\#\mathcal{F}_n = \delta_n^{-(d-1)}$. We will now create "good" families of cubes $\mathcal{G}_n \subset \mathcal{F}_n$ and functions u_n inductively as follows.

Let $\delta_0 = s_0$, $\mathcal{G}_0 = \{Q_0\}$ and let $\hat{u}_0$ be the harmonic extension into Ω of the boundary data $u_0 \in C_0^{1,\alpha}\left(\varphi\left(\frac{1}{100}Q_0\right)\right)$. Given $Q \in \mathcal{G}_n$, subdivide it into $\left(\frac{\delta_n}{\delta_{n+1}}\right)^{d-1}$ cubes. For each of these new cubes Q', place it into $\mathcal{G}_{n+1}$ if

$$\left(\frac{1}{|\varphi(Q)|}\Big|\int_{\varphi(Q)} \partial_\nu u_n\Big|^p d\mathcal{H}^{n-1}\right)^{\frac{1}{p}} \leq K_{n+1} e^{-\beta n}. \tag{3}$$

Note that in particular, for any $Q' \in \mathcal{G}_{n+1}$, its parent cube $Q \in \mathcal{F}_n$, $Q \supset Q'$, must be in $\mathcal{G}_n$. Now given u_n and $\mathcal{G}_{n+1}$, define the boundary datum

$$u_{n+1} = u_n + \sum_{Q \in \mathcal{G}_{n+1}} \partial_\nu u_n(X_Q) \tilde{h}^Q_{\varepsilon_{n+1}},$$

for $\tilde{h}^Q_{\varepsilon_{n+1}}$ as in Corollary 4. In particular, (4) in Corollary 4 tells us that for $X = (x, x_d) \in \partial\Omega$,

$$\sum_{\substack{Q \in \mathcal{G}_{n+1} \\ |x - x_Q| > r}} |\nabla \tilde{h}^Q_{\varepsilon_{n+1}}(X)| \leq C e^{-(d-\frac{1}{2})M} \frac{\delta_{n+1}}{r} \qquad \text{if } r > C\delta_{n+1}. \tag{4}$$

This, combined with the defining property (3) of cubes $Q \in \mathcal{G}_{n+1}$ and the continuity of $\partial_\nu u_n$ yields the following estimate for each $X \in \partial\Omega$:

$$|\nabla u_{n+1}(X) - \nabla u_n(X)| \lesssim_d K_{n+1} e^{-\beta n} \varepsilon_{n+1}^{-d} \qquad \text{for } \delta_{n+1} \text{ sufficiently small.} \tag{5}$$

In addition, we can inductively show that for each n we have

$$\|\partial_\nu u_n\|_{L^p\left(\varphi\left(\bigcup_{\mathcal{G}_n} Q\right)\right)} \leq A e^{-\beta n}, \tag{6}$$

by decomposing the cubes $Q \in \mathcal{G}_{n+1}$ into "type I" cubes Q with $|\partial_\nu u_{n+1}(X_Q)| > e^{-4\beta(n+1)}$ and "type II" cubes Q with $|\partial_\nu u_{n+1}(X_Q)| \leq e^{-4\beta(n+1)}$, and treating each one separately. We refer to [BW] or [Wa] for the details.

Thus, for each $X \in \partial\Omega$ we have

$$\sum_n |\nabla u_{n+1}(X) - \nabla u_n(X)| \lesssim_d \sum_n K_{n+1} \varepsilon_{n+1}^{-d} e^{-\beta n},$$

and so, as long as we choose K_n, ε_n so that the sum on the right-hand side is finite, the harmonic extensions $\hat{u}_n$ of the boundary data u_n converge in $C^1(\bar{\Omega})$ to a limiting harmonic function $\hat{u} \in C^1(\bar{\Omega})$ with boundary datum u.

Now let us check that u satisfies the desired property (2). Indeed, we have

$$\begin{aligned}
\mathcal{H}^{d-1}(\varphi(Q_0) \cap \{u \neq 0\}) &\leq \sum_n \mathcal{H}^{d-1}(\{u_{n+1} \neq u_n\}) \\
&\leq \sum_n \mathcal{H}^{d-1}\left(\varphi(B_{\varepsilon_{n+1}\ell(Q)}(x_Q))\right) \\
&\lesssim_d \sum_n \delta_{n+1}^{-(d-1)} \varepsilon_{n+1}^{d-1} \ell(Q)^{d-1} \\
&\lesssim_d \sum_n \varepsilon_{n+1}^{d-1} \\
&\leq \frac{|\varphi(Q_0)|}{10}.
\end{aligned}$$

provided that we redefine ε_n appropriately if necessary. Moreover, we have

$$\begin{aligned}\mathcal{H}^{d-1}(\varphi(Q_0)\cap\{\partial_\nu u\neq 0\}) &\leq \sum_n \mathcal{H}^{d-1}\Big(\varphi(\bigcup_{\mathcal{G}_n} Q)\setminus\varphi(\bigcup_{\mathcal{G}_{n+1}} Q)\Big)\\ &\leq \sum_n \sum_{Q\in\mathcal{G}_n\setminus\mathcal{G}_{n+1}} \mathcal{H}^{d-1}(\varphi(Q))\\ &\overset{(6)}{\leq} \sum_n e^{\beta np}K_{n+1}^{-p}\int_{\varphi\big(\bigcup_{\mathcal{G}_n} Q\big)} |\partial_\nu u_n|^p\\ &\leq \sum_n A^p K_{n+1}^{-p}\\ &\leq \frac{|\varphi(Q_0)|}{10}\end{aligned}$$

for K_n further amended if necessary. This completes the proof. □

1.5 Proof that Theorem 5 implies Theorem 1

Firstly, suppose in addition that D is bounded. Then we may choose our coordinate system so that $0\in\partial D$, $\mathbb{R}^{d-1}=T_0\partial D$ and $D\subset\mathbb{R}^d_+$. We may find $s_0>0$ small enough and $\Omega\in\mathscr{D}_{2M}$ such that $\partial\Omega\cap\mathbf{B}_{s_0}=\partial D\cap\mathbf{B}_{s_0}$ and $D\subset\Omega$. We can then apply Theorem 5 and restrict the resulting map to D to conclude.

If D is unbounded, first of all apply the proof of Case 1 to $D\cap\mathbf{B}$ for some ball $\mathbf{B}\subset\mathbb{R}^d$ centered at a point $X_0\in\partial D$ to yield a harmonic map $v_1\in C^1(D\cap\bar{\mathbf{B}})$. Then find a harmonic map $v_2\in C^1(\bar{\Omega}\setminus\mathbf{B})$ that agrees with v_1 on $D\cap\partial\mathbf{B}$ via the Kelvin transform $X\mapsto\frac{X-X_0}{|X-X_0|^2}$.

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2 Unique continuation at the boundary for harmonic functions in special Lipschitz domains

After X. Tolsa [T]

A summary written by Giovanni Covi

Abstract

Consider a function u harmonic in a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with small Lipschitz constant, continuous in $\overline{\Omega}$, and such that $u = 0$ in $\Sigma \subset \partial\Omega$ and $\partial_\nu u = 0$ in $E \subset \Sigma$, with Σ relatively open and E of positive measure. This paper shows that, under the stated assumptions, u must vanish identically on Ω.

2.1 Introduction

The work [BW] showed that there exists a nontrivial function u which is harmonic in $\mathbb{R}^n_+$ with $n \geq 3$, C^1 up to the boundary, and vanishes together with its normal derivative on a set $E \subset \partial\mathbb{R}^n$ with positive measure. This is still true if $\mathbb{R}^n_+$ is substituted by a $C^{1,\alpha}$ domain Ω ([W]), but it is unknown whether the conditions $u = 0$ in an open set $\Sigma \subset \partial\Omega$ and $\partial_\nu u = 0$ in $E \subset \Sigma$, where E has positive measure, suffice to deduce $u \equiv 0$ in Ω when Ω is Lipschitz and the dimension is arbitrary ([L, AEK]). The paper [T] we study here proves the conjecture for Lipschitz domains with small local Lipschitz constant:

Theorem 1 (Theorem 1.1 from [T])**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, let B be a ball centered in $\partial\Omega$, and assume $\Sigma := B \cap \partial\Omega$ is a Lipschitz graph with slope at most τ_0, where $\tau_0 > 0$ is a small constant depending only on n. Let $u \in C(\overline{\Omega})$ be such that*

$$\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ in } \Sigma \quad \text{and} \quad \partial_\nu u = 0 \text{ in } E,$$

where $E \subset \Sigma$ has positive measure. Then $u \equiv 0$ in $\overline{\Omega}$.

The proof is based on the doubling property of the L^2 averages of the harmonic function u (see Lemma 3), which in turn is derived from a key result (see Lemma 2) studying the behaviour of Almgren's frequency function at points close to $\partial\Omega$.

2.2 Preliminaries

2.2.1 Frequency functions

Let u as in the statement of Theorem 1, and extend it by 0 in $\mathbb{R}^n \setminus \Omega$. For $x \in \mathbb{R}^n$ and $r > 0$ denote

$$h(x,r) := \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u^2 d\sigma,$$

where σ indicates the surface measure. The measure in $\mathbb{R}^n$ will be indicated by m. If $B(x,r)$ intersects Ω, the Almgren frequency function is defined by

$$F(x,r) := r\partial_r \log h(x,r).$$

Assume $x \in \mathbb{R}^n$ and the closed interval $I \subset (0,\infty)$ are such that $B(x,r) \cap \Omega \neq \emptyset$ and $\overline{B(x,r)} \subset 2B$ for all $r \in I$. Then $h(x,\cdot)$ is of class C^1 and $F(x,\cdot)$ is absolutely continuous in I (check [T, Lemmas 2.1, 2.2]), and $\partial_r h(x,r)$, $F(x,r)$, $\partial_r F(x,r)$ can be explicitly computed from u and ∇u. In particular, $h(x,r)$ is non-decreasing with respect to r, and if $(y-x)\cdot\nu(y) \geq 0$ for σ-a.e. $y \in B(x,r) \cap \partial\Omega$, then $\partial_r F(x,r) \geq 0$.

We call a closed interval $I \subset (0,\infty)$ admissible for $x \in \mathbb{R}^n$ if $h(x,r) > 0$ and $\partial_r F(x,r) \geq 0$ for a.e. $r \in I$. If I is admissible for x, and $a > 1$ and $r \in I$ are such that $ar \in I$, then by [T, Lemma 2.3]

$$F(x,r) \leq \log_a \frac{h(x,ar)}{h(x,r)} \leq F(x,ar). \tag{1}$$

Assume I is admissible for $x, y \in \mathbb{R}^n$, and $r, 2(1+\gamma^{1/2})r \in I$, where $r > 0$ and γ is a small constant. If $|x-y| \leq \gamma r$ and $B(x,5r) \cap \partial\Omega \subset \Sigma$, then by [T, Lemma 2.4] there exists a constant $C > 0$ such that

$$F(y,r) \leq (1 + C\gamma^{1/2})F(x, 2(1+\gamma^{1/2})r) + C\gamma^{1/2}. \tag{2}$$

2.2.2 Geometric constructions and definitions

Let B_0 be any ball centered in Σ such that $M^2 B_0 \subset B$ for some $M > 1$, and define $\Sigma_0 := \partial\Omega \cap B_0$. Fix a coordinate system in which Σ_0 is a Lipschitz graph (with small slope τ_0) and $\Omega \cap MB_0$ is above the graph. Let H_0 be the horizontal plane through the origin, and Π the orthogonal projection to H_0.

We consider a Whitney decomposition of Ω by means of a family $\mathcal{W}$ of dyadic cubes in $\mathbb{R}^n$ with disjoint interiors such that $\mathrm{dist}(Q, \partial\Omega) \approx \ell(Q)$ for all $Q \in \mathcal{W}$, where $\ell(Q)$ is the length of the side of Q and x_Q is the center.

To each cube $Q \in \mathcal{W}$ we associate a cylinder $\mathcal{C}(Q) := \Pi^{-1}\Pi(Q)$.

For all $k \in \mathbb{N}$ and all $R \in \mathcal{W}$ we consider a family $\mathcal{D}^k_{\mathcal{W}}(R) \subset \mathcal{W}$ satisfying

$$\bigcup_Q \Pi(Q) = \Pi(R), \quad \ell(Q) = 2^{-k}\ell(R), \quad Q \cap Q' = \emptyset, \quad Q \text{ is under } R$$

for all $Q, Q' \in \mathcal{D}^k_{\mathcal{W}}(R)$. We also let $\mathcal{D}_{\mathcal{W}}(R) := \bigcup_{k\in\mathbb{N}} \mathcal{D}^k_{\mathcal{W}}(R)$.

After a translation, we can make sure to find a special cube $R_0 \in \mathcal{W}$ such that $\Pi(B_0) \subset \Pi(R_0)$ and $R_0 \subset \frac{M}{2}B_0$. Of course it holds $B_0 \subset \mathcal{C}(R_0)$. We also let $\Pi_{\Sigma_0} : \mathcal{C}(R_0) \to R_{\Sigma_0} := \mathcal{C}(R_0) \cap \Sigma_0$ be the projection on R_{Σ_0} in the direction perpendicular to H_0, and consider the measure $\mu := m_{n-1} \circ \Pi^{-1}_{\Sigma_0}$.

If A, K, N_0 are positive constants and $j \in \mathbb{N}$, define

$$\mathcal{G}_K(R) := \{Q \in \mathcal{D}^K_{\mathcal{W}}(R) : F(x_Q, A\ell(Q)) \leq F(x_R, A\ell(R))/2\}$$

and also $T_j := \bigcup_{Q\in\mathcal{T}'_j} \Pi_{\Sigma_0}(Q)$, $\mathcal{T}_j := \Pi_{\Sigma_0}(\mathcal{T}'_j)$, where

$$\mathcal{T}'_j := \{Q \in \mathcal{D}^{jK}_{\mathcal{W}}(R_0) : F(x_Q, A\ell(Q)) \leq N_0\}.$$

Moreover, we let $\widetilde{\mathcal{D}}_j(R_{\Sigma_0}) := \Pi_{\Sigma_0}(\mathcal{D}^{jK}_{\mathcal{W}}(R_0))$ and $\widetilde{\mathcal{D}}(R_{\Sigma_0}) := \bigcup_{j\in\mathbb{N}} \widetilde{\mathcal{D}}_j(R_{\Sigma_0})$. If $R \in \widetilde{\mathcal{D}}_j(R_{\Sigma_0})$, let $R' \in \mathcal{D}_{jK}(R_{\Sigma_0})$ be such that $R = \Pi_{\Sigma_0}(R')$, and define the good set

$$G(R) := \begin{cases} \bigcup_{Q'\in\mathcal{G}_K(R')} \Pi_{\Sigma_0}(Q') & \text{if } F(x_{R'} A\ell(R')) \geq N_0 \\ R & \text{otherwise,} \end{cases}$$

and for all $j \geq J \geq 0$ the functions $f_j := \sum_{R\in\widetilde{\mathcal{D}}_j(R_{\Sigma_0})} f_R$, where

$$f_R := \begin{cases} \frac{\mu(R)}{\mu(G(R))}\chi_{G(R)} - \chi_R & \text{if } \nexists \widetilde{R} \in \bigcup_{j\geq J} \mathcal{T}_j : R \subset \widetilde{R} \\ 0 & \text{otherwise.} \end{cases}$$

Probabilistic considerations (check [E]) ensure that for μ-a.e. $x \in R_{\Sigma_0}$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=J+1}^{m} f_j(x) = 0. \tag{3}$$

2.2.3 The Key Lemma

Assume that T, A are two positive constants, $M > T, A$ and τ_0 is small enough (depending on T, A). If $x \in \Omega$ and $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ are such that $\operatorname{dist}(x, R) \leq T\ell(R)$ and $\operatorname{dist}(x, \partial\Omega) \geq T^{-1}\ell(R)$, then the interval $(0, A\ell(R))$ is admissible for x (check [T, Remark 3.3]). According to the properties of frequency functions, for this to be true one has to verify the condition $(y-x)\cdot\nu(y) \geq 0$ for σ-a.e. $y \in B(x,r)\cap\partial\Omega$, which is granted by the smallness of τ_0.

Proving the admissibility of intervals allows us to use the results of formulas (1) and (2), which compare different values of the frequency function. Eventually, this lets us bound the set where the frequency function is large:

Lemma 2 (Key Lemma 3.1 from [T])**.** *Let $N_0 > 1$ be fixed and large enough. There exists a constant $\delta_0 > 0$ such that for all A, $M = M(A)$ big enough, $\tau_0 = \tau_0(A)$ small enough, and $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ satisfying $F(x_R, A\ell(R)) \geq N_0$ there exists $K = K(A)$ such that*

1. $m_{n-1}(\bigcup_{Q\in\mathcal{G}_K(R)}\Pi(Q)) \geq \delta_0 m_{n-1}(\Pi(R))$,
2. $F(x_Q, A\ell(Q)) \leq (1 + CA^{-1/2})F(x_R, A\ell(R))$ *for all* $Q \in \mathcal{D}^K_{\mathcal{W}}(R)$.

2.3 Proof of Theorem 1

We want to show that, if $u \in C(\overline{\Omega})$ is harmonic in Ω, not identically 0, and vanishes in Σ, then there is no $E \subset \Sigma$ with positive measure where $\partial_\nu u = 0$.

Since $|u|$ is subharmonic, for all $x \in \mathbb{R}^n$ and all small enough $r > 0$

$$h(x,r) = \frac{1}{\sigma(\partial B(x,r))}\int_{\partial B(x,r)} |u|^2 d\sigma \lesssim \left(\frac{1}{m(B(x,2r))}\int_{B(x,2r)} |u| dm\right)^2.$$

On the other hand, since $h(x,\cdot)$ is non-decreasing,

$$\left(\frac{1}{m(B(x,12r))}\int_{B(x,12r)} |u| dm\right)^2 \leq \frac{1}{m(B(x,12r))}\int_{B(x,12r)} |u|^2 dm \lesssim h(x,12).$$

Therefore,

$$\liminf_{r\to0}\frac{h(x,12r)^{1/2}}{h(x,r)^{1/2}} \gtrsim \liminf_{r\to0}\frac{m(B(x,2r))\int_{B(x,12r)}|u|dm}{m(B(x,12r))\int_{B(x,2r)}|u|dm}. \tag{4}$$

Assume for the sake of contradiction that there exists $E \subset \Sigma$ with positive measure such that $\partial_\nu u = 0$ holds in E. Then by [AE, Lemma 0.2] the right hand side of (4) diverges whenever $x \in \Sigma$ is a density point for E, which in turn implies

$$\lim_{r \to 0} \frac{h(x, 12r)}{h(x, r)} = \infty. \tag{5}$$

However, we have the following:

Lemma 3 (Lemma 4.1 from [T])**.** *For σ-almost every $x \in \Sigma_0 \cap \mathcal{C}(R_0)$,*

$$\liminf_{r \to 0} \frac{h(x, 12r)}{h(x, r)} < \infty. \tag{6}$$

Since B_0 is arbitrary, (6) holds almost everywhere in Σ, and the density set of E has vanishing measure by (5). Since this is a contradiction, the proof will be complete as soon as we prove the doubling property from Lemma 3.

Proof of Lemma 2. If $x \in T_j$, then there exists $Q \in \mathcal{T}_j'$ such that $x \in \Pi_{\Sigma_0}(Q)$, and thus $F(x_Q, A\ell(Q)) \le N_0$. Using this, formula (1) and the subharmonicity of $|u|$, for A large enough we have

$$\frac{h(x, A\ell(Q)/2)}{h(x, A\ell(Q)/24)} \lesssim \frac{h(x_Q, A\ell(Q))}{h(x_Q, A\ell(Q)/48)} \le 48^{N_0},$$

with $\ell(Q) \approx 2^{-jK}\ell(R_0)$. Therefore, formula (6) holds for all $x \in \limsup_{j\to\infty} T_j$, and by the mutual absolute continuity of μ and σ we only need to prove $\mu(R_{\Sigma_0} \setminus \limsup_{j\to\infty} T_j) = 0$. It suffices to show $\mu(R_{\Sigma_0} \setminus \bigcup_{j \ge J} T_j) = 0$ for all $J \in \mathbb{N}$. To this end, assume $x \in R_{\Sigma_0} \setminus \bigcup_{j \ge J} T_j$. For each $j \ge J$ there exists $Q_j' \in \mathcal{D}_{\mathcal{W}}^{jK}(R_0)$ such that $x \in Q_j := \Pi_{\Sigma_0}(Q_j')$. However, since $x \notin T_j$ for all $j \ge J$, necessarily $Q_j' \notin \mathcal{T}_j'$, and thus $F(x_{Q_j'}, A\ell(Q_j')) > N_0$. This means that Lemma 2 can be applied, and so we deduce

1. $\mu(G(Q_j)) = m_{n-1}(\bigcup_{Q \in \mathcal{G}_K(Q_j')} \Pi(Q_j')) \ge \delta_0 m_{n-1}(\Pi(Q_j')) = \delta_0 \mu(Q_j)$,

2. $F(x_{Q_{j+1}'}, A\ell(Q_{j+1}')) \le (1 + CA^{-1/2}) F(x_{Q_j'}, A\ell(Q_j'))$.

Moreover, if $x \in G(Q_j)$, then $Q_{j+1}' \in \mathcal{G}_K(Q_j')$ by the definition of G, and

3. $F(x_{Q_{j+1}'}, A\ell(Q_{j+1}')) \le \frac{1}{2} F(x_{Q_j'}, A\ell(Q_j'))$.

If (3) holds for x, then for m large enough $\sum_{j=J+1}^{m} \chi_{G(Q_j)}(x) \geq \delta_0 m/2$, where we used the first estimate above. This ensures that we can use the third estimate at least $\delta_0 m/2$ times, and we get

$$F(x_{Q'_{m+1}}, A\ell(Q'_{m+1})) \leq 2^{-\delta_0 m/2}(1 + CA^{-1/2})^m.$$

If A is large enough, clearly $F(x_{Q'_m}, A\ell(Q'_m)) \to 0$ as $m \to \infty$, which is absurd because $x \notin \bigcup_{j\geq J} T_j$. Thus (3) does not hold for x, and $\mu(R_{\Sigma_0} \setminus \bigcup_{j\geq J} T_j) = 0$. □

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3 Monotonicity Properties of Variational Integrals, A_p Weights and Unique Continuation

After N. Garofalo and F.H. Lin [GL]

A summary written by Valentina Ciccone

Abstract

Following [GL] we study unique continuation for solutions of equation (3).

3.1 Settings

Assume that Ω is an open and connected subset of $\mathbb{R}^n$, $n \geq 3$, and that $A(x)$ is a symmetric $n \times n$ matrix-valued function on Ω. Moreover, assume that A satisfies the following

(i) there exists $\Gamma > 0$ such that for every $x, y \in \Omega$ it holds

$$|a_{ij}(x) - a_{ij}(y)| \leq \Gamma |x - y| \quad i, j = 1, ..., n \ , \tag{1}$$

where $a_{ij}(\cdot)$ denotes the entry in the $i-$th row and $j-$th column of $A(\cdot)$;

(ii) there exists $\lambda \in (0, 1)$ s.t. for all $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$ it holds that

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda^{-1} |\xi|^2. \tag{2}$$

Let u be the solution of

$$Lu = \mathrm{div}(A(x)\nabla u(x)) = 0 \quad \text{on} \ \ \Omega \ . \tag{3}$$

That is, $u \in H^{1,2}_{loc}(\Omega)$ and for all $\phi \in H^{1,2}_0(\Omega)$ it holds that

$$\int_\Omega \langle A(x)\nabla u(x), \nabla \phi(x) \rangle dx = 0 \ .$$

It is known that if A satisfies the hypothesis (i) and (ii) then $u \in H^{2,2}_{loc}(\Omega)$. Also throughout we will assume that Ω strictly contains $\overline{B}_2$, namely the closure of the ball centered at the origin and of radius 2.

3.2 Overview of the main results

The first main result of the paper is the following.

Theorem 1. *Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of* (3).

(a.) If $u \neq 0$ there exist a $p > 1$ and a constant $C > 0$ such that for all ball B_R centered at the origin and such that $B_{2R} \subset B_1$ it holds that

$$\left(\frac{1}{|B_R|}\int_{B_R}|u|dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|u|^{-1/(p-1)}dx\right)^{p-1} \leq C \ . \tag{4}$$

Here, C and p depend on u, Γ, λ, n but not on B_R.

(b.) If u is not identically equal to a constant there exist a $q > 1$ and a constant $B > 0$ such that for any ball B_R as in (a.) it holds that

$$\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{-1/(q-1)}dx\right)^{q-1} \leq B \ . \tag{5}$$

Here, B and q depend on u, Γ, λ, n but not on B_R.

Theorem 1 is telling us that, under the considered assumptions, $|u|$ and $|\nabla u|$ are, respectively, A_p and A_q weights of Muchenhoupt in B_1.

Our focus for the summer school is on the following two results. The first concerns strong unique continuation.
We recall that a function $u \in L^2_{loc}(\Omega)$ *vanishes of infinity order at* $x_0 \in \Omega$ if for $R > 0$ sufficiently small it holds that

$$\int_{|x-x_0|<R} u^2 dx = O(R^N) \quad for\ all\ N \in \mathbb{N} \ .$$

Theorem 2. *Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution of* (3).

(i) If u vanishes of infinite order at x_0 then u is identically zero on Ω.

(ii) $|\nabla u|$ cannot vanish of infinite order at $x_0 \in \Omega$ unless u is identically equal to a constant on Ω.

The following is needed in the proof of Theorem 1 and Theorem 2 and is a further main result of the paper.

Theorem 3 (Doubling condition)**.** *Let $u \in H^{1,2}_{loc}(\Omega)$ be a solution to* (3)*. Then there exists a constant $C > 0$, which depends on u, Γ, λ, n, such that for any ball B_R, such that $B_{2R} \subset B_1$, we have that*

$$\int_{B_{2R}} u^2 dx \leq C \int_{B_R} u^2 dx. \tag{6}$$

To get a rough intuition of some of the ideas behind the proof of Theorem 3 we consider the following simplified setting. Assume that $L = \Delta$, that is, the Laplace operator in $\mathbb{R}^n$, and, therefore, that u is an harmonic function on Ω. For a ball B_r centered at the origin define the quantity

$$H(r) = \int_{\partial B_r} u^2 dH_{n-1}$$

where dH_{n-1} is the $(n-1)-$dimensional Hausdorff measure on ∂B_r. $H(r)$ can be shown to be related to the integral

$$D(r) = \int_{B_r} |\nabla u|^2 dx \ .$$

Set $N(r) = rD(r)/H(r)$. The key claim is the following

$$N(r) \text{ is a decreasing function of r.}$$

This, for the case $L = \Delta$, was observed by [Al]. He called $N(r)$ the *frequency* of u.

The idea is to extend the outlined approach to the general setting of (3). To this end, suitable modifications of $H(r)$ and $D(r)$ are introduced and it is shown that if u solves (3) then a *modified generalized frequency* is a monotonically non-decreasing function of r. All of this will be used to prove Theorem 3.

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4 The stability for the Cauchy problem for elliptic equations

After G. Alessandrini, L. Rondi, E.Rosset, and S. Vessella [1]

A summary written by Dimas de Albuquerque

Abstract

We study the ill posed Cauchy problem for elliptic equations of divergence form. More precisely, we provide stability results for solutions of such equations, and the results obtained are based on a central technique: the three spheres inequality.

4.1 Introduction

A problem in partial differential equations is well posed when the following three conditions are satisfied: the problem has a solution, the solution is unique, and the solution depends continuously on the data (stability).

As it turns out, the Cauchy problem for Laplace's equation is not well posed, and this comes from the fact that this problem does not satisfy the third condition. In [2], Hadamard provided his now classical example of instability, which indicates the above. More precisely, he considered the following problem:

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2;\ y > 0\} \\ u(x,0) = 0 \\ u_y(x,0) = A_n \sin(nx) \end{cases} \tag{1}$$

The solution to (1) is given by $u_n(x,y) = A_n \sin(nx)\sinh(ny)$. As a result, if $A_n = n^{-1}$, for example, it follows that $(u_n)_y(x,0) \to 0$ uniformly in x as $n \to \infty$, whereas $|u_n(x,y)| \to \infty$ as $n \to \infty$ for any positive y.

This issue can be fixed provided an a priori bound (on some norm) for the solution u is known from the beginning. Under this extra assumption, the problem becomes well posed, and the problem of interest is to obtain the so called stability estimates, which consist on bounding some other norm of u by corresponding bounds on the Cauchy data.

In order to proper state the Cauchy problem we'll work with and the results associated to it, we need some definitions. These are mainly related to the regularity of the domain and operators we shall consider. In what follows, Ω is a bounded open connected set of $\mathbb{R}^n$.

Let Σ be an open subset of $\partial\Omega$, and let $\Sigma' = \partial\Omega \backslash \Sigma$. For $P \in \Sigma$, set

$$r(P) = \text{dist}(P, \Sigma') \;\; ; \;\; \rho(P) = \min\left\{\rho_0, \frac{r(P)M_0}{\sqrt{1+M_0^2}}\right\} \tag{2}$$

Definition 1. *An open subset $\Sigma \subset \partial\Omega$ is said to be an open Lipschitz portion of $\partial\Omega$ with constants $M_0, \rho_0 > 0$ if for every $P \in \partial\Omega$ there exists a rigid motion taking P to the origin, such that*

$$\Omega \cap \Gamma_{\frac{\rho(P)}{M_0},\rho(P)} = \{(x', x_n) \in \mathbb{R}^n \mid x_n > Z(x')\} \tag{3}$$

where $Z : B'_{\frac{\rho_0}{M_0}} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function satisfying

$$Z(0) = 0 \;\; ; \;\; \|Z\|_{C^{0,1}} \leq M_0\rho_0 \tag{4}$$

Here $\Gamma_{a,b}(x)$ is the cylinder $\{(y', y_n) \in \mathbb{R}^n \| x' - y'| < a \,\text{and}\, |y_n - x_n| < b\}$.

Definition 2. *The portion Σ has size at least $\rho_1 \in (0, \rho_0]$ if there exists $P \in \Sigma$ such that $\rho(P) \geq \rho_1$.*

The elliptic operators that we are going to consider are of the form

$$Lu = \text{div}(A\nabla u) + cu \tag{5}$$

where A is a symmetric matrix satisfying the ellipticity condition: $K^{-1}|y|^2 \leq A(x)x \cdot y \leq K|y|^2 \; \forall \; x, y \in \mathbb{R}^n (K \geq 1)$. Moreover, we'll assume that A has Lipschitz regularity:

$$|A(x) - A(y)| \leq \frac{L}{\rho_0}|x - y| \tag{6}$$

and the lower order coefficient is essentially bounded:

$$\|c\|_{L^\infty(\mathbb{R}^n)} \leq \frac{\kappa}{\rho_0^2} \tag{7}$$

The Cauchy problem that will be on our minds is the following: given $f \in L^2(\mathbb{R}^n), F \in L^2(\mathbb{R}^n; \mathbb{R}^n), g \in H^{\frac{1}{2}}(\Sigma), \psi \in H^{-\frac{1}{2}}(\Sigma)$, we would like to find $u \in H^1(\Omega)$ such that $u|\Sigma = g$ in the trace sense and

$$\int_{\Omega}(A\nabla u\cdot\nabla\phi - cu\phi)\,dx = \int_{\Sigma}\psi\phi - \int_{\Omega} f\phi - F\cdot\nabla\phi\,dx\ \forall\ \phi\in H^1_{co}(\Omega\cup\Sigma) \quad (8)$$

Note that the above is a generalization of the rigorous weak formulation of the Cauchy problem given by

$$\begin{cases} \operatorname{div}(A\nabla u) + cu = f \text{ in } \Omega \\ u = g \text{ in } \Sigma \\ A\nabla u\cdot\nu = \psi \text{ in } \Sigma \end{cases} \quad (9)$$

4.2 Main result

In what follows, we are going to assume that:

$$\|f\|_{L^2(\mathbb{R}^n)} + \frac{1}{\rho_0}\|F\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \leq \frac{\epsilon}{\rho_0^2} \quad (10)$$

and also the following bounds on the Cauchy data:

$$\|g\|_{H^{\frac{1}{2}}(\Sigma)} + \rho_0\|\psi\|_{H^{-\frac{1}{2}}(\Sigma)} \leq \eta \quad (11)$$

Bearing the above in mind, we have the:

Theorem 3 (Stability in the interior)**.** *Let $u\in H^1(\Omega)$ be a (weak) solution to (8), and Σ be a Lipschitz portion of Ω satisfying the conditions in Definition 1 and Definition 2. Suppose also that we assume the a priori bound:*

$$\|u\|_{L^2(\Omega)} \leq E_0 \quad (12)$$

There exists $\overline{h}$ such that for every $0 < h < \overline{h}$, and for every $G\subset\Omega$ such that:

$$dist(G,\partial\Omega)\geq h \ \ ; \ \ dist(P,G) < \frac{\rho_1}{8M_0} \quad (13)$$

where $P\in\Sigma$ is the point appearing in Definition 2, we have:

$$\|u\|_{L^2(G)} \leq C(\epsilon+\eta)^{\delta}(E_0+\epsilon+\eta)^{1-\delta} \quad (14)$$

for some constantes $\delta\in(0,1)$ and $C>0$.

4.3 Ideas of the proof

The proof is strongly based on a single building block: the three spheres inequality. The proof of (14) happens in four steps:

(i) Three spheres inequality

We prove the above mentioned three spheres inequality, which for our situation comes modelled as follows:

Theorem 4 (Three spheres inequality). *Let $u \in H^1(B_R)$ be a weak solution to the inhomogeneous elliptic equation*

$$div(A\nabla u) + cu = f + divF \text{ in } B_R \tag{15}$$

Then, for every $r_1 < r_2 < r_3 < R$, there exist constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\|u\|_{L^2(B_{r_2})} \leq C(\|u\|_{L^2(B_{r_1})} + \epsilon)^{\alpha}(\|u\|_{L^2(B_{r_3})} + \epsilon)^{1-\alpha} \tag{16}$$

The proof for this fact in turn happens again in four steps, where we obtain several versions of three spheres inequality, each corresponding to an equation which is a particular case of (15). These equations are as follows:

- (Homogeneous in pure principal part) $\text{div}(A\nabla u) = 0$ in B_R
- (Complete homogeneous equation) $\text{div}(A\nabla u) + cu = 0$ in B_R
- (Complete equation with right hand side) (15)

It should be noted however that the three above results are only guaranteed with the restriction that the radii must satisfy $r_1 < r_2 < \frac{r_3}{K} \leq r_3$. Nevertheless, the last result above will be enough for us to prove the second step below, and using the second step itself we remove the restriction on the radii and conclude the proof of Theorem 4.

(ii) Estimates of propagation of smallness

As a consequence from the proof of (i), we obtain estimates on how smallness propagates for solutions of (15), that is, we assume that u is small in some ball $B_{r_0}(x_0)$ inside the domain, and then we estimate how small u is in some larger connected open set $G \subset \Omega$.

(iii) Extension

Given u solution to a Cauchy problem with data on a portion Σ of Ω, we extend u to a larger open set containing Ω obtaining a new function $\tilde{u}$. The extension is such that $\tilde{u}$ also solves an equation in the augmented domain, but now with a different right hand side, which in turn is controlled by the original Cauchy data.

(iv) Conclusion

We apply the estimates of propagation of smallness to the extended function on the new open set.

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5 Quantitative propagation of smallness for solutions of elliptic equations

After Alexander Logunov and Eugenia Malinnikova [LM1]

A summary written by Gautam Neelakantan Memana

Abstract

In this section we show that smallness for solutions to ellitpic PDE in divergence form with Lipschitz coefficents can be propogated to a larger set by proving a logarithmic convexity property like the classical three spheres theorem.

5.1 Introduction

It is a classical result that $log\ |f|$ of a holomorphic function f on $\mathbb{C}$ is subharmonic (and hence for gradients of real valued harmonic functions in $\mathbb{R}^2$). But this does not hold in higher dimensions. However, for a harmonic function (also solutions to reasonable uniformly elliptic PDE $Lu = 0$) we have a logarithmic convexity property, known as the three spheres theorem. The theorem claims that

$$\underset{\mathcal{B}}{sup}\ |u| \leq C \left(\underset{\frac{1}{2}B}{sup}\ |u| \right)^{\gamma} \left(\underset{2B}{sup}\ |u| \right)^{1-\gamma}, \tag{1}$$

where $B = \{x \in \mathbb{R}^n; |x| \leq 1\}$, the constants $C > 0$ and $\gamma \in (0, 1)$ depend only on the elliptic operator L and do not depend on u.

The main aim of this paper is to prove the the three spheres lemma for much more general sets than spheres, which will imply the propogation of smallness of a solution to an elliptic PDE in divergence form with Lipschitz coefficients. Moreover, the paper proves a similar three spheres theorem for gradients of the solutions, which was not even known for a harmonic function in $\mathbb{R}^n, n \geq 3$.

The setting.

Definition 1. *The Hausdorff content of a set $E \subset \mathbb{R}^n$ is*

$$\mathcal{C}^d_{\mathcal{H}}(E) = inf \left\{ \sum_j r_j^d : E \subset \cup_j B(x_j, r_j) \right\}, \tag{2}$$

and the Hausdorff dimension of E is defined as

$$dim_{\mathcal{H}}(E) = inf \left\{ d : \mathcal{C}^d_{\mathcal{H}}(E) = 0 \right\}.$$

One of the reasons why the Hausdorff content is used over Hausdorff measure in the main theorem is that it is always finite for bounded sets. Moreover, the Hausdorff content of order n is equivalent to the lebesgue measure.

Definition 2. *For a cube Q in $\mathbb{R}^n$ let $s(Q)$ denote its side length and let tQ be the cube with same centre as Q and such that $s(tQ) = ts(Q)$. Suppose that $20nQ \subset \Omega$. We define the doubling index of a function u in the cube Q by*

$$N(u, Q) = \sup_{x \in Q, r \leq s(Q)} log \frac{sup_{B(x,10nr)} |u|}{sup_{B(x,r)} |u|}.$$

This defintion is slightly different from the standard definition of doubling index for balls as we take the supremum over all cubes contained in Q. The definition implies the following useful estimate. Let q be a subcube of Q and $K = \frac{s(Q)}{s(q)} \geq 2$. Then

$$\sup_q |u| \geq cK^{-CN} \sup_Q |u|, \tag{3}$$

where $N = N(u, Q)$, c and C depend only on n. Assume that u is a solution of an elliptic equation in the diveregence form in a bounded domain $\Omega \subset \mathbb{R}^n$,

$$div(A \nabla u) = 0, \tag{4}$$

where $A(x) = [a_{ij}(x)]_{1 \leq i,j \leq n}$ is a symmetric uniformly elliptic matrix with Lipschitz entries,

$$\Lambda_1^{-1} ||\zeta||^2 \leq \langle A\zeta, \zeta \rangle \leq \Lambda_1 ||\zeta||^2, \quad |a_{ij}(x) - a_{ij}(y)| \leq \Lambda_2 |x - y|. \tag{5}$$

Let m, δ, ρ be positive numbers. Suppose a set $E \subset \Omega$ satisfies

$$\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(E) > m \ (see\ (2)), \quad dist(E, \partial\Omega) > \rho.$$

Let κ be a subset of Ω with $dist(\kappa, \partial\Omega) > \rho$. Then, the main result the paper is the following

Theorem 3. *There exist $C, \gamma > 0$, depending on $m, \delta, \rho, A, \Omega$ only such that*

$$\sup_{\kappa} |u| \leq C\big(\sup_{E} |u|\big)^{\gamma}\big(\sup_{\Omega} |u|\big)^{1-\gamma} \tag{6}$$

for any solution u of $div(A\nabla u) = 0$ in Ω.

Remark 4. *Now, if $\sup_{\Omega} |u| = 1$ and $\sup_{E} = \varepsilon$, then (6) can be rewritten as*

$$\sup_{\mathcal{K}} |u| \leq C\varepsilon^{\gamma}. \tag{7}$$

The techniques used to prove the above theorem were also used in the seminal papers [L1], [L2], [LM2].

Main tool. The technique of using doubling indices developed in [L1], [L2], [LM2] is the important idea in the proof (see 2). It is very useful in estimating the Hausdorff measure of zero sets of elliptic equations. The following lemma about the doubling index is crucially used in the proof of the main theorem

Lemma 5. *([L2]) Let u be a solution $div(A\nabla u) = 0$ in Ω. There exist positive constants s_0, N_0, B_0 that depend only A, Ω such that if Q is a small enough cube contained in Ω with sidelength $< s_0$, and Q is divided into B^n equal cubes subcubes with $B > B_0$, then the number of subcubes q with $N(u, q) \geq max\left(\frac{1}{2}N(u, Q), N_0\right)$ is less than B^{n-1-c}, where c depends only on the dimension n.*

5.2 Reformulation of the main theorem

Theorem 3 is a local result, so we can formulate an equivalent local version of the same.

Proposition 6. *Let Ω be a bounded domain in $\mathbb{R}^n$, A satisfy (5) and δ and m be positive. There exists $C, \gamma > 0$, depending on A, Ω, m and δ such that the following holds. Suppose that u is a solution to $div(A\nabla u) = 0$ in $\Omega \supset (10n^2)Q$ and $E \subset \frac{1}{20n}Q$ satisfy $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(E) \geq ms(Q)^{n-1+\delta}$, then*

$$\sup_{Q} |u| \leq C \left(\sup_{E} |u| \right)^{\gamma} \left(\sup_{10n^2 Q} |u| \right)^{1-\gamma}. \tag{8}$$

The constants $10n$ and $20n^2$ are only for technical purposes. Once we prove the above proposition we can get the main theorem from standard arguments. First, find a suitable cube Q with $10n^2 \subset\subset \Omega$ and $\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(\frac{1}{20n})Q \cap E > 0$. Now, with the help of Proposition 6 we can propogate the smallnes from E to the cube Q. Then, with the help of the three spheres theorem and the standard Harnack chain argument allows us to propogate the smallnes from Q onto the whole of $\mathcal{K} \subset\subset \Omega$.

Then, it just remians to prove the Proposition 6 and it follows from the following lemma. All the important ideas of estimates of zero sets, sub level sets and the estimates for doubling index are used in the proof of the lemma.

Lemma 7. *Suppose that $div(A\nabla u) - 0$ and $sup_Q |u| = 1$. Let $N = N(u, Q) \geq 1$. Set as above*

$$E_a = \{x \in \frac{1}{2}Q : |u(x)| < e^{-a}\}.$$

Then

$$\mathcal{C}_{\mathcal{H}}^{n-1+\delta}(E_a) < Ce^{-\beta a/N} s(Q)^{n-1+\delta},$$

for some $C, \beta > 0$ that depend on A, δ only.

For the talk I will concentrate on trying to prove this lemma.

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6 Nodal sets of eigenfunctions on Riemannian manifolds

After H. Donnelly and C. Feffermann[DF]

A summary written by Germán Miranda

Abstract

We present two results about nodal domains of eigenfunctions on compact Riemannian manifolds. First result gives an upper bound of the vanishing order of the eigenfunctions in a Riemannian manifold with smooth metric. The second main theorem proves Yau's conjecture in the case of real-analytic compact Riemannian manifolds.

6.1 Introduction

We consider M to be a n-dimensional compact connected Riemannian manifold with C^∞ Riemannian metric (in the second result, real-analyticity will be required). Let $-\Delta$ be the Laplace-Beltrami operator on M, then Δ is a negative definite, self-adjoint and elliptic operator. Moreover, its spectrum consists of negative isolated eigenvalues of finite multiplicity accumulating only to $-\infty$. We assume that M has no boundary.

Definition 1. *Let F be a real eigenfunction of $-\Delta$ with eigenvalue $\lambda > 0$, the **nodal set** N of F is defined as follows*

$$N := \{x \in M : F(x) = 0\}. \tag{1}$$

The first theorem follows the ideas introduced in [A] to give a vanishing order upper bound of Laplace-Beltrami eigenfunctions.

Theorem 2. *Let M be a n-dimensional compact connected Riemannian manifold with C^∞ Riemannian metric, and F be a real eigenfunction of $-\Delta$, then F vanishes at most to order $c\sqrt{\lambda}$, for any point in M, where c is a suitable positive constant.*

The second main result is the proof of Yau's conjecture for real analytic manifolds.

Theorem 3. *Let M be a real analytic n-dimensional compact connected Riemannian manifold with real analytic metric, then the nodal set N of a real eigenfunction F of $-\Delta$ with eigenvalue λ sastisfies*

$$c_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(N) \leq c_2\sqrt{\lambda},$$

where c_1, c_2 are positive constants and $\mathcal{H}^{n-1}(N)$ denotes the $n-1$ dimensional Hausdorff measure of N.

Below, we will present the main tools and ideas needed to prove Theorem 2 and Theorem 3.

6.2 Proof sketch of Theorem 2

The main idea of the proof is the usage of inequalities relating the growth of the eigenfunctions on large balls to their vanishing order in small balls. We will first get local results which will be extended using the compactness of M.

As before, let F be an eigenfunction of $-\Delta$ with eigenvalue $\lambda > 0$. We consider some geodesic ball $B(p, h_0)$ and work on geodesic polar coordinates (r, t).

Using a suitable local conformal change in the metric and volume element, and doing partial integrations in the radial and spherical variables one can obtain the following stronger version of a Carleman estimate. Let u be a smooth function supported in $\frac{\delta}{2} < r < h$, with $h < h_0$ small enough,

$$\int\int \bar{r}^{2(2-\beta)}|(\Delta+\lambda)u|^2 r^{-1} dr dt$$

$$\geq B_9\beta^2 \int\int \bar{r}^{2-2\beta}u^2r^{-1}drdt + C_9\delta\beta^2\int\int \bar{r}^{-1-2\beta}u^2r^{-1}drdt, \qquad (2)$$

where $\bar{r}$ is a weight function comparable to the geodesic distance r from p in $B(p, h_0)$ (related to the conformal change mentioned before) and $\beta > a_1\sqrt{\lambda} + a_2$ with a_1, a_2 being sufficiently large constants. Jacobi fields' theory guarantees that all the constants appearing in (2) depend only on h_0 and an upper bound for the absolute values of the sectional curvatures in $B(p, h_0)$.

In order to apply equation (2) to F, we introduce a cut off function supported in an annulus $\delta\left(1 - \frac{1}{10\beta}\right) < \bar{r} < \frac{2}{3}h$, and define $u = \theta F$. Assuming also that $\beta > a_3 \log\left(\max_{r\le h}|F|/\max_{h/10\le r\le h/5}|F|\right)$ and using standard elliptic theory to bound the L^∞ norm of F by a multiple of its L^2 norm, one gets

$$D_1\beta^3\delta^{-2\beta}\max_{(1-\frac{1}{\beta})\delta\le\bar r\le(1+\frac{1}{\beta})\delta}|F|^2+(D_2\lambda+D_3)\left(\frac{h}{2}\right)^{2(2-\beta)}\max_{h/4\le\bar r\le 3h/4}|F|^2$$
$$\ge (D_4\lambda+D_5)^{-n/2}\left(\frac{h}{3}\right)^{2(2-\beta)}\beta^2\max_{h/12\le\bar r\le h/4}|F|^2. \qquad (3)$$

The additional assumption on β allows us to absorb the second term on the left hand side on the right hand side and obtain

$$D_1\beta^3\delta^{-2\beta}\max_{(1-\frac{1}{\beta})\delta\le\bar r\le(1+\frac{1}{\beta})\delta}|F|^2\ge\frac{1}{2}(D_4\lambda+D_5)^{-n/2}\left(\frac{h}{3}\right)^{2(2-\beta)}\beta^2\max_{h/12\le\bar r\le h/4}|F|^2,$$

which applying standard estimates leads to

$$\max_{r\le\delta}|F|\ge(C_{13}\delta)^{D_{13}\beta}\max_{h/10\le r\le h/5}|F|. \qquad (4)$$

By similar arguments, if $\beta > a_1\sqrt{\lambda}+a_2+a_3\log\left(\max_{r\le h}|F|/\max_{r\le h/5}|F|\right)$, $|F|\le 1$ for $r\le h$, and $\max_{r\le h/5}|F|\ge\exp(-D_{15}\sqrt{\lambda}-C_{14})$ we have

$$\max_{r\le h/10}|F|\ge\exp(-D_{16}\sqrt{\lambda}-C_{15}). \qquad (5)$$

Finally, normalizing F to have $\|F\|_\infty=1$, we obtain that $|F|\le 1$ and $F(x_0)=1$ for some $x_0\in M$. Since M is connected, for an arbitrary $x\in M$ we can construct a chain of overlapping balls joining x_0 and x with radius $h/5$, whose centers at separated at most $h/10$. By compactness of M this chain has a finite number of balls. Using (5) inductively and replacing h by $h/20$ we obtain

$$\max_{B(x,h/200)}|F|\ge\exp(-C_4\sqrt{\lambda}-C_5),$$

for any $x\in M$. This inequality tells us that $\beta > a_4\sqrt{\lambda}+a_5$ is enough to fulfill the conditions we had to obtain (5) for any point $x\in M$. Combining (4) and (5) for an arbitrary point $x\in M$ we have for $\delta < ah$,

$$\max_{B(x,\delta)}|F|\ge(C_6\delta)^{C_7\sqrt{\lambda}+C_8}\max_{B(x,h/5)\setminus B(x,h/10)}|F|. \qquad (6)$$

Theorem 2 from this inequality follows from this inequality.

6.3 Proof sketch of Theorem 3

To prove Theorem 3 the main idea will be that an eigenfunction F of $-\Delta$, with eigenvalue λ, on a real analytic manifold behaves like a polynomial of degree $c_3\sqrt{\lambda}$.

• **Upper bound:** Let $P(x)$ be a non-zero polynomial of degree $c_3\sqrt{\lambda}$ defined on $\mathbb{R}^n$. Let $V := \{|x| < 1 : P(x) = 0\}$ and $\mathcal{L}$ be the set of lines in $\mathbb{R}^n$ intersecting $|x| < 1$, then we have that

$$\mathcal{H}^{n-1}(V) \leq \int_{\mathcal{L}} |L \cap V| d\mu(L),$$

where $L \in \mathcal{L}$ and $d\mu$ is a measure on $\mathcal{L}$. It is clear that $|L \cap V| \leq c_3\sqrt{\lambda}$ almost everywhere ($|L \cap V|$ denotes the cardinality of $L \cap V$). Hence, $\mathcal{H}^{n-1}(V)$ is bounded by a a multiple of $\sqrt{\lambda}$.

F is not a polynomial but since $-\Delta$ is an elliptic operator with analytic coefficients, if we assume that our metric continues analytically into the complex ball $|z| < 2$ we can extend F to an analytic function to $|z| < 1$ in $\mathbb{C}^n$. Moreover, we get the estimate

$$\max_{|z|<1} |F(z)| \leq e^{c_5\sqrt{\lambda}} \max_{|x|<2} |F(x)|. \tag{7}$$

Using an argument similar to the one used to prove Theorem 2, one can show that

$$\max_{|x|<2} |F(x)| \leq e^{c_6\sqrt{\lambda}} \max_{|x|<1/5} |F(x)|,$$

and combining the two previous equations we obtain an important growth condition

$$\max_{|z|<1} |F(z)| \leq e^{c_4\sqrt{\lambda}} \max_{|x|<1/5} |F(x)|. \tag{8}$$

Integral geometry methods can be used to prove the upper bound. Again we will use different coordinate patches where our assumptions are fulfilled and then use compactness of M to see that we need only a finite number of them to cover our manifold.

• **Lower bound:** By a maximum principle argument, we know that every ball of radius $d_1/\sqrt{\lambda}$ contains a zero of F. Thus, we can get a family of pairwise disjoint balls $B_\nu = B(x_\nu, d_2/\sqrt{\lambda})$ covering a fixed portion of M, with F vanishing at the centers x_ν. The number of balls is at least of magnitude $d_3\lambda^{n/2}$.

The methods presented below can be discussed in a more intuitive way for harmonic functions on $\mathbb{R}^n$ because the arguments are similar to the ones used for eigenfunctions F on B_ν. In our case, one can see that B_ν is sufficiently small with respect to the operator $\Delta + \lambda$, and this implies that $\mathcal{H}^{n-1}(B_\nu \cap N) \geq d_5 \lambda^{-(n-1)/2}$ provided

$$\int_{\mathcal{Q}_\nu} |F|^2 \leq c_9 \int_{B_\nu} |F|^2, \tag{9}$$

where $\mathcal{Q}_\nu$ is a cube containing the double of B_ν and with side $a_2\sqrt{\lambda}$. Note that (9) follows from the fact that a real analytic function G satisfying a growth estimate like (8) and non-negative for real $x \in \mathcal{Q}$, $|x_j| \leq 1$, where $\mathcal{Q}$ is the standard cube centered at the origin, satisfies the following inequality

$$|\log G(x) - \log \mathrm{Av}_{\mathcal{Q}_\nu} G| \leq d_8, \tag{10}$$

where $x \in \mathcal{Q}_\nu \setminus S$ and $S \subset \mathcal{Q}$ is a set of measure less than ϵ. The proof of this is done by reducing it to the case when G is a polynomial and using induction on the dimension n. The base case $n = 1$ turns out to be more interesting than the induction step, since the weak type $(1, 1)$ inequality for the Hilbert transform is the key of the proof.

Applying (10) to $G = F^2$ we obtain (9) for at least half of the balls. Then

$$\mathcal{H}^{n-1}(N) \geq \sum_{\mathcal{Q}_\nu \in \mathcal{S}} \mathcal{H}^{n-1}(N \cap B_\nu) \geq E_9\sqrt{\lambda},$$

where $\mathcal{S}$ is the set of cubes $\mathcal{Q}_\nu$ such that B_ν satisfies (10). This gives the desired lower bound.

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7 Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in the dimensions two and three

After A.Logunov, E.Malinnikova [1]

A summary written by Fred Lin

Abstract

We study the Hausdorff measure of the zero sets of the eigenfunction u of Δ_M: $\Delta_M u + \lambda u = 0$. In [1], with a new combinatoric argument, they gave a upper bound in two dimension $\mathcal{H}^1(\{u = 0\}) \lesssim \lambda^{\frac{3}{4}-\beta}$ and lower bound in three dimension $\mathcal{H}^2(\{u = 0\}) \gtrsim \lambda^{\alpha}$.

7.1 Introduction

Let Δ_M be the Laplace operator on a compact n-dimensional Riemannian manifold without boundary. Yau conjectured that the Hausdorff measure of Nodal sets $E_\lambda := \{u_\lambda = 0\}$ of Laplace eigenfunctions $u_\lambda, \Delta_M u + \lambda u = 0$, satisfy the following bound

$$\mathcal{H}^{n-1}(E_\lambda) \sim \lambda^{\frac{1}{2}}$$

The conjecture was proved by Donnelly and Fefferman [2] under the assumption that the Riemannian metric is real-analytic. As for the non-analytic setting, in dimension two, Donnelly and Fefferman [3] established the bound

$$\mathcal{H}^1(E_\lambda) \lesssim \lambda^{\frac{3}{4}}$$

Following the idea of Donnelly and Fefferman, with a new combinatoric argument, Logunov and Malinnikova improved the upper bound a bit (This upper bound is independent to the polynomial upper bound introduce later.)

$$\mathcal{H}^1(E_\lambda) \lesssim \lambda^{\frac{3}{4}-\beta}$$

As for the lower bound, using the same combinatoric argument, in dimension three, Logunov and Malinnikova showed that

$$\mathcal{H}^2(E_\lambda) \gtrsim \lambda^{\alpha}$$

7.2 Lower bound in three dimension

Here's a well-known trick. Let u be the solution to Δ_M: $\Delta_M u + \lambda u = 0$, consider $h(\xi, t) := u(\xi)e^{\lambda^{\frac{1}{2}}t}$, then h is harmonic on the product manifold $\mathcal{M} := M \times \mathbb{R}$. We work in a local geodesic coordinate, then the metric is locally equivalent to the Euclidean one. Here we Define an important quantity: **Doubling index**. For a function h and a cube q, we define doubling index $N(h, q)$ by

$$\int_{lq} |h(x)|^2 dx = 2^{N(h,q)} \int_q |h(x)|^2 dx$$

We can establish some properties for this doubling index, for example, L^∞ estimate, monotonicity of doubling index and estimate doubling index by wavelength. Most of the properties of doubling index we take from other paper such as [3]. We also define maximal doubling index $\widetilde{N}(h, q) := \sup_{q' \subseteq q} N(h, q)$. Next, we will establish a local Nodal set estimate

$$\mathcal{H}^{n-1}\{|x| \leq \frac{r}{2}, u(x) = 0\} \gtrsim r^{n-1} N^{2-n}$$

The last lemma we need is a estimation on the number of cubes with large doubling index. Suppose we partition the cube q into B equal subcubes, then at least half of these subcubes q' satisfy $\widetilde{N}(h, q') \leq \frac{\widetilde{N}(h,q)}{B^\delta}$, where δ is a constant only depend on dimension.

The combinatoric argument is as following, we divide q into Y subcubes in the first step. Let $N_0 := \widetilde{N}(h, q)$. We can show that if Y is large enough, then at least one subcube q' will have $\widetilde{N}(h, q') \leq \frac{N_0}{2}$, and the other bounded by N_0. Continue this process j times, then the number of cubes whose maximal doubling index bounded by $\frac{N_0}{2^{k_0}}$ is

$$\sum_{k \geq k_0} \binom{j}{k} 1^k \cdot (Y-1)^{j-1}$$

Let $\xi_1, ..., \xi_j$ be i.i.d random variable such that $P(\xi_1 = 1) = \frac{1}{Y}$ and $P(\xi_1 = 0) = 1 - \frac{1}{Y}$. Then since the expextation is $\frac{1}{Y}$ and by the law of large number, we have

$$P(\frac{\sum_{i=1}^j \xi_i}{j} > \frac{1}{2Y}) \xrightarrow{j \to \infty} 1$$

Hence, for j large enough, we have

$$\frac{1}{2} \le P(\sum_{i=1}^{j} \xi_i \ge \frac{j}{2Y}) = \sum_{k \ge \frac{j}{2Y}} \binom{j}{k} \frac{(Y-1)^{j-k}}{Y^j}$$

That is, half of the subcubes whose $\widetilde{N}$ is bounded by $N_0/2^{\frac{j}{2Y}}$. Actually, we can take δ to be the constant such that $Y^\delta \le 2^{\frac{1}{4Y}}$, then $2^{\frac{j}{2Y}} \ge (Y^j)^\delta$, then we have the desired result.

Now combine every estimate together. For $Q \subseteq M$ where M is a three dimension manifold. Let $\widetilde{Q} := Q \times I$ where the side length of I is equal to the side length of Q. By Doubling index estimate, we have $\widetilde{N}(h, \widetilde{Q}) \lesssim \lambda^{\frac{1}{2}}$ for Q small enough. We then partition $\widetilde{Q}$ into B small subcubes with side length $\lambda^{-\frac{1}{2}}$. Then $|\widetilde{q}| \sim (\lambda^{-\frac{1}{2}})^4$ and $B \sim \frac{1}{|\widetilde{q}|} \sim \lambda^2$. And by combinatoric argument above, half of the small cube have doubling index bounded by $\frac{\lambda^{\frac{1}{2}}}{B^\delta} \sim \lambda^{\frac{1}{2}-2\delta}$. In the scale of $\lambda^{-\frac{1}{2}}$, the doubling index of h is comparable to doubling index of u. Let q' be the projection of $\widetilde{q}$ on M. Then by the local nodal estimate, we have

$$\mathcal{H}^2(\{u=0\} \cap q') \gtrsim r^2 N^{-1} \gtrsim (\lambda^{-\frac{1}{2}})^2 \cdot (\lambda^{\frac{1}{2}-2\delta})^{-1} = \lambda^{-\frac{3}{2}+2\delta}$$

Note that now we are in the projection to a three dimension manifold. $|q'| \sim (\lambda^{-\frac{1}{2}})^3$. The number of such cubes are $\frac{1}{2} c \lambda^{\frac{3}{2}}$. Hence,

$$\mathcal{H}^2(\{u=0\}) \gtrsim (\lambda^{-\frac{3}{2}+2\delta}) \cdot \lambda^{\frac{3}{2}} = \lambda^{2\delta}$$

7.3 Upper bound in two dimension

The argument is roughly the same to the previous case. We first need a local lower estimate for nodal set from [3]. For a square q with side length $\lambda^{-\frac{1}{4}}$, we have

$$\mathcal{H}^1(\{u=0\} \cap q) \lesssim \widetilde{N}(u, 100q)^{\frac{1}{2}}$$

Then we can play the similar combinatoric argument. Let Q be a cube and we can partition Q into small cube q of side length $\lambda^{-\frac{1}{4}}$. Again, consider the harmonic extension $h(\xi, x) := e^{\lambda^{\frac{1}{2}}\xi} u(x)$ of u on $\widetilde{Q} = Q \times I$. Choose an integer j such that $Y^j \sim \lambda^{\frac{3}{4}}$. Now we partition $\widetilde{Q}$ into Y^j subcubes with sidelength $\lambda^{-\frac{1}{4}}$. (This make sense since $(\lambda^{-\frac{1}{4}})^3 \cdot \lambda^{\frac{3}{4}} = 1$.) Then with the fact that there are $\lambda^{\frac{1}{4}}$ $\widetilde{q}$ project to the same q and $N_0 \lesssim \lambda^{\frac{1}{2}}$, we may play the same combinatoric argument as above.

$$\begin{aligned}\mathcal{H}^1(\{u=0\}\cap Q) &\lesssim \sum_{q\subseteq Q} \widetilde{N}^{\frac{1}{2}}(u,100q) \lesssim \lambda^{-\frac{1}{4}} \sum_{\widetilde{q}\subseteq \widetilde{Q}} \widetilde{N}^{\frac{1}{2}}(h,100\widetilde{q}) \\ &\lesssim \lambda^{-\frac{1}{4}}(\sum_{k=0}^{j} \binom{j}{k} (Y-1)^k (\frac{N_0}{2^{j-k}})^{\frac{1}{2}}) \\ &= \lambda^{-\frac{1}{4}}(\sum_{k=0}^{j} \binom{j}{k} (Y-1)^k \lambda^{\frac{1}{4}} (2^{-\frac{1}{2}})^{j-k}) \\ &= [(Y-1)+2^{-\frac{1}{2}}]^j\end{aligned}$$

Notice that $Y^j \sim \lambda^{\frac{3}{4}}$ and we may take a constant δ such that $Y-1+2^{-\frac{1}{2}} = Y^{1-\delta}$. Hence

$$(Y-1+2^{-\frac{1}{2}})^j = \lambda^{\frac{3}{4}(1-\delta)} = \lambda^{\frac{3}{4}-\beta}$$

which is the desired result.

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8 Nodal sets of Laplace eigenfunctions: polynomial upper bounds for Hausdorff measure

After A. Logunov [Lo]

A summary written by Josep M. Gallegos

Abstract

We prove a polynomial upper bound for the $(n-1)$-Hausdorff measure of the zero set of eigenfunctions of n-manifolds.

8.1 Introduction

The main motivation of this work is to show the following upper bound on the measure of the zero sets of eigenfunctions of compact C^∞ manifolds.

Theorem 1. *Let (W, g) be a compact C^∞ smooth Riemannian n-manifold without boundary. For a Laplace eigenfunction φ on W with $\Delta\varphi + \lambda\varphi = 0$ define its nodal set $Z_\varphi := \{\varphi = 0\}$. There exist $C = C(W, g)$ and α, depending only on the dimension n of W, such that*

$$H^{n-1}(Z_\varphi) \leq C\lambda^\alpha$$

where H^{n-1} is the $(n-1)$-Hausdorff measure.

This result is a weak version of the upper bound in Yau's conjecture, which states that α should be $1/2$ for every dimension n.

8.2 Laplace eigenfunctions and harmonic functions

Although our motivation concerns eigenfunctions in manifolds, we will only work with solutions of divergence form elliptic PDEs in $\mathbb{R}^n$. Indeed, we will use a standard trick to pass from Laplace eigenfunctions on the manifold W to harmonic functions in the product manifold $M = W \times \mathbb{R}$, by

$$u(x,t) := \varphi(x)\exp(\sqrt{\lambda}t), \quad x \in W,\ t \in \mathbb{R}.$$

Note that $Z_u = Z_\varphi \times \mathbb{R}$, so it suffices to understand the behavior of nodal sets of harmonic functions.

Also, to work in Euclidean space, we will only consider small geodesic balls $B_g(O,R)$ and identify the Laplace operator of the manifold with a divergence form elliptic operator in a fixed domain in $\mathbb{R}^n$.

Now we introduce a very useful quantity for the study of growth properties of solutions of divergence form elliptic PDEs with Lipschitz coefficients (which is the case of the function u once we consider it in $\mathbb{R}^n$).

Definition 2. *The doubling index of a ball $B(x,r) \subset \mathbb{R}^n$ is*

$$N(x,r) = \log_2 \left(\frac{\sup_{2B} |u|}{\sup_B |u|} \right).$$

The doubling index of a cube $Q \subset \mathbb{R}^n$ is

$$N(Q) = \sup_{\substack{x \in Q, \\ r \in (0, \operatorname{diam} Q)}} N(x,r).$$

Note that the definitions of doubling indices for balls and cubes are different! The doubling index of cubes is monotone which makes it more convenient, but the doubling index of balls also satisfies some *almost monotonicity* properties related to those of the *frequency function* (see [GL]).

8.3 Outline of results

In order to prove Theorem 1, we will prove the following result.

Theorem 3. *There exist positive numbers $r = r(M,g,O)$, $C = C(M,g,O)$, and $\alpha = \alpha(n)$ such that for any harmonic function u on M and any cube $Q \subset B(O,r)$,*

$$H^{n-1}(Z_u \cap Q) \leq C \, (\operatorname{diam} Q)^{n-1} \, N_u^{\alpha}(Q),$$

where $N_u(Q)$ is the doubling index of Q for the function u.

The proof of Theorem 1 follows from the *Donnelly-Fefferman doubling index estimate* (see [DF]) which bounds the doubling index of $u(x,t) = \varphi(x) \exp(\sqrt{\lambda} t)$ by $C_1 \sqrt{\lambda}$, Theorem 3, and a finite covering of the manifold W by balls.

For the proof of Theorem 3, we require two (very important on its own) lemmas: the *simplex* and *hyperplane* lemmas. These lemmas exploit some *additivity* properties of the doubling index.

Let $x_1, \dots, x_{n+1}$ be vertices of simplex S in $\mathbb{R}^n$ and x_0 its barycenter. Assume the simplex is not degenerated: there exists $a > 0$ such that width of S divided by its diameter is larger than a.

Lemma 4 (Simplex lemma)**.** *Let B_i be balls with centers at x_i and radii not greater than $\frac{K}{2}\operatorname{diam}(S), i = 1, \dots, n+1$, where $K = K(a, n)$. There exist positive numbers $c = c(a, n)$, $C = C(a, n) \geq K$, $r = r(M, g, O, a)$, $N_0 = N_0(M, g, O, a)$ such that if $S \subset B(O, r)$ and if $N(B_i) > N$ for each $x_i, i = 1 \dots n+1$, where N is a number greater than N_0, then $N(x_0, C\operatorname{diam}(S)) > N(1 + c)$.*

This lemma states that if the doubling index of small balls centered at the vertices of a non-degenerate simplex are all large ($N(B_i) > N$), then the doubling index of a large enough ball B centered at the barycenter of the simplex must be even larger ($N(B) > (1 + c)N$).

Lemma 5 (Hyperplane lemma)**.** *Let Q be a cube $[-R, R]^n$ in $\mathbb{R}^n$. Divide Q into $(2A+1)^n$ equal subcubes q_i with side-length $\frac{2R}{2A+1}$. Consider the cubes $q_{i,0}$ that have nonempty intersection with the hyperplane $x_n = 0$. Suppose that for each $q_{i,0}$ there exists $x_i \in q_{i,0}$ and $r_i < 10\operatorname{diam}(q_{i,0})$ such that $N(x_i, r_i) > N$, where N is a given positive number. Then there exist $A_0 = A_0(n), R_0 = R_0(M, g, O)$, $N_0 = N_0(M, g, O)$ such that if $A > A_0, N > N_0, R < R_0$ then $N(Q) > 2N$.*

Let's discuss what this lemma states. Suppose you have a cube Q divided into many, equally sized, small cubes $(q_i)_i$. Consider the set of cubes that touch the hyperplane $\{x_n = 0\}$ and suppose all these cubes have large doubling index (at least N). Then the original big cube must have much larger doubling index (at least $2N$).

Combining both lemmas, we can prove a result on the number of subcubes of a cube with large doubling index, which is crucial for the proof of Theorem 3.

Theorem 6. *There exist constants $c > 0$, an integer A depending on the dimension d only, and positive numbers $N_0 = N_0(M, g, O)$, $r = r(M, g, O)$ such that for any cube $Q \subset B(O, r)$ the following holds: if we partition Q into A^n equal subcubes, then the number of subcubes with doubling index greater than* $\max(N(Q)/(1 + c), N_0)$ *is less than $\frac{1}{2}A^{n-1}$.*

The proof of Theorem 3 is now a consequence of Theorem 6 using as starting point the *Hardt-Simon exponential bound* (see [HS]) which states that

$$H^{n-1}(Z_u \cap Q) \leq C_1 N(Q)^{C_2 N(Q)}.$$

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9 Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture

After A. Logunov [L1]

A summary written by Lars Becker

Abstract

Let u be a harmonic function or an eigenfunction of the Laplace operator on $\mathbb{R}^n$. We prove lower bounds for the $n-1$ dimensional Hausdorff measure of the zero set of u.

9.1 Introduction

Let M be a C^∞-manifold of dimension n and let B be a geodesic ball on M of radius 1. We have the following theorem:

Theorem 1. *There exists a constant $c > 0$, depending on M and B only, such that for every harmonic function h on M that vanishes at the center of B the following estimate holds:*

$$H^{n-1}(\{h = 0\} \cap B) \geq c\,.$$

Consider an eigenfunction u of the Laplace operator on M with eigenvalue $-\lambda$. Then $h(x,t) = u(x)\exp(\sqrt{\lambda}t)$ is harmonic on $B \times [-1,1]$. By the Harnack inequality the zero set of h is $c/\sqrt{\lambda}$ dense in $B \times [-1,1]$. Combining this with a scaled version of Theorem 1 one obtains the following lower bound for the measure of the zero set of u:

Theorem 2. *There exists $c > 0$ and λ_0, depending on M and B only, such that if $\lambda > \lambda_0$ and u solves $-\Delta u = \lambda u$ then*

$$H^{n-1}(\{u = 0\} \cap B) \geq c\sqrt{\lambda}\,.$$

For simplicity we will prove Theorem 1 only for $M = \mathbb{R}^n$.

9.2 The frequency function

For a harmonic function u we set $H(x,r) = \int_{\partial B(x,r)} |u|^2 \, \mathrm{d}H^{n-1}$ and define the frequency function

$$\beta(x,r) = \frac{rH'(x,r)}{2H(x,r)} = \frac{r}{2} \log(H(x,r))' \,. \tag{1}$$

The frequency function β is increasing in r, see [M] Theorem 2.1. Using this one can find a layer of size $\sim 1/\log^2 N$ where it is comparable to N:

Lemma 3. *Let $B(p,2r) \subset B$ and assume that $\beta(p,r) > 10$. Then there exists $s \in [r, \frac{3}{2}r]$ and $N \geq 10$ with*

$$N \leq \beta(p,t) \leq eN$$

for all

$$t \in I := (s(1 - \frac{1}{1000 \log^2(N)}), s(1 + \frac{1}{1000\log^2(N)}))\,.$$

9.3 Estimates near a maximum on a sphere

Let p, r, s and I be as in Lemma 3. In most of the proof we will work on the spherical layer $\{|y-p| \in I\}$. We now collect some estimates that hold there. Let x be a maximizer of $|u|$ on $\partial B(p,s)$ and define $K = |u(x)|$. Fix $A = 10^6$, $\delta \in [\frac{1}{A \log^{100} N}, \frac{1}{A\log^2 N}]$, $s_{-\delta} = s(1-\delta)$ and $s_\delta = s(1+\delta)$.

By (1) we have for all $r_1 \leq r_2 \in I$ that

$$\left(\frac{r_2}{r_1}\right)^{2N} \leq \frac{H(p,r_2)}{H(p,r_1)} \leq \left(\frac{r_2}{r_1}\right)^{2eN} \,.$$

Using the mean value property of harmonic functions and the estimate of an L^2-norm by an L^∞-norm this inequality implies estimates for the suprema of $|u|$ on balls:

Lemma 4. *There exists $c > 0$ and $C > 0$ such that*

$$\sup_{B(p,s_{-\delta})} |u| \leq CK2^{-c\delta N}$$

and

$$\sup_{B(p,s_\delta)} |u| \leq CK2^{c\delta N}\,.$$

The doubling index $\mathcal{N}(x,r)$ is defined by

$$2^{\mathcal{N}(x,r)} = \frac{\sup_{B(x,2r)}|u|}{\sup_{B(x,r)}|u|}\,.$$

We will use without proof the following fact from [L2]: For every $\varepsilon > 0$ there exists $C > 0$ such that for every $2r_1 \leq r_2$ with $B(x, r_2) \subset B$ it holds that

$$\left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_1)(1-\varepsilon)-C} \leq \frac{\sup_{B(x,r_2)}|u|}{\sup_{B(x,r_1)}|u|} \leq \left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_2)(1+\varepsilon)+C}\,. \qquad (2)$$

Combining this with the previous lemma one obtains:

Lemma 5. *There exists $C > 0$ such that*

$$\sup_{B(x,\delta s)} |u| \leq K 2^{C\delta N + C}$$

and for any $\tilde{x}$ with $|x - \tilde{x}| \leq \frac{\delta}{4}s$

$$\mathcal{N}(\tilde{x}, \frac{\delta}{4}s) \leq C\delta N + C$$

and

$$\sup_{B(\tilde{x},\frac{\delta s}{10N})} |u| \geq K 2^{-C\delta N \log N - C}\,.$$

9.4 The doubling index

Given a cube Q we call

$$N(Q) = \sup_{x\in Q, r\leq \operatorname{diam} Q} \log \frac{\sup_{B(x,10nr)}|u|}{\sup_{B(x,r)}|u|}$$

the doubling index of Q. We will need to find many cubes with small doubling index. Iterating a result from [L1] one obtains the following statement:

Theorem 6. *There exist constants $c_1, c_2, C, N_0 > 0$ and a positive integer A_0, depending on the dimension only, such that for any cube $Q \subset B$ the following holds: If we partition Q into A^n equal subcubes, where $A > A_0$, then the number of subcubes with doubling index greater than*

$$\max\left(N(Q)2^{-c_1 \log(A)/\log\log(A)}, N_0\right)$$

is less then CA^{n-1-c_2}.

9.5 A tunnel with controlled growth

The next theorem is the main ingredient in the proof of Theorem 1.

Theorem 7. *For $B(p,2r) \subset B$ and every harmonic function u on B the following holds: If $\beta(x,r)$ is sufficiently large then there is a number N with*

$$\beta(p,r)/e \leq N \leq \beta(p,\frac{3}{2}r)$$

and at least $\lfloor\sqrt{N}\rfloor^{n-1}2^{c_3 \log N/\log\log N}$ disjoint balls $B(x_i, \frac{r}{\sqrt{N}}) \subset B(p,2r)$ with $u(x_i)=0$.

We sketch the proof of Theorem 7. Fix s, N from Lemma 3 and let x be a maximizer of $|u|$ on $\partial B(p,s)$. We will always assume that N is sufficiently large. Put $\delta = \frac{1}{10^8 n^2 \log^2 N}$. Consider a point $\tilde{x} \in \partial B(p,(1-\delta)s)$ with $|x-\tilde{x}| = \delta s$, i.e. the nearest point to x on $\partial B(p,(1-\delta)s)$. Let T be a box such that x and $\tilde{x}$ are the centers of opposite sites of T and T has one side of length $|x-\tilde{x}|$ and $n-1$ sides of length $\frac{|x-\tilde{x}|}{\lfloor\log N\rfloor^4}$. We divide T into $\lfloor\sqrt{N}\rfloor^{n-1}$ boxes T_i such that each has one side of length $|x-\tilde{x}|$ and $n-1$ sides of length $\frac{|x-\tilde{x}|}{\lfloor\log N\rfloor^4\lfloor\sqrt{N}\rfloor}$. Finally we decompose each T_i into cubes $q_{i,t}$, $t=1,\ldots,\lfloor\log N\rfloor^4\lfloor\sqrt{N}\rfloor$, arranged so that $d(x,q_{i,t}) \geq d(x,q_{i,t+1})$. The boxes T_i are called tunnels.

Using Lemma 4 and Lemma 5 one can show that there exist $c, C > 0$ such that for all i

$$\sup_{\frac{1}{2}q_{i,\lfloor\log N\rfloor^4\lfloor\sqrt{N}\rfloor}} |u| \geq 2^{c\frac{N}{\log^2 N}-C} \sup_{\frac{1}{2}q_{i,1}} |u|. \tag{3}$$

We partition T into equal cubes Q_i, $i=1,\ldots,\lfloor\log N\rfloor^4$ with side length $\frac{|x-\tilde{x}|}{\lfloor\log N\rfloor^4}$. From Lemma 5 and (2) it follows that $N(Q_i) \leq N$ for all i. Applying Theorem 6 with $A = \lfloor\sqrt{N}\rfloor$ to all cubes Q_i then yields that the total number of cubes $q_{i,t}$ with $N(q_{i,t}) > \max\left(\frac{N}{2^{c_1\log N/\log\log N}}, N_0\right)$ is at most $C\lfloor\sqrt{N}\rfloor^{n-1-c_2}\lfloor\log N\rfloor^4 < \frac{1}{2}\lfloor\sqrt{N}\rfloor^{n-1}$. We conclude that at least half of the tunnels T_i have the following property:

$$N(q_{i,t}) \leq \max\left(\frac{N}{2^{c_1\log N/\log\log N}}, N_0\right)$$

for all $t = 1, \ldots, \lfloor \log N \rfloor^4 \lfloor \sqrt{N} \rfloor$. We call such tunnels good. For each good tunnel we have

$$\log \frac{\sup_{\frac{1}{2}q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \le \log \frac{\sup_{4q_{i,t+1}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \le \frac{N}{2^{c_1 \log N / \log \log N}}$$

for $t = 1, \ldots, \lfloor \log N \rfloor^4 \lfloor \sqrt{N} \rfloor - 1$. If u has no sign change in $\overline{q_{i,t}} \cup \overline{q_{i,t+1}}$ then the same estimate holds with C on the right hand side, by the Harnack inequality. On the other hand, by (3) we have that

$$\log \frac{\sup_{\frac{1}{2}q_{i,\lfloor \log N \rfloor^4 \lfloor \sqrt{N} \rfloor}} |u|}{\sup_{\frac{1}{2}q_{i,t}} |u|} \ge c \frac{N}{\log^2 N} - C \, .$$

Combining this one finds that there exists $c_3 > 0$ such that if T_i is a good tunnel, then there are at least $2^{c_3 \log N / \log \log N}$ closed cubes $q_{i,t}$ that contain a zero of u. Denote by $\tilde{x}_{i,t}$ a zero of u in $q_{i,t}$. The cubes $q_{i,t}$ have diameter $\sim r/(\sqrt{N} \log^6 N)$, hence each ball $B(\tilde{x}_{i,t}, \frac{r}{\sqrt{N}})$ intersects at most $C \lfloor \log N \rfloor^{6n}$ other such balls. Choosing a maximal disjoint collection we thus find at least $2^{c_3' \log N / \log \log N}$ balls $B(\tilde{x}_{i,t}, \frac{r}{\sqrt{N}})$ with $u(x_i) = 0$.

9.6 Conclusion of the proof of Theorem 1

In this section we complete the proof of Theorem 1. Define the function

$$F(N) = \inf \frac{H^{n-1}(\{u = 0\} \cap B(x, \rho))}{\rho^{n-1}} \, ,$$

where the infimum is taken over all balls $B(x, \rho) \subset B$ and all harmonic functions u on B such that $u(x) = 0$ and $N(B) \le N$. Here we denote by $N(B)$ the supremum of $\beta(x, r)$ over all $B(x, r) \subset B$.

Theorem 8. *There exists $c > 0$ with $F(N) \ge c$ for all positive N.*

Proof. We fix u, ρ where the infimum is almost attained: $H^{n-1}(\{u = 0\} \cap B(x, \rho)) \le 2F(N)\rho^{n-1}$. In [LM] it is shown that

$$\frac{H^{n-1}(\{u = 0\} \cap B(x, \rho))}{\rho^{n-1}} \ge \frac{c_1}{\beta(x, \frac{\rho}{2})^{n-1}} \ge \frac{c_2}{N^{n-1}}$$

if $N(B(x, \frac{\rho}{2})) \le N$, by inscribing a ball of radius $\sim \frac{\rho}{N}$ in $B(x, \frac{\rho}{2})$ where u is positive and a ball of the same radius where u is negative.

This estimate is good enough if N is bounded, hence we may assume that N is sufficiently large. We will show that $\beta(x, \frac{\rho}{2})$ is bounded, which completes the proof by the last display. If $\beta(x, \frac{\rho}{2})$ is sufficiently large, then we can apply Theorem 7 for the ball $B(x, 2r) = B(x, \rho)$ and find a number $\tilde{N} \geq \beta(x, \frac{\rho}{2})$ and $\lfloor \sqrt{\tilde{N}} \rfloor^{n-1} 2^{c_3 \log \tilde{N} / \log \log \tilde{N}}$ disjoint balls $B(x_i, r/\sqrt{\tilde{N}})$ contained in $B(x, 2r)$ such that $u(x_i) = 0$. For each i we know that

$$H^{n-1}(\{u = 0\} \cap B(x_i, r/\sqrt{\tilde{N}})) \geq F(N) \left(r/\sqrt{\tilde{N}} \right)^{n-1} ,$$

hence

$$H^{n-1}(\{u = 0\} \cap B(x, \rho)) \geq 2^{c_3 \log \tilde{N} / \log \log \tilde{N} - c_4} F(N) \rho^{n-1} .$$

This contradicts our choice of u and ρ if $\beta(x, \frac{\rho}{2})$ is sufficiently large. □

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10 The sharp upper bound for the area of the nodal sets of Dirichlet Laplace eigenfunctions

After A. Logunov, E. Malinnikova, N. Nadirashvili and F. Nazarov [LMNN]

A summary written by Mario Stipčić

Abstract

It is conjectured that the Hausdorff measure of the zero set of the Dirichlet Laplace eigenfunction in a domain Ω with eigenvalue λ can be bounded with $C(\Omega)\sqrt{\lambda}$. This paper proves that for a bounded domain with C^1 boundary. In fact, it strengthens this result to the Lipschitz domain (and a non-analytic boundary).

10.1 Introduction

Let u_λ be a solution of the equation $\Delta u_\lambda + \lambda u_\lambda = 0$ and $Z(u_\lambda)$ be its zero set. If we were able to estimate the Hausdorff measure of this set, it would be reasonable to expect

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \sim \sqrt{\lambda}, \tag{1}$$

where the constants from the upper and lower bound depend only on the manifold where the Laplace operator is defined. The main result of these authors is the upper bound of this type, given the Euclidean Laplace operator and the Dirichlet boundary condition.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let u_λ satisfy $\Delta u_\lambda + \lambda u_\lambda = 0$ and $u_\lambda|_{\partial\Omega} = 0$. Then,*

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \leq C(\Omega)\sqrt{\lambda}.$$

Donnelly and Fefferman [DF] proved (1), where they observed compact connected manifolds and the domains with real C^∞-smooth boundary. From their result, one can obtain the lower bound $\mathcal{H}^{n-1}(Z(u_\lambda)) \geq c(\Omega)\sqrt{\lambda}$ given the setting from Theorem 1, for a sufficiently large λ and while not assuming the Dirichlet boundary condition.

In comparison with the previous work of Logunov and Malinnikova, this paper generalizes the result by working with a non-analytic boundary $\partial\Omega$. Specifically, the authors turn their attention to Lipschitz domains, defined as follows.

Definition 2. *Let Ω be a domain in $\mathbb{R}^d$, $\tau \in (0,1)$, and let $B = B(x,r)$ be a ball centered on $\partial\Omega$. We say that $\partial\Omega$ is τ-Lipschitz in B if there is an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ and a function $f : B^{d-1}(0,r) \to \mathbb{R}$ such that $T(0) = x$, f is a Lipschitz function with the Lipschitz constant bounded by τ, $f(0) = 0$, and*

$$\Omega \cap B = T\Big(\{(y', y'') \in B^d(0,r) \subset \mathbb{R}^{d-1} \times \mathbb{R} : y'' > f(y')\}\Big).$$

In this case we write $\partial\Omega \cap B \in Lip(\tau)$.

We say that Ω is a Lipschitz domain with local Lipschitz constant τ if there exists $r > 0$ such that $\partial\Omega \cap B(x,r) \in Lip(\tau)$ for any $x \in \partial\Omega$.

In several instances throughout the proof, as one observes a local property, it is possible to assume that the operator T is the identity, and $x + \varepsilon e_d \in \Omega$ for each $0 < \varepsilon < r$, where e_d is the unit vector in the direction of the last coordinate axis.

The following theorem is a restatement of Theorem 1 in the context of a Lipschitz domain. We can readily verify that it is also its generalization, which is why we can focus on proving this statement.

Theorem 3. *For each n, there exists $\tau_n > 0$ such that the following statement holds. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with local Lipschitz constant τ_n and let u_λ satisfy $\Delta u_\lambda + \lambda u_\lambda = 0$ and $u_\lambda|_{\partial\Omega} = 0$. Then,*

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \leq C(\Omega)\sqrt{\lambda}.$$

10.2 Useful estimates

Let us state a few estimates that give us the control of the numerical values that are a part of the proof of Theorem 3.

10.2.1 Doubling index and its (almost) monotonicity

A doubling index $N_h(x,r)$, of a non-zero harmonic function $h \in C(\overline{\Omega})$ in $B(x,r)$, $x \in \overline{\Omega}, r > 0$ is defined as

$$N_h(x,r) = \log \frac{\int_{B(x,2r)\cap\Omega} h^2(y)dy}{\int_{B(x,r)\cap\Omega} h^2(y)dy}.$$

When $B(x,r) \subset B(x,R) \subset \Omega$, one can show the monotonicity property

$$N_h(x,r) \leq N_h(x,R). \tag{2}$$

We also have the almost monotonicity property in the Lipschitz domain.

Lemma 4. *For any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that if $\tau < \tau_\varepsilon$, $\partial\Omega \cap B(x,R) \in Lip(\tau)$, $x \in \partial\Omega$, and $h \in C(\overline{\Omega})$ is a non-zero harmonic function in Ω that vanishes on $\partial\Omega \cap B(x,R)$, then, for any $x_0 \in \overline{\Omega} \cap B(x, \frac{R}{4})$ and $r < \frac{R}{16}$,*

$$N_h(x_0,r) \leq (1+\varepsilon)N_h(x_0,2r).$$

10.2.2 Three ball inequalities

When $B(x,4r) \subset \Omega$, the inequality (2) implies

$$\sup_{B(x,3r/2)} |h| \leq 2^d (\sup_{B(x,r)} |h|)^{1/2} (\sup_{B(x,4r)} |h|)^{1/2}.$$

In case of the τ-Lipschitz boundary, the similar inequality follows by the application of Lemma 4.

Theorem 5. *Let B be a ball centered on $\partial\Omega$ such that $\partial\Omega \cap B \in Lip(\tau)$, where τ is small enough. Then for any function $h \in C(\overline{\Omega})$ harmonic in Ω and vanishing on $\partial\Omega \cap B$, we have*

$$\sup_{\frac{3}{2}B_0\cap\Omega} |h| \leq 3^d (\sup_{B_0\cap\Omega} |h|)^{1/3} (\sup_{4B_0\cap\Omega} |h|)^{2/3}$$

for any ball B_0 with the center in $\overline{\Omega} \cap \frac{1}{4}B$ and such that $16B_0 \subset B$.

10.2.3 Estimates of the Hausdorff measure of a restricted zero set

The desired inequality from Theorem 3 will follow from a somewhat different estimate, by observing the zero set localized on the ball and by the upper bound expressed with the doubling index.

Theorem 6. *Let $\Omega \subset \mathbb{R}^d$, $x \in \partial\Omega$ and $r > 0$ be such that $\partial\Omega \cap B(x, 128r) \in Lip(\tau)$, where τ is small enough. Then there exists C such that*

$$\mathcal{H}^{d-1}(Z(h) \cap B(x, r)) \leq C(N_h(x, 4r) + 1)r^{d+1}. \tag{3}$$

for any non-zero function $h \in C(\overline{\Omega})$ that is harmonic in Ω and vanishes on $\partial\Omega \cap B(x, 128r)$.

The same inequality has been obtained by Donnelly and Fefferman in [DF] with the only assumption on Ω being $\overline{B}(x, 8r) \subset \Omega$ and without the requirement on the value of h on $\partial\Omega$.

Interestingly, the inequality (3) is obtained by introducing the maximal doubling index of h in the closed cube Q. In particular, if $\partial\Omega \cap B \in Lip(\tau)$, B is a ball centered on $\partial\Omega$, $h \in C(\overline{\Omega})$ is a function harmonic in Ω and equals zero on $\partial\Omega \cap B$, for a closed cube $Q \subset \frac{1}{32}B$, $Q \cap \Omega \neq \emptyset$ we define

$$N_h^*(Q) = \sup_{\substack{x \in Q \cap \overline{\Omega}, \\ \frac{\operatorname{diam}(Q)}{2} \leq r \leq \operatorname{diam}(Q)}} N_h(x, r).$$

For a particularly chosen number N, we can obtain an estimate for the Hausdorff measure of a zero set restricted to the cube,

$$\mathcal{H}^{d-1}(Z(h) \cap Q) \leq C \max\{N_h^*(Q), N\} s(Q)^{d-1}, \tag{4}$$

where $s(Q)$ is a side length of the cube Q.

10.3 A sketch proof of Theorem 3

The proofs of Theorems 6 and 3 involve introducing the construction of additional objects in the following way.

10.3.1 Boundary and inner cubes

Let $k \geq 3$, let B be a ball centered in ∂Q and let $Q \subset B$ be a cube with sides parallel to the coordinate axes. Let us denote with π a projection onto the hyperplane $\mathbb{R}^{d-1} \times \{0\}$, and with $\{q_1, \dots, q_{k(d-1)}\}$ a family of cubes of side length $2^{-k}s(Q)$ such that they partition $\pi(Q)$ (we allow the nontrivial intersection of their boundaries). A triple

$$(Q, \mathcal{B}_k(Q), \mathcal{I}_k(Q))$$

is called a standard construction and defined as follows. Each $\pi^{-1}(q_i)$ is a superset of a unique cube of a side length of $2^{-k}s(Q)$ and a center in $\partial\Omega \cap Q$; these cubes form a family $\mathcal{B}_k(Q)$ of boundary cubes. Also, each set $(\pi^{-1}(q_i) \cap \Omega \cap Q) \backslash (\cup_{Q' \in \mathcal{B}_k(Q)} Q')$ can be covered with at most 2^k cubes with the same side length $2^{-k}s(Q)$ and such that those cubes are also subsets of the same set. The union of all of the covers is a family of inner cubes $\mathcal{I}_k(Q)$.

This construction helps us prove Theorem 6. Once we cover the ball $B(x, r)$ from (3) with cubes inside $B(x, 2r)$, we apply (4) for the boundary cubes, and Lemma 4 with a variant of Theorem 6 already proven in [DF] for the inner cubes. The maximal doubling index for a boundary cube that arose from (4) is controlled with the hyperplane lemmas. Depending if the index is large enough, we can obtain a boundary cube of a smaller maximal doubling index, or such that it does not intersect the zero set of h.

10.3.2 The harmonic extension of u_λ

We will turn our attention to the extended function of u_λ, defined as

$$h(x,t) = u_\lambda(x)e^{\sqrt{\lambda}t}.$$

Clearly, h is a harmonic function with a domain $\Omega \times \mathbb{R}$, and its zero set is $Z(h) = Z(u_\lambda) \times \mathbb{R}$. One of the reasons of studying h as the next step is that, while estimating its doubling index in the ball, we can obtain the multiplication factor $\sqrt{\lambda}$ required in the estimate from Theorem 3.

The goal is to obtain a cover of a bounded set $\Omega \times [-1, 1] \subseteq \mathbb{R}^{n+1}$ with open balls for which we can apply Theorem 6. We choose the sufficiently small balls B_j, $1 \leq j \leq J$, centered in $\partial\Omega \times [-1, 1]$ and that cover the closed neighbourhood of the boundary $\partial\Omega \times [-1, 1]$. Also, we select small enough balls B'_k, $1 \leq k \leq K$, that cover $(\Omega \times [-1, 1]) \backslash (\cup_{j=1}^{J} B_j)$. The boundary balls

B_j, just as the inner balls B'_k, have the same radii, respectively, r_1 and r_2, that depend only on the radius from the Definition 2. Similarly, J and K depend on the same radius, as well as Ω and the dimension n.

For each $B(x,r) \in \{B_j, B'_k : 1 \le j \le J, 1 \le k \le k\}$ we can show

$$N_h(x, 4r) \le C_0(r, \Omega)\sqrt{\lambda}.$$

This inequality and the estimate (3) give us

$$\mathcal{H}^n(Z(h) \cap (\Omega \times [-1,1])) \le \sum_{j=1}^{J} \mathcal{H}^n(Z(h) \cap B_j) + \sum_{k=1}^{K} \mathcal{H}^n(Z(h) \cap B'_k)$$

$$\le C(C_0\sqrt{\lambda} + 1)(Jr_1^n + Kr_2^n) \le C_1\sqrt{\lambda}.$$

This gives

$$\mathcal{H}^{n-1}(Z(u_\lambda) \cap \Omega_0) = \mathcal{H}^n(Z(h) \cap (\Omega \times [-1,1])) \le C_1\sqrt{\lambda},$$

which is the inequality required in Theorem 3.

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11 On the continuity behaviour of the variational eigenvalues of the p-Laplace operator

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A summary written by Sebastian Gietl

Abstract

We study the continuity behaviour of the variational eigenvalues of the p-Laplace operator for varying p. We gonna see that that for an arbitrary bounded domain we just have continuity from the right and in the end will charaterize the continuity from the left.

11.1 Introduction

Throughout this summary let Ω be a bounded, open and connected subset of $\mathbb{R}^n$ and $1 < p < \infty$. We are interested in the (non linear eigenvalue) problem

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

We define

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{2}$$

as the minimum of the *Rayleigh quotient*. When we interpret problem 1 in the weak sense with $\lambda = \lambda_p$ it is equivalent to the minimization problem given by 2. For $p = 2$ the value λ_p is the smallest possible eigenvalue of the Laplace operator. For arbitrary p this leads to following generalized notion of eigenvalues.

Definition 1. *We say $\lambda \in \mathbb{R}$ is an eigenvalue if there exists a continuous $u \in W_0^{1,p}(\Omega)$ with $u \neq 0$ s.t.*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx = \lambda \int_\Omega |u|^{p-2}uv\, dx \qquad \forall v \in W_0^{1,p}(\Omega)\,.$$

The function u is called an eigenfunction.

The eigenfunctions corresponding to λ_p are essentially unique (i.e. they are constant multiples of each other). We will denote by u_p the unique eigenfunction corresponding to λ_p which is positive and normalized $||u_p||_{L^p} = 1$.

11.2 Limit from the right

In order to prove the continuity from the right we need the following proposition.

Proposition 2. *We have* $p\lambda_p^{\frac{1}{p}} \leq s\lambda_s^{\frac{1}{s}}$*, when* $1 < p < s < \infty$.

Proof. Choose any $\psi \in C_c^\infty(\Omega)$ and define $\phi = \psi^{\frac{s}{p}}$. Then plugging in ϕ in the Rayleight quotient together with the Hölder inequality gives us

$$\begin{aligned} \lambda_p^{\frac{1}{p}} &\leq \frac{(\int_\Omega |\nabla\phi|^p dx)^{\frac{1}{p}}}{(\int_\Omega |\phi|^p dx)^{\frac{1}{p}}} = \frac{s}{p}\frac{(\int_\Omega |\psi|^{s-p}|\nabla\psi|^p dx)^{\frac{1}{p}}}{(\int_\Omega |\psi|^s dx)^{\frac{1}{p}}} \\ &\leq \frac{s}{p}\frac{((\int_\Omega |\psi|^s dx)^{1-\frac{p}{s}}(\int_\Omega |\nabla\psi|^s dx)^{\frac{p}{s}})^{\frac{1}{p}}}{(\int_\Omega |\psi|^s dx)^{\frac{1}{p}}} = \frac{s}{p}\frac{(\int_\Omega |\nabla\psi|^s dx)^{\frac{1}{s}}}{(\int_\Omega |\psi|^s dx)^{\frac{1}{s}}}. \end{aligned}$$

Taking the infimum over all admissible ψ yields the claim. □

Now we can proof the continuity from the right.

Theorem 3. *We have* $\lim_{s\to p^-} \lambda_s \leq \lambda_p = \lim_{s\to p^+} \lambda_s$.

Proof. By Proposition 2 the quantity $s\lambda_s^{\frac{1}{s}}$ is a monoton function in s, so it has one-sided limits, therefore also the one-sided limits of λ_s exist. So we have $\lim_{s\to p^-} \lambda_s \leq \lambda_p \leq \lim_{s\to p^+} \lambda_s$. Now consider $\phi \in C_c^\infty(\Omega)$, then we have

$$\lim_{s\to p^+} \lambda_s \leq \lim_{s\to p^+} \frac{\int_\Omega |\nabla\phi|^s dx}{\int_\Omega |\phi|^s dx} = \frac{\int_\Omega |\nabla\phi|^p dx}{\int_\Omega |\phi|^p dx}.$$

Taking the infimum over all admissible ϕ gives us $\lim_{s\to p^+} \lambda_s \leq \lambda_p$ and therefore also $\lim_{s\to p^+} \lambda_s = \lambda_p$ □

11.3 A domain with $\lim_{s\to p^-} \lambda_s < \lambda_p$

In order to find a domain with $\lim_{s\to p^-} \lambda_s < \lambda_p$ we need a little bit of preparation. For any open set $A \subseteq \mathbb{R}^n$ and any compact set $K \subseteq A$ we define

$$cap_p(K, A) = \inf_\phi \int_A |\nabla\phi|^p dx,$$

where the infimum is taken over all $\phi \in C_c^\infty(\Omega)$ s.t $\phi \geq 1$ in K, and call it the $p-$ *capacity* of (K, A).

Now let $x_0 \in \partial\Omega$, we consider the auxiliary quantity

$$\gamma_p(x_0, r) = \frac{cap_p(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))}{cap_p(\overline{B_r(x_0)}, B_{2r}(x_0))}$$

and define the Wiener integral $W_p(x_0) = \int_0^1 \gamma_p(x_0, r)^{\frac{1}{p-1}} r^{-1} dr$. The point x_0 is called *p-regular* when $W_p(x_0) = \infty$ and *p-irregular* otherwise. We need the following two results. The Wiener criterion tells us that $\lim_{x \to x_0} u_p(x) = 0$ and the Kellog property gives us that the set of all irregular boundary points has zero p-capacity.

Lemma 4. *Suppose that $1 < p \leq n$. Then there is a compact set $F_p \subseteq [0,1)^n$ s.t. $cap_p F_p > 0$ and $cap_s F_p = 0$, when $s < p$. Moreover F_p can be constructed as a Cantor set.*

Now let $Q = (-1,1)^n$ and consider $\Omega = Q \setminus F_p$, therefore F_p lies in the boundary of Ω. We want to show that for the domain Ω it holds that $\lim_{s \to p^-} \lambda_s < \lambda_p$.

To this end we will show that for $s < p$ we have $\lambda_s^Q = \lambda_s^\Omega$ and $\lambda_p^Q < \lambda_p^\Omega$, then because Q is regular we get $\lim_{s \to p^-} \lambda_s^\Omega = \lim_{s \to p^-} \lambda_s^Q < \lambda_p^\Omega$.

Now we want to see that $\lambda_s^Q = \lambda_s^\Omega$. We always have that $\lambda_s^Q \leq \lambda_s^\Omega$. We gonna show that because $cap_s F_p = 0$ it holds that $u_s^Q \in W_0^{1,s}(\Omega)$, which means that u_s^Q is a admissable function for the Rayleigh quotient so we have $\lambda_s^\Omega \leq \lambda_s^Q$.

Because $cap_s F_p = 0$ it follows that for any $\epsilon > 0$ there is $\phi_\epsilon \in C_c^\infty(Q)$ s.t. $0 \leq \phi_\epsilon \leq 1$, the function $\phi_\epsilon = 1$ in an open neighborhood of F_p and $||\nabla \phi_\epsilon||_s < \epsilon$. From this we derive that $(1 - \phi_\epsilon) u_s^Q \in W_0^{1,s}(\Omega)$ and

$$||u_s^Q - (1 - \phi_\epsilon) u_s^Q||_{W_0^{1,s}(\Omega)} = ||\phi_\epsilon u_s^Q||_{W_0^{1,s}(\Omega)} \to 0, \; \epsilon \to 0.$$

Therefore u_s^Q lies in $W_0^{1,s}(\Omega)$.

It remains to show that $\lambda_p^Q < \lambda_p^\Omega$. By the Kellog property the p-capacity of those points in $\partial\Omega$ which are p-irregular is zero, but by construction the p-capacity of $F_p \subseteq \partial\Omega$ is bigger than zero. So there has to be a point $x_0 \in F_p$ which is regular, by the Wiener criterion this implies $\lim_{x \to x_0} u_p^\Omega(x) = 0$. Therfore we can modify u_p^Ω (in a neighborhood of x_0) s.t the modified function lies in $W_0^{1,p}$ and its L^p-norm gets strictly bigger than that of u_p^Ω which implies, by using the Rayleigh quotient, that $\lambda_p^Q < \lambda_p^\Omega$.

11.4 A characterization for continuity from the left

We want to prove the following theorem.

Theorem 5. *We have,* $\lim_{s \to p^-} \lambda_s = \lambda_p$ *if and only if*

$$\lim_{s \to p^-} \int_\Omega |\nabla u_s - \nabla u_p|^s \, dx = 0. \tag{3}$$

The necessity of 3 for the continuity from the left was proven in [PL], however for the other direction it was just proven that for each increasing sequence (p_k) of real numbers converging to p there is a subsequence and a function u in

$$W_0^{1,p-}(\Omega) = W^{1,p}(\Omega) \cap \left(\cap_{1<s<p} W_0^{1,s}(\Omega) \right).$$

s.t. the integral in 3 converges. Now the trick in [DM] is to show that also for this larger space similar things as for the space $W_0^{1,p}$ hold. Which enable us to show that all those limits must coincide and $u = u_p$.

We have the following facts. The space $W_0^{1,p-}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \subseteq W_0^{1,p-}(\Omega)$ and $(\int_\Omega |\nabla u|^p \, dx)^{1/p}$ is a norm on $W_0^{1,p-}(\Omega)$ equivalent to the one induced by $W^{1,p}(\Omega)$. When we define

$$\underline{\lambda}_p = \inf_{u \in W_0^{1,p-}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

then there exist a unique $\underline{u}_p \in W_0^{1,p-}(\Omega)$ s.t.

$$\underline{u}_p \geq 0 \text{ a.e. in } \Omega \,, \qquad \int_\Omega \underline{u}_p^p \, dx = 1 \,, \qquad \int_\Omega |\nabla \underline{u}_p|^p \, dx = \underline{\lambda}_p. \tag{4}$$

Similary to Proposition 2 we have that for $1 < s < p < \infty$ it holds that

$$s \underline{\lambda}_s^{1/s} \leq s \lambda_s^{1/s} < p \underline{\lambda}_p^{1/p} \leq p \lambda_p^{1/p}. \tag{5}$$

With those facts we are ready to prove theorem 5

Proof. Now assume $\lim_{s \to p^-} \lambda_s = \lambda_p$. Let (p_k) be a sequence strictly increasing to p and let $1 < t < p$. We can assume that (u_{p_k}) is weakly convergent to some $u \in W_0^{1,t}(\Omega)$, because up to a subsequence $p_k > t$ and u_{p_k} is uniformly bounded in $W_0^{1,t}(\Omega)$. It follows $u \in \bigcap_{1<s<p} W_0^{1,s}(\Omega)$. Moreover, it holds $u \geq 0$ a.e. in $\Omega\,, \int_\Omega u^p\, dx = 1$ and for every $s < p$,

$$\int_\Omega |\nabla u|^s\, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_{p_k}|^s\, dx \leq \mathcal{L}^n(\Omega)^{1-\frac{s}{p}} \left(\lim_{k\to\infty} \lambda_{p_k}\right)^{\frac{s}{p}} .$$

By the arbitrariness of s, we infer that $u \in W^{1,p}(\Omega)$, hence $u \in W_0^{1,p-}(\Omega)$, with $\underline{\lambda}_p \leq \int_\Omega |\nabla u|^p\, dx \leq \lim_{k\to\infty} \lambda_{p_k}$. It follows, since by the inequalities in 5 it is clear that $\lim_{s\to p^-} \lambda_s \leq \underline{\lambda}_p$, that

$$\int_\Omega |\nabla u_p|^p dx = \lambda_p = \lim_{s\to p^-} \lambda_s = \underline{\lambda}_p = \int_\Omega |\nabla u|^p\, dx\,.$$

Because the properties 4 determine the function uniquely we have $u = u_p$. Therefore $\lim_{s\to p^-} u_s = u_p$ weakly in $W_0^{1,t}(\Omega)$ for any $t < p$. In particular, it holds $\lim_{s\to p^-} \int_\Omega |\frac{u_s+u_p}{2}|^s\, dx = 1\,$, whence $\liminf_{s\to p^-} \int_\Omega |\frac{\nabla u_s+\nabla u_p}{2}|^s\, dx \geq \lambda_p\,$. Using Clarkson's inequality we can derive

$$\lim_{s\to p^-} \int_\Omega |\nabla u_s - \nabla u_p|^s\ dx = 0.$$

When we assume 3, by using the Hölder inequality, it follows for any $t < p$

$$\int_\Omega |\nabla u_p|^t dx = \lim_{s\to p^-} \int_\Omega \nabla |u_s|^t dx \leq \mathcal{L}^n(\Omega)^{1-\frac{t}{p}} \left(\lim_{s\to p^-} \lambda_s\right)^{\frac{t}{p}} .$$

Now taking the limit $t \to p$ gives us $\lambda_p \leq \lim_{s\to p^-} \lambda_s$, which because of theorem 3 implies the continuity from the left. □

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12 A nodal domain property for the p-Laplacian

After M. Cuesta, D.G. De Figueiredo and J.-P. Gossez [CFG00]

A summary written by Lars Niedorf

Abstract

We show that any eigenfunction $u \in W_0^{1,p}(\Omega)$ associated with the second eigenvalue of the p-Laplacian on some bounded domain $\Omega \subseteq \mathbb{R}^N$ admits exactly two nodal domains.

12.1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Courant's nodal domain theorem [CH, p. 451] states that the zero set of any eigenfunction $u \in H_0^1(\Omega)$ associated with the k-th eigenvalue (counted with multiplicity) of the Laplacian $-\Delta$ divides the domain Ω in at most k connected components, the nodal domains. For the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Anane and Tsouli [AT, Cor. 1] proved that any eigenfunction $u \in H_0^1(\Omega)$ associated with an eigenvalue λ smaller than the k-th eigenvalue of Δ_p, the number of associated nodal domains is smaller than k.

For the second eigenvalue λ_2 of the Laplacian, Courant's theorem implies that any eigenfunction u admits exactly two nodal domains. In [CFG00], Cuesta, De Figueiredo and Gossez proved an extension of that result for eigenfunctions of the p-Laplacian. In fact, they proved a more general result for points on a certain curve $\mathcal{C}$ along the Fučik spectrum $\Sigma_p \subseteq \mathbb{R}^2$ of the p-Laplacian, which they constructed in a previous work [CFG99].

12.2 Preliminaries

Let $1 < p < \infty$. The *Fučik spectrum* $\Sigma_p \subseteq \mathbb{R}^2$ of the p-Laplacian on $W_0^{1,p}(\Omega)$ is the set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta_p u = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

has a nontrivial solution $u \in W^{1,p}(\Omega)$. Then $\lambda \geq 0$ is an eigenvalue of the p-Laplacian if and only if $(\lambda, \lambda) \in \Sigma_p$. Integrating the first equation of (1) against u shows that the first eigenvalue λ_1 of the p-Laplacian is given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}. \tag{2}$$

The eigenvalue λ_1 is simple, and admits an eigenfunction $\varphi_1 \in W_0^{1,p} \cap C^1(\Omega)$ with $\varphi_1 > 0$ on Ω and $\int_\Omega \varphi_1^p = 1$. Thus the Fučik spectrum Σ_p clearly contains the two lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$. Those two lines are isolated from the rest of the Fučik spectrum Σ_p [CFG99, Prop. 3.4].

The construction of the curve $\mathcal{C}$ along the Fučik spectrum in [CFG99] is carried out as follows. For $s \geq 0$, we consider the functional $J_s : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_s(u) = \int_\Omega |\nabla u|^p - s \int_\Omega (u^+)^p, \quad u \in W_0^{1,p}(\Omega),$$

and its restriction $\widetilde{J}_s$ to the C^1 Banach manifold

$$S = \Big\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p = 1 \Big\}.$$

In the sense of the Lagrange multiplier theorem, we consider critical points of $\widetilde{J}_s$. A function $u \in S$ is called a *critical point* of $\widetilde{J}_s$ if the (Fréchet) derivatives satisfy $J_s'(u) = tI'(u)$ for some $t \in \mathbb{R}$, i.e.,

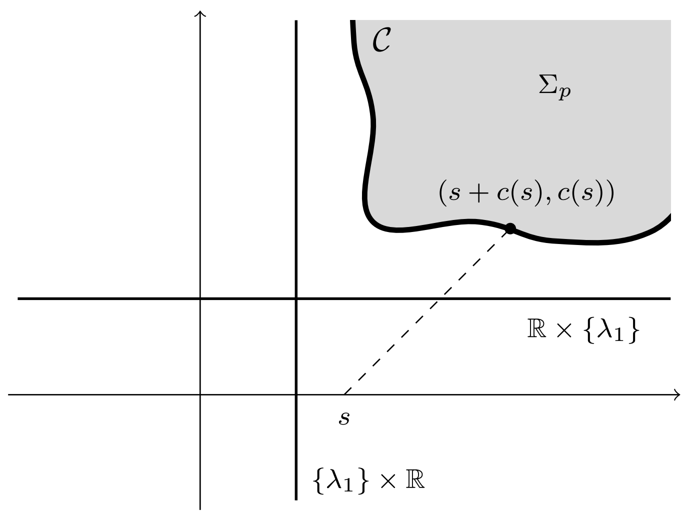
$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v - s \int_\Omega (u^+)^{p-1} v = t \int_\Omega |u|^{p-2} uv \tag{3}$$

for all $v \in W_0^{1,p}(\Omega)$. This is equivalent to (1) for $\alpha = s + t$ and $\beta = t$, i.e., $(s+t, t) \in \Sigma_p$. Taking $v = u$ in (3) yields that necessarily $t = \widetilde{J}_s(u)$ whenever u is a critical point. As an upshot, points in Σ_p on the parallel to the diagonal passing through $(s, 0)$ are exactly those of the form $(s + \widetilde{J}_s(u), \widetilde{J}_s(u))$ with u being a critical point of $\widetilde{J}_s(u)$ (cf. Lemma 2.1 of [CFG99]).

To construct the curve $\mathcal{C}$, the idea of [CFG99] is to find critical values of $\widetilde{J}_s$ by a mountain pass type theorem [CFG99, Prop. 2.5], which asserts that a critical value of $\widetilde{J}_s$ is given by the real number

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} \widetilde{J}_s(u) > \lambda_1,$$

where Γ is the set of all continuous paths in S going from $-\varphi_1$ to φ_1, with φ_1 being the eigenfunction associated with λ_1 from above. Varying $s \geq 0$ yields a curve $s \mapsto (s + c(s), c(s))$. The curve $\mathcal{C}$ is given by all those points $(s + c(s), c(s))$, together with their symmetric counterparts $(c(s), s + c(s))$ with respect to the diagonal, which then of course also lie in Σ_p.



12.3 Results

The main results of [CFG00] are the following.

Theorem 1. *Let $u \in W^{1,p}(\Omega)$ be a nontrivial solution of (1) with $(\alpha, \beta) \in \mathcal{C}$. Then u admits exactly two nodal domains.*

Corollary 2. *Any eigenfunction $u \in W_0^{1,p}(\Omega)$ associated with the second eigenvalue λ_2 of the p-Laplacian admits exactly two nodal domains.*

Since $\mathcal{C}$ passes in particular through the point (λ_2, λ_2), cf. [CFG99, Thm 3.1], Corollary 2 is an immediate consequence of Theorem 1.

12.4 Structure of the proof

The proof works by contradiction. Suppose that Ω admits at least three nodal domains $\Omega_1, \Omega_2, \Omega_3$ such that $u > 0$ on $\Omega_1 \cup \Omega_3$ and $u < 0$ on Ω_2. We can assume $\alpha \geq \beta$ without loss of generality by passing from u to $-u$ if necessary. Let $s = \alpha - \beta$. In view of the mountain pass type theorem in [CFG99, Prop. 2.5], we seek to construct a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma} \widetilde{J}_s(u) < \beta = c(s). \tag{4}$$

This would lead to a contradiction by the definition of $c(s)$. The construction of that path works in two steps.

Step 1

Using the fact that the decomposition of Ω into nodal domains arises from the zero set of a C^1 function, the implicit function theorem provides a connected open subset

$$\Omega_2 \subsetneq \widetilde{\Omega}_2 \subseteq \Omega$$

which is disjoint of Ω_1 or Ω_3 [CFG00, Claim 3.1]. Since $\Omega_2 \subsetneq \widetilde{\Omega}_2$, we have $\lambda_1(\widetilde{\Omega}_2) < \lambda_1(\Omega_2)$ [CFG99, Lem. 5.7], where $\lambda_1(\mathcal{O})$ denotes the first eigenvalue associated with the p-Laplacian on a bounded domain $\mathcal{O}$. Hence, since Ω_1 and Ω_2 are nodal domains, we have

$$\lambda_1(\Omega_1) = \alpha \quad \text{and} \quad \lambda_1(\widetilde{\Omega}_2) < \lambda_1(\Omega_2) = \beta$$

Then we pass to two new open disjoint subsets $\widetilde{\Omega}_1$ and $\widetilde{\widetilde{\Omega}}_2$ by decreasing $\widetilde{\Omega}_2$ and increasing Ω_1 a little bit so that

$$\lambda_1(\widetilde{\Omega}_1) < \alpha \quad \text{and} \quad \lambda_1(\widetilde{\widetilde{\Omega}}_2) < \beta.$$

Let v_1 and v_2 be the positive eigenfunctions associated with those eigenvalues. Put $v = v_1 - v_2$. Then, by (2),

$$\frac{\int_\Omega |\nabla v^+|^p}{\int_\Omega |v^+|^p} < \alpha \quad \text{and} \quad \frac{\int_\Omega |\nabla v^-|^p}{\int_\Omega |v^-|^p} < \beta.$$

Step 2

In the second step, we use the function v to construct the desired path γ from $-\varphi_1$ to φ_1 satisfying (4). The path γ is given by

$$-\varphi_1 \overset{-\nu}{\rightsquigarrow} \frac{-v^-}{\|v^-\|_p} \overset{\gamma_3}{\rightsquigarrow} \frac{v}{\|v\|_p} \overset{\gamma_1}{\rightsquigarrow} \frac{v^+}{\|v^+\|_p} \overset{\gamma_2}{\rightsquigarrow} \frac{v^-}{\|v^-\|_p} \overset{\nu}{\rightsquigarrow} \varphi_1. \tag{5}$$

The paths $\gamma_1, \gamma_2, \gamma_3$ are given by a convex combination of the respective endpoints. For instance,

$$\gamma_1(t) = \frac{tv + (1-t)v^+}{\|tv + (1-t)v^+\|_p}, \quad t \in [0,1].$$

Then the functional $\widetilde{J}_s$ evaluated on each of the paths $\gamma_1, \gamma_2, \gamma_3$ is $< \beta$, with being $< \beta - s$ at the endpoint $v^-/\|v^-\|$. On the other hand, by Lemma 3.6 of [CFG99], any connected component of the sublevel set

$$\mathcal{O} = \{u \in S : \widetilde{J}_s(u) < \beta - s\}$$

contains a critical point, which can only be $\pm\varphi_1$, since by Theorem 3.1 of [CFG99], the point $(s+c(s), c(s))$ is the first non-trivial point on the parallel to the diagonal through $(s, 0)$ (see the picture). Using the fact that any connected component of $\mathcal{O}$ is pathwise connected [CFG99, Lem. 3.5] yields a path ν from $v^-/\|v^-\|_p$ to $-\varphi_1$ or φ_1 (say φ_1 without loss of generality). In particular, the functional $\widetilde{J}_s$ evaluated on the path ν is $< \beta - s$. Since $|\widetilde{J}_s(u) - \widetilde{J}_s(-u)| \leq s$ for all $u \in S$,

$$\widetilde{J}_s(-u) \leq J_s(u) + s < \beta - s + s = \beta.$$

Hence $\widetilde{J}_s$ stays $< \beta$ along the path $-\nu$ from $-\varphi_1$ to $-v^-/\|v^-\|_p$. As an upshot, the functional $\widetilde{J}_s$ along the path γ in (5) stays at level $< \beta$, which is (4), and finishes the proof.

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13 Nodal Domains via Diffusion Processes

After Georgiev and Muckherjee [2]

A summary written by Jacob Denson

Abstract

We describe the theory of Itô diffusions on compact manifolds, and it's application by Georgiev and Muckherjee to the study of the geometry of nodal sets to the Laplace-Beltrami operator on manifolds.

Let M^d be a compact Riemannian manifold, and let $e_\lambda \in C^\infty(M)$ be an eigenfunction for the Laplacian, such that $\Delta e_\lambda = -\lambda^2 e_\lambda$. Our goal is to use the theory of stochastic diffusions to study asymptotic properties of *nodal domains* D_λ, open connected components of $\{x \in M : e_\lambda(x) \neq 0\}$, as $\lambda \to \infty$. In particular, we show it is impossible to fit D_λ in a $O(\lambda^{-1})$ tubular neighborhood of a 'flat' embedded surface $\Sigma_\lambda \subset M$. Steinerberger [6] gave the first version of this result using diffusion. Here we discuss a more robust result, using similar techniques, due to Georgiev and Muckherjee [2].

Theorem 1. *Let M be a Riemannian manifold. Then there exists $c > 0$ such that for any $\lambda > 0$, if Σ_λ is a smooth surface in M of dimension k such that for any $x \in M$ with $d(x, \Sigma_\lambda) \leq \lambda^{-1}$, there exists a unique point on Σ_λ closest to x, then no nodal domain D_λ is contained in a $c \cdot \lambda^{-1}$ neighborhood of Σ_λ for all $\lambda > 0$.*

Why should we expect the theory of diffusions to give us information about nodal sets? A major reason is that eigenfunctions to the Laplace-Beltrami operator behave well under the heat equation, i.e. if $e^{t\Delta}$ are the propogators for the heat equation $\partial_t = \Delta$ on M, then $(e^{t\Delta} e_\lambda)(x) = e^{-\lambda^2 t} e_\lambda(x)$. The heat equation mathematically describes the distribution of particles diffusing through a medium in which they are subject to random molecular bombardments. The theory of *diffusions* in probability gives an alternate mathematical model of this situation, so it is reasonable that applying the theory will bring light upon the theory of eigenfunctions. In particular, we will see that it gives us a theory of *exit times*, that give us a way to study the rate of propogation of a diffusion process.

13.1 Probabilistic Tools

Let us begin by introducing the probabilistic machinery required to describe Itô diffusions. We work over a fixed probability space S, and study a *continuous stochastic process* on S, valued in a space M. There are three useful ways to think of such a process. The first is as a Borel-measurable function X from S to $C([0,\infty), M)$, intuitively, a random continuous path on M. The second is as a family of variables $\{X_t : t \in [0,\infty)\}$. The third is as a *law* which describes how random variables in the 'future' depend on random variables in the past, which leads to the study of the conditional operators $\mathbb{E}^x[f(X)]$, defined for $x \in M$ and a random statistic $f(X)$ associated with the process X, which give the average value of the statistic given that we start the process at the state x, i.e. we let $X_0 = x$, and then let the process evolve according to the law defining the process.

The most basic Itô diffusion is *Brownian motion*. A one-dimensional Brownian motion B is a continuous stochastic process on $\mathbb{R}$ such that for any interval $I = [t,s]$, the increments $\Delta_I B = B_s - B_t$ are mean zero, variance $s - t$ Gaussian random variables, and for any almost disjoint family of intervals $\{I_1, \dots, I_n\}$ in $[0,\infty)$, the random variables $\{\Delta_{I_1} B, \dots, \Delta_{I_n} B\}$ are independent. A Brownian motion on $\mathbb{R}^d$ is precisely a continuous process whose coordinates are independent one-dimensional Brownian motions.

By tweaking Brownian motion locally, we end up with a more general *Itô diffusion*. Suppose that for each $x \in \mathbb{R}^d$, we are given a $d \times d$ positive semidefinite symmetric matrix $A(x)$. Then we obtain a continuous process X defined by the law given by the *stochastic differential equation*

$$dX = A(X)dB.$$

The formal definition of this differential equation is quite technical, but for our purposes, the equation means that there a Brownian motion B such that

$$X_{t+\delta} = X_t + A(X_t) \cdot [B_{t+\delta} - B_t] + o(\delta),$$

where the $o(\delta)$ term is a random variable with mean $o(\delta)$, and with L^3 norm $O(\delta)$. As one might expect, one can analogously define Itô diffusions on a compact manifold M given a section A of $\text{Hom}(TM)$, which will satisfy analogous formulae. Thus the diffusion acts like Brownian motion, except that instead of acting radially, it spreads out unevenly from a point x in the directions dictated by the extent of the matrix $A(x)$.

Now we connect diffusions to semielliptic differential operators. For any diffusion X, we can associate such an operator L, known as the *generator* of the diffusion, and this is a one to one correspondence. As an example, Brownian motion on $\mathbb{R}^d$ has $\Delta/2$ as it's generator. This motivates us to *define* Brownian motion on a manifold M as a process generated by $\Delta/2$, i.e. half the Laplace-Beltrami operator. This correspondence becomes useful in several scenarios. First, for any $f \in C^2(M)$, we have

$$\lim_{t \to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} = (Lf)(x).$$

To see how L emerges, use the approximation $X_t = x + A(x)B_t + o(t)$ and expand $f(X_t)$ in a Taylor series about x. On a sample by sample basis, the first order terms in the expansion of $f(X_t)$ will behave badly, on the order of $O(t^{1/2})$, since Brownian motion is non-differentiable everywhere. But thankfully these terms vanish in the expectation since B is highly oscillatory. The second order terms will involve squares of Brownian motion, and the fact that $\mathbb{E}^0[B_t^2] = t$ implies that these terms will have expected value $O(t)$. Higher order terms will be $O(t^{3/2})$, and thus not affect the value of the limit as $t \to 0$. *Dynkin's formula* follows formally from this calculation by time-homogeneity and the fundamental theorem of calculus, implying that for any time T (possibly a *random time*, provided it is an 'integral stopping time'),

$$\mathbb{E}^x[f(X_T)] = f(x) + \mathbb{E}^x \left[\int_0^T (Lf)(X_s)\, ds \right].$$

To test Dynkin's formula, if B is a Brownian motion on $\mathbb{R}^n$, T is the first time that B exits an open ball of radius R about the origin (an *exit time*, which will always be an integrable stopping time for any bounded open set), and if $f(y) = |y|^2$, then $f(X_T) = R^2$ and $(\Delta/2)f = n$, so

$$R^2 = \mathbb{E}^x[f(X_T)] = f(x) + \mathbb{E}^x \left[\int_0^T n\, ds \right] = n \cdot \mathbb{E}^x[T].$$

Thus we see that on average Brownian motion diffuses a distance $O(R)$ in R^2 units of time. It is interesting to note that if R is sufficiently small, one can perform a similar calculation for Brownian motion on a Riemannian manifold with the exit time of a geodesic ball. If x is fixed, and $f(y) = d(x,y)^2$ is the geodesic distance to x, then one can show (see Section 2.4 of [5]) that

$(\Delta/2)f(x)-n$ is proportional to the *rate of volume expansion* in the manifold at the point x. Thus we expect Brownian motion to diffuse more slowly in positively curved regions (with a negative rate of expansion), and faster in regions of negative curvature (with positive rate of expansion).

The correspondence also applies in the reverse manner, giving the *Feynman-Kac formula* and it's variants; if f is a function on M, and we define $u(x,t) = \mathbb{E}^x[f(X_t)]$, then u is the solution to the partial differential equation $\partial_t u = Lu$ on M, with initial conditions f. We can also consider boundary value problems. If D is a bounded region of $\mathbb{R}^d$, and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the *exit time* for D, then the Dynkin formula tells us that the unique solution to the Dirichlet problem $Lv = -h$ on D with boundary conditions ϕ on ∂D is given by

$$v(x) = \mathbb{E}^x[\phi(X_{\tau_D})] + \mathbb{E}^x\left[\int_0^{\tau_D} h(X_t)\,dt\right].$$

One can also solve the heat equation $\partial_t u = Lu$ with absorbing boundary conditions $u(x,t) = 0$ for $x \in \partial D$, and initial condition $u(x,0) = f(x)$ by setting $u(x,t) = \mathbb{E}^x[f(X_t)\chi_t]$, where $\chi_t = 1$ if $t < \tau_D$, and $\chi_t = 0$ otherwise (we kill paths that reach the boundary and are 'absorbed'). There is also a way to consider solutions to the heat equation with an insulating boundary, i.e. finding a solution to $\partial_t u = \Delta u$ such that $\partial u/\partial \eta = 0$ on ∂D by considering a *reflecting Brownian motion* which 'bounces off the boundary' instead of being killed.

13.2 Nodal Sets Via Brownian Motion

Let us use the theory we have introduced to prove Theorem 1. Let e_λ be an eigenfunction, and D_λ a nodal domain of e_λ. We may assume without loss of generality that e_λ is positive on D_λ. Consider two solutions $p(x,t)$ and $u(x,t)$ to the heat equation on D_λ, with initial conditions $p(x,0) = 0$, $u(x,0) = e_\lambda(x)$, and with boundary conditions $p(x,t) = 1$ and $u(x,t) = 0$ for $x \in \partial D_\lambda$. The Feynman-Kac formula tells us that

$$p(x,t) = 1 - \mathbb{E}^x[\chi_t] = \mathbb{P}^x[t > \tau_\lambda] \quad \text{and} \quad u(x,t) = \mathbb{E}^x[e_\lambda(B_t)\chi_t].$$

where $\chi_t = \mathbb{I}(t \leq \tau_\lambda)$, and $\tau_\lambda = \inf\{t > 0 : B_t \in D_\lambda^c\}$ is the exit time of D_λ. If $x_0 \in D_\lambda$ maximizes e_λ on D_λ, then

$$e^{-\lambda^2 t} e_\lambda(x) = u(x,t) = \mathbb{E}^x[e_\lambda(B_t)\chi_t] \leq e_\lambda(x_0)\mathbb{E}^x[\chi_t] = e_\lambda(x_0)(1 - p(x,t)).$$

In particular, $p(x_0,t) \leq 1 - e^{-\lambda^2 t}$, so $\mathbb{P}^{x_0}[\tau_\lambda \geq t\lambda^{-2}] \geq e^{-t}$. Thus $\mathbb{E}[\tau_\lambda] \gtrsim \lambda^{-2}$, and the heuristic diffusion rate of Brownian motion therefore leads us to believe that x_0 lies roughly $\gtrsim \lambda^{-1}$ from ∂D_λ. If we had $M = \mathbb{R}^d$, this would immediately yield a contradiction if D_λ was contained in a $c \cdot \lambda^{-1}$ neighborhood of a k dimensional plane Σ, because we have a strong quantification of this heuristic; if B' is the projection of Brownian motion onto the $d-k$ dimensional plane normal to Σ, then a result due to Kent [3] implies that for any $\varepsilon > 0$, there is $c > 0$ such that

$$\mathbb{P}^0 \left(\sup_{0 \leq t \leq \lambda^{-2}} |B'_t| \leq c \cdot \lambda^{-1} \right) \leq \varepsilon.$$

Letting c corresponding to $\varepsilon = (1/2)e^{-1}$ gives a contradiction, since then

$$e^{-1} \leq \mathbb{P}^{x_0}[\tau_\lambda \geq \lambda^{-2}] \leq \mathbb{P}^0 \left(\sup_{0 \leq t \leq \lambda^{-2}} |B'_t| \leq c \cdot \lambda^{-1} \right) \leq (1/2)e^{-1}.$$

Extending this to the non-Euclidean setting is not too difficult. Given a general k dimensional hypersurface Σ on a manifold M satisfying the assumptions of the result we are trying to prove, we suppose that D_λ is contained in a $c \cdot \lambda^{-1}$ neighborhood U_λ of Σ_λ. The flatness assumptions imply that we can find a normal coordinate system ϕ on a $2c \cdot \lambda^{-1}$ neighborhood of Σ_λ using geodesics. Then $\phi(\Sigma_\lambda)$ is a k dimensional plane, and $\phi(U_\lambda)$ is a $c \cdot \lambda^{-1}$ neighborhood of this plane. Because the Euclidean and Riemannian metrics are comparable, Brownian motion on M should behave in coordinates analogously to Brownian motion on $\mathbb{R}^d$. The rate of diffusion of both processes leads us to believe the behaviour should be similar up to times $c \cdot \lambda^{-2}$, before which both Brownian motions are highly unlikely to leave the neighborhoods U_λ and $\phi(U_\lambda)$. Thus our proof can be completed by an application of the following 'comparison result' for hitting times, which is Theorem 2.2 of [2].

Theorem 2. *Let M^d be a compact Riemannian manifold, and consider an open geodesic ball $B \subset M$ around a point x_0 with radius r smaller than the injectivity radius of M. Let (U, ϕ) be a chart on M with $B \subset U$, and suppose that the metric of M is comparable to the Euclidean metric in the coordinates ϕ. Fix a compact set $K \subset B$. Let B^1 be a Brownian motion on M, and B^2 be a Brownian motion on $\mathbb{R}^d$. If τ_1 denotes the time that B^1 exits K, and τ_2 denotes the time that B^2 exits $\phi(K)$, then for any $c > 0$, there exists $C > 0$ such that*

$$(1/C) \cdot \mathbb{P}(\tau_2 \leq cr^2) \leq \mathbb{P}(\tau_1 \leq cr^2) \leq C \cdot \mathbb{P}(\tau_2 \leq cr^2).$$

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14 Large time behavior of solutions of Trudinger's equation

After R. Hynd and E. Lindgren [HL]

A summary written by Ilseok Lee

Abstract

We investigate the large time behavior of the solution of Trudinger's equation with Dirichlet boundary condition using monotonicity properties corresponding to Poincare inequality and dual Poincare inequality.

14.1 Introduction

For $p \in (1, \infty)$, let $v : \Omega \times (0, \infty) \to \mathbb{R}$ be a solution of Trudinger's equation with Dirichlet boundary condition

$$\begin{cases} \partial_t(|v|^{p-2}v) = \Delta_p v & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times [0,\infty) \\ v = g & \text{on } \Omega \times \{0\} \end{cases} \tag{1}$$

where $g : \Omega \to \mathbb{R}$ is a given initial value function. The large time behavior of the solution will be investigated by studying its compactness and monotonicity properties and by considering the homogeneity of Trudinger's equation.

The solution of (1) has the following monotonicity property

$$\frac{d}{dt}\Big[\frac{\|Dv(\cdot,t)\|^p_{L^p(\Omega)}}{\|v(\cdot,t)\|^p_{L^p(\Omega)}}\Big] \leq 0. \tag{2}$$

Thus it would be expected that the flow (1) is related to the Poincare inequality

$$\lambda_p \|u\|^p_{L^p(\Omega)} \leq \|Du\|^p_{L^p(\Omega)} \quad \text{for } u \in W^{1,p}_0(\Omega) \tag{3}$$

where λ_p is the largest constant c satisfying $c\|u\|^p_{L^p(\Omega)} \leq \|Du\|^p_{L^p}$ for each $u \in W^{1,p}_0(\Omega)$. Extremal functions satisfy (3) with equality. Also note that a function $u \in W^{1,p}_0(\Omega)$ is extremal for (3) if and only if u satisfies

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

Another monotonicity property of the solution of (1) is

$$\frac{d}{dt}\Big[\frac{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{L^q(\Omega)}}{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{W^{-1,q}(\Omega)}}\Big] \le 0 \tag{5}$$

where $q = \frac{p}{p-1}$ is the Holder conjugate of p. (5) implies that (1) improves how $|v(\cdot,t)|^{p-2}v(\cdot,t)$ satisfies the inequality

$$\mu_p \|f\|^q_{W^{-1,q}(\Omega)} \le \|f\|^q_{L^q(\Omega)} \quad \text{for } f \in L^q(\Omega) \tag{6}$$

as t increases where $\mu_p := \lambda_p^{\frac{1}{p-1}}$. (6) is called the dual Poincare inequality as equality holds if and only if $f = |u|^{p-2}u$ where u is extremal for the Poincare inequality (3).

Definition 1. *Assume $g \in L^p(\Omega)$. A weak solution of* (1) *is a function $v : \Omega \times [0,\infty) \to \mathbb{R}$ that satisfies:*
(i) $v \in L^\infty([0,\infty); L^p(\Omega)) \cap L^p([0,\infty); W^{1,p}_0(\Omega))$;
(ii) $\int_0^\infty \int_\Omega |v|^{p-2}v\psi_t dxdt = \int_0^\infty \int_\Omega |Dv|^{p-2}Dv \cdot D\psi dxdt$ for each $\psi \in C^\infty_c(\Omega \times (0,\infty))$;
(iii) $v(\cdot,0) = g$.

Our main result regarding the large time behavior of the solution of (1) is as follows.

Theorem 2. *(i) Assume v is a weak solution of* (1)*. Then the limit*

$$u := \lim_{t\to\infty} e^{(\frac{\lambda_p}{p-1})t} v(\cdot,t) \tag{7}$$

exists in $L^p(\Omega)$ and u is extremal for (3)*. If $u \not\equiv 0$, then $v(\cdot,t) \not\equiv 0$ for all $t \ge 0$ and $\mu_p = \lim_{t\to\infty} \frac{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{L^q(\Omega)}}{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{W^{-1,q}(\Omega)}}$.*

(ii) There is a weak solution v of (1) *such that the limit* (7) *exists in $W^{1,p}_0(\Omega)$. If $u \not\equiv 0$, $\lambda_p = \lim_{t\to\infty} \frac{\|Dv(\cdot,t)\|^p_{L^p(\Omega)}}{\|v(\cdot,t)\|^p_{L^p(\Omega)}}$.*

14.2 Sketch of proof of the theorem 2

To claim (i) of the theorem 2, we need a lemmas and a proposition.

Proposition 3. *Assume* $\{v^k\}_{k\in\mathbb{N}}$ *is a sequence of weak solutions of* (1) *with* $v^k(\cdot,0)=g^k$ *and* $\sup_{k\in\mathbb{N}}\int_\Omega|g^k|^p dx<\infty$. *There is a subsequence* $\{v^{k_j}\}_{j\in\mathbb{N}}$ *and* v *satisfying (i) of the definition 1 such that*

$$v^{k_j}\to v \quad \text{in } L^p_{loc}([0,\infty);W^{1,p}_0(\Omega)) \quad \text{and} \quad v^{k_j}\to v \quad \text{in } C_{loc}((0,\infty);L^p(\Omega))$$

$$\partial_t(|v^{k_j}|^{p-2}v^{k_j})\to\partial_t(|v|^{p-2}v) \quad \text{in } L^q_{loc}([0,\infty);W^{-1,q}(\Omega))$$

Moreover, v *is a weak solution of* (1) *with* $v(\cdot,0)=g$, *where* $|g|^{p-2}g$ *is a weak limit of* $\{|g^{k_j}|^{p-2}g^{k_j}\}_{j\in\mathbb{N}}$ *in* $L^q(\Omega)$.

Proof. See Proposition 2.8 of [HL]. □

Lemma 4. *Suppose that for* $l,C>0$, *select* $\delta=\delta(l,C)>0$ *such that* v *is a weak solution of* (1) *which satisfies*

(i) $e^{p(\frac{\lambda_p}{p-1})t}\|v(\cdot,t)\|^p_{L^p(\Omega)}\geq l$ *for* $t\geq 0$, *(ii)* $\|v(\cdot,0)\|^p_{L^p(\Omega)}\leq C$,

(iii) $\|v^+(\cdot,0)\|^p_{L^p(\Omega)}\geq\frac{1}{2}l$, *and* *(iv)* $\frac{\||v(\cdot,0)|^{p-2}v(\cdot,0)\|^q_{L^q(\Omega)}}{\||v(\cdot,0)|^{p-2}v(\cdot,0)\|^q_{W^{-1,q}(\Omega)}}<\mu_p+\delta$.

Then $e^{p(\frac{\lambda_p}{p-1})t}\|v^+(\cdot,t)\|^p_{L^p(\Omega)}\geq\frac{1}{2}l$ *for* $t\geq 0$.

Proof. See Corollary 3.3 of [HL]. □

By Corollary 2.3 of [HL], $S:=\lim_{\tau\to\infty}e^{p(\frac{\lambda_p}{p-1})\tau}\|v(\cdot,\tau)\|^p_{L^p(\Omega)}$ exists. It is trivial if $S=0$. Thus suppose that $S>0$. For an increasing sequence $\{s_k\}_{k\in\mathbb{N}}$ of positive numbers converging to ∞, define $v^k(x,t):=e^{(\frac{\lambda_p}{p-1})s_k}v(x,t+s_k)$ for $(x,t)\in\Omega\times[0,\infty)$. Then v^k is a weak solution of (1) for each $k\in\mathbb{N}$. Also $\|v^k(\cdot,0)\|^p_{L^p(\Omega)}=e^{p(\frac{\lambda_p}{p-1})s_k}\|v^k(\cdot,s_k)\|^p_{L^p(\Omega)}\leq\|g\|^p_{L^p(\Omega)}$. Then the proposition 3 implies the existence of a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}}$ and weak solution v^∞ such that

$$v^{k_j}\to v^\infty \quad \text{in } C_{loc}((0,\infty);L^p(\Omega)) \text{ and } L^p_{loc}([0,\infty);W^{1,p}_0(\Omega)) \tag{8}$$

as $j\to\infty$. Furthermore, $v^\infty(\cdot,0)=u$ where $|v^{k_j}(\cdot,0)|^{p-2}v^{k_j}(\cdot,0)\rightharpoonup|u|^{p-2}u$ in $L^p(\Omega)$. This weak convergence implies that

$$S=\lim_{j\to\infty}\int_\Omega||v^{k_j}(x,0)|^{p-2}v^{k_j}(x,0)|^q dx\geq\int_\Omega||u|^{p-2}u|^q dx=\|u\|^p_{L^p(\Omega)} \tag{9}$$

(8) implies that for all $t>0$,

$$S=e^{p(\frac{\lambda_p}{p-1})t}\lim_{j\to\infty}\|v^{k_j}(\cdot,t)\|^p_{L^p(\Omega)}=e^{p(\frac{\lambda_p}{p-1})t}\|v^\infty(\cdot,t)\|^p_{L^p(\Omega)}. \tag{10}$$

Differentiating (10) in time gives

$$0 = (\frac{p}{p-1})e^{p(\frac{\lambda_p}{p-1})t}(\lambda_p \|v^\infty(\cdot,t)\|^p_{L^p(\Omega)} - \|Dv^\infty(\cdot,t)\|^p_{L^p(\Omega)})$$

for almost every $t \geq 0$. As in the proof of Corollary 2.11 of [HL], this actually holds for every $t \geq 0$. Thus $v^\infty(x,t) = e^{-p(\frac{\lambda_p}{p-1})t}u$ for each $t \geq 0$ and u is an extremal for (3) and $S = \|u\|^p_{L^p(\Omega)}$.

As the collection of extremals of the Poincare inequality (3) is one dimensional (see [SS]), it must be either $u > 0$ or $u < 0$ in Ω. Without loss of generality, suppose $u > 0$. Applying the lemma 4 with $l = S > 0$ and $C := \|g\|^p_{L^p(\Omega)}$. From above results, there exists j^* such that v^{k_j} satisfies hypotheses (ii)-(iv) in the lemma 4 for each $j \geq j^*$. For (i), since S is the infimum of $e^{p(\frac{\lambda_p}{p-1})\tau}\|v(\cdot,\tau)\|^p_{L^p(\Omega)}$ over over $\tau > 0$. Thus

$$e^{p(\frac{\lambda_p}{p-1})(t+s_{k_j})}\|v^+(\cdot,t+s_{k_j})\|^p_{L^p(\Omega)} \geq \frac{1}{2}S \tag{11}$$

for every $t \geq 0$ and each $j \geq j^*$.

Now suppose there is another increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ of positive numbers converging to ∞ such that $\lim_{m\to\infty} e^{(\frac{\lambda_p}{p-1})t_m}v(\cdot,t_m) = -u$ in $L^p(\Omega)$. Select a subsequence $\{t_{m_j}\}_{j\in\mathbb{N}}$ such that $t_{m_j} > s_{k_j}$ for all $j \in \mathbb{N}$. Substituting $t = t_{m_j} - s_{k_j}$ in (11) gives $e^{p(\frac{\lambda_p}{p-1})t_{m_j}}\|v^+(\cdot,t_{m_j})\|^p_{L^p(\Omega)} \geq \frac{1}{2}S$. Then $j \to \infty$ gives $\|(-u)^+\|^p_{L^p(\Omega)} \geq \frac{1}{2}S$ which is a contradiction to u being a positive function.

As S is independent of the sequence $\{s_k\}_{k\in\mathbb{N}}$, the limit $\lim_{t\to\infty} e^{(\frac{\lambda_p}{p-1})t}v(\cdot,t) = u$ exists in $L^p(\Omega)$ and

$$\lim_{t\to\infty} \frac{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{L^q(\Omega)}}{\||v(\cdot,t)|^{p-2}v(\cdot,t)\|^q_{W^{-1,q}(\Omega)}} = \frac{\||u|^{p-2}u\|^q_{L^q(\Omega)}}{\||u|^{p-2}u\|^q_{W^{-1,q}(\Omega)}} = \mu_p.$$

To prove (ii) of the theorem 2, assume v is a weak solution of (1) satisfying

$$\|Dv(\cdot,t)\|^p_{L^p(\Omega)} \leq \|Dv(\cdot,s)\|^p_{L^p(\Omega)} \tag{12}$$

for a.e. $t, s \in (0,\infty)$ with $t \geq s$. Let $\{s_k\}_{k\in\mathbb{N}}$ and v^k as above for each $k \in \mathbb{N}$. From the proof of (i) above,

$$\lim_{k\to\infty} v^k(\cdot,t) = e^{-(\frac{\lambda_p}{p-1})t}u \tag{13}$$

exists in $L^p(\Omega)$ for each time $t \geq 0$ where u is an extremal of (3). Applying Corollary 2.11 of [HL] to v^k, $\{\|v^k(\cdot,t)\|_{W_0^{1,p}(\Omega)}\}_{k\in\mathbb{N}}$ is bounded for each $t \geq 0$ which implies that (13) holds weakly in $W_0^{1,p}(\Omega)$ for all $t \geq 0$. Then by (8), (13) holds strongly in $W_0^{1,p}(\Omega)$ for almost every $t \geq 0$ for a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}}$. By (12), $t \mapsto \|Dv^k(\cdot,t)\|^p_{L^p(\Omega)}$ is nonincreasing for each $k \in \mathbb{N}$ in $t \in [0,\infty)$. Applying Helly's theorem (Lemma 3.3.3 in [AGS]), there is a subsequence (again labeled) $\{v^{k_j}\}_{j\in\mathbb{N}}$ such that $h(t) := \lim_{j\to\infty}\|Dv^{k_j}(\cdot,t)\|^p_{L^p(\Omega)}$ holds for every $t \geq 0$. Then (13) implies that $h(t) = \|e^{-(\frac{\lambda_p}{p-1})t}Du\|^p_{L^p(\Omega)}$ for a.e. $t \geq 0$ and $h(t) \geq \|e^{-(\frac{\lambda_p}{p-1})t}Du\|^p_{L^p(\Omega)}$ for all $t \geq 0$.

Repeating the steps of part 4 of Proposition 2.8 in [HL], $\lim_{j\to\infty}\|Dv^{k_j}(\cdot,t)\|^p_{L^p(\Omega)} = \|e^{-(\frac{\lambda_p}{p-1})t}Du\|^p_{L^p(\Omega)}$ for every $t \geq 0$. As (13) holds weakly in $W_0^{1,p}(\Omega)$ at $t = 0$, we have $\lim_{j\to\infty} e^{(\frac{\lambda_p}{p-1})s_{k_j}}v(\cdot,s_{k_j}) = u$ in $W_0^{1,p}(\Omega)$. Since $\{s_{k_j}\}_{j\in\mathbb{N}}$ was any subsequence of arbitrary sequence $\{s_k\}_{k\in\mathbb{N}}$, $\lim_{t\to\infty} e^{(\frac{\lambda_p}{p-1})t}v(\cdot,t) = u$ in $W_0^{1,p}(\Omega)$. Finally, if u does not vanish identically then $v(\cdot,t)$ does not vanish identically for all $t \geq 0$ and

$$\lim_{t\to\infty}\frac{\|Dv(\cdot,t)\|^p_{L^p(\Omega)}}{\|v(\cdot,t)\|^p_{L^p(\Omega)}} = \frac{\|Du\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}} = \lambda_p.$$

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15 Localization of low-lying eigenfunctions of vibrating systems and landscape function

After M. Filoche and S. Mayboroda [FM]

A summary written by Mingyi Hou

Abstract

It is observed that low-lying eigenfunctions of the Schrödinger operator concentrate on small subregions of the original domain. This also happens to a wider range of elliptic operators. We will use a newly defined landscape function to explain this phenomena.

15.1 Introduction

A puzzling feature found by physicists is that for certain vibrating systems standing waves or vibrations are restricted in a small subregion of their original domain even in the absence of confining force or potential. One famous example is the so-called Anderson localization discovered in the '50s. Mathematically it is the localization phenomena shown by low-lying eigenfunctions of the Schrödinger operator. And it has been experimentally demonstrated in optic or electromagnetic systems.

In general, a vibrating system is governed by a wave equation associated to a suitable elliptic operator L. In the sequel, we consider a positive self-adjoint elliptic differential operator L and look at its eigenfunctions

$$L\psi = \lambda\psi, \quad \psi|_{\partial\Omega} = 0. \tag{1}$$

An amazing fact is that this localization phenomenon can be characterized by the *landscape function* which is defined to be the solution to the following Dirichlet problem

$$Lu = 1, \quad u|_{\partial\Omega} = 0. \tag{2}$$

The essential idea is to look for all "valleys" of the landscape and use that to partition the domain. Then the value of the landscape function on the boundary of each subdomain is relatively small. Moreover, eigenfunctions are controlled by this landscape function through the following inequality

$$|\varphi(x)| \leq \lambda u(x), \quad \forall x \in \Omega, \tag{3}$$

here φ is a normalized eigenfunction, i.e. $\|\varphi\|_\infty = 1$, and λ is the corresponding eigenvalue. Hence restricted to a subdomain D, the eigenfunction φ, roughly speaking, satisfies $L\varphi = \lambda\varphi$ on D but with relatively small boundary data on ∂D. We will show that this controls the behavior of φ. If φ is localized in a subdomain D it has to be like an eigenfunction of L on D in the sense λ almost coincides with an eigenvalue of L on D.

15.2 Discussion on valley networks

In this section, we discuss how to find "valleys". For now assume a positive landscape function exists. In the original paper, the authors use streamlines to define valleys, i.e. all flow lines from saddle points to local minimums. However, there are two problems here. The first is we may not have enough smoothness for the landscape function. If only requiring bounded measurable coefficients we can only expect Hölder continuity. To handle the non-smooth case, we may mollify the landscape and simply consider valleys from the mollified landscape. Since our original landscape is continuous the value should stay small in the valley. The Second point is that even if we have smoothness there would be problems with the geometry, e.g. the landscape is not necessarily a Morse function. There may exist degenerate critical points or even flat parts. One possibility is to use the shortest height minimizing path between two local minimums. While other technicalities would arise.

Another way to work around this problem is to use level sets. We can imagine filling water in the landscape. Then mountains will emerge and be separated by water. While water would also cover some fine structures of the landscape. Therefore we should carefully choose a level so that we don't get a trivial partition, e.g. there is only one mountain. As you will see in later sections, different partitions won't affect our two main results.

Besides, inequality (3) implies for an eigenvalue λ only the part where $u(x) \leq 1/\lambda$ is meaningful. Hence we will define *effective valley network* $\mathcal{N}(\lambda) := \{x \in \Omega : u(x) \in \mathcal{N} \text{ and } u(x) \leq 1/\lambda\}$ where $\mathcal{N}$ is the valley network discussed above. The connected components of the complement of $\mathcal{N}(\lambda)$ give us the possible localization regions.

15.3 Constraints from the landscape

In this section we give rigorous proof for the inequality (3) in a more general setting, i.e. including operators with order higher than 2. Assume Ω a bounded open set in $\mathbb{R}^n$ and L an elliptic differential operator associated to a symmetric positive bilinear form B. For example the Laplacian $L = -\Delta$ with $B[u,v] = \int_\Omega \nabla u \nabla v \, dx$ and the Hamiltonian $L = -\Delta + V(x)$, $0 \leq V(x) \leq C$, with $B[u,v] = \int_\Omega \nabla u \nabla v + Vuv \, dx$. In general, L is of order $2m$, $m \in \mathbb{N}$, defined in the weak sense:

$$\int_\Omega Luv := B[u,v], \quad \text{for } u, v \in H_0^m(\Omega), \tag{4}$$

where $H_0^m(\Omega)$ is the Sobolev space of functions given by the completion of $C_c^\infty(\Omega)$ in the norm

$$\|u\|_{H_0^m(\Omega)} := \|\nabla^m u\|_{L^2(\Omega)}.$$

Recall that the Lax-Milgram theorem ensures well-posedness, i.e. for every $f \in (H_0^m(\Omega))^* =: H^{-m}(\Omega)$ the boundary value problem

$$Lu = f, \quad u \in H_0^m(\Omega), \tag{5}$$

has a unique solution in the weak sense.

We also define the Green function of L by

$$L_x G(x,y) = \delta_y(x), \quad \text{where } G(\cdot, y) \in H_0^m(\Omega) \quad \text{for all } y \in \Omega, \tag{6}$$

in the weak sense, so that

$$\int_\Omega L_x G(x,y) v(x) \, dx = v(y), \quad y \in \Omega, \tag{7}$$

for every $v \in H_0^m(\Omega)$. The existence of such a function is again guaranteed by the Lax-Milgram theorem. It's easy to see that for a self-adjoint elliptic operator, or equivalently a symmetric bilinear form, the Green function is symmetric, i.e., $G(x,y) = G(y,x)$.

The Fredholm alternative provides the framework to consider the eigenvalue problem so that equation (2) makes sense. Now we can formulate the theorem

Theorem 1. *Let Ω be a bounded open set, L be a self-adjoint operator on Ω with bounded coercive bilinear form, and assume $\varphi \in H^m(\Omega)$ is an eigenfunction of L and λ is the corresponding eigenvalue. Then*

$$\frac{|\varphi(x)|}{||\varphi||_{L^\infty(\Omega)}} \leq \lambda u(x), \ \textit{for all } x \in \Omega, \tag{8}$$

provided that $\varphi \in L^\infty(\Omega)$, with

$$u(x) = \int_\Omega |G(x,y)|dy, \quad x \in \Omega. \tag{9}$$

If the Green function is non-negative, then u is the solution to the boundary value problem

$$Lu = 1, \quad u \in H_0^m(\Omega). \tag{10}$$

Proof. Just observe

$$\varphi(x) = \int_\Omega \varphi(y) L_y G(x,y)\ dy = \int_\Omega L_y \varphi(y) G(x,y)\ dy = \int_\Omega \lambda\varphi(y) G(x,y)\ dy. \tag{11}$$

□

Remark 2. *In particular, equation (9) is the general definition of a landscape function.*

Remark 3. *The Green function is positive in Ω and eigenfunctions are bounded if the maximum principle holds for L.*

15.4 Localization on subdomains

In this section, we show how an eigenfunction is constrained by relatively small boundary values. Consider a subregion D of the landscape of u defined in section 2. By construction, u is relatively small along ∂D. Since we are working with Sobolev functions we should understand this in the following way: a function $v \in H^m(D)$ satisfies $v \leq 0$ on ∂D if $v^+ := \max\{v, 0\} \in H_0^m$.

An eigenfunction φ of L in Ω with eigenvalue λ satisfies $L\varphi = \lambda\varphi$ in D with the same data as φ on ∂D. The smallness of this boundary data is interpreted in the following way: define the norm of the boundary data by $\varepsilon = \|v\|_{L^2(D)}$, where $v \in H^m(D)$ is such that $\varphi - v \in H_0^m(D)$ and $Lv = 0$ on D. Such v exists by the classical weak existence theory. If a maximum principle holds for L then ε will be bounded by the norm of boundary data given by λu which is relatively small. Furthermore, we can show the following theorem.

Theorem 4. *Let Ω, L, φ, λ be as above. Suppose D is a subset of Ω and denote by ε the norm of the boundary data of φ on ∂D. Then either λ is an eigenvalue of L in D or*

$$\|\varphi\|_{L^2(D)} \leq \left(1 + \frac{\lambda}{\min_{\lambda_k(D)}\{|\lambda - \lambda_k(D)|\}}\right)\varepsilon, \tag{12}$$

where the minimum is taken over all eigenvalues of L in D.

Proof. Suppose λ is not an eigenvalue of D. Let v be as above and define $w := \varphi - v$ then

$$(L - \lambda)w = \lambda v \quad \text{on } D. \tag{13}$$

Claim

$$\|w\|_{L^2(D)} \leq \max_{\lambda_k(D)}\left\{\frac{1}{|\lambda - \lambda_k(D)|}\right\}\|\lambda v\|_{L^2(D)}. \tag{14}$$

In our setting the eigenvalues of L are real, positive, at most countable, and eigenfunctions of L, $\{\psi_{k,D}\}_k$, form an orthonormal basis of $L^2(D)$. Moreover, $\{\psi_{k,D}\}_k$ form an orthogonal basis of $H_0^m(D)$. Also for $f \in H_0^m(D)$ we can write

$$f = \sum_k c_k(f)\psi_k, \quad c_k(f) = \int_D f\psi_k \, dx. \tag{15}$$

This series converges both in $L^2(D)$ and $H_0^m(D)$, and with norm $\|f\|_{L^2(D)} = (\sum_k c_k(f)^2)^{1/2}$. These arguments follow from standard functional analysis theory.

Therefore we have for λ not in the spectrum of L on D

$$\|(L-\lambda)w\|_{L^2(D)} = \left\|\sum_k c_k((L-\lambda)w)\psi_k\right\|_{L^2(D)}, \tag{16}$$

where

$$c_k((L-\lambda)w) = \int_D (L-\lambda)w\psi_k \, dx = \int_D w(L-\lambda)\psi_k \, dx \tag{17}$$

$$= (\lambda_k(D) - \lambda)\int_D w\psi_k \, dx = (\lambda_k - \lambda)c_k(w). \tag{18}$$

Hence

$$\|(L-\lambda)w\|_{L^2(D)} = \left\| \sum_k (\lambda_k(D)-\lambda)c_k(w)\psi_k \right\|_{L^2(D)} = \left(\sum_k (\lambda_k-\lambda)^2 c_k(w)^2 \right)^{1/2}$$
$$\geq \min_{\lambda_k(D)} |\lambda_k(D)-\lambda| \left(\sum_k c_k(w)^2 \right)^{1/2} = \min_{\lambda_k(D)} |\lambda_k(D)-\lambda| \|w\|_{L^2(D)}$$

which proves the claim. Absorbing λ and replacing $\|v\|_{L^2(D)}$ by ε finishes the proof. □

In conclusion, on one hand, if λ is far away from the spectrum of L in D, the L^2 norm of the eigenfunction is smaller than a quantity of order ε. On the other hand, an eigenfunction can only be substantial in the subregion when its eigenvalue almost coincides with one of the local eigenvalues of the operator L in D. Consequently, an eigenfunction with a small eigenvalue can cross the boundary of two adjacent subregions only if they have similar local eigenvalues. Another case of delocalization is that, when λ increases, the valley network will shrink so that subregions can be connected.

It is worth noting that this theorem is only an upper bound so it gives us a restriction on where the localization can happen. Whether it happens or not still depends on the operator itself.

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16 A variation estimate for eigenfunctions of Schrödinger operators

After S. Steinerberger [S]

A summary written by Bruno Poggi

Abstract

A localized variation estimate is shown for the Dirichlet eigenfunctions of Schrödinger operators $L = -\Delta + V$ with non-negative potentials on bounded smooth domains, via the use of a probabilistic interpretation. This estimate is used to give an implicit description of a confining landscape. The relationship between this landscape and the Filoche-Mayboroda landscape function u is analyzed, and certain refinements of the landscape function are considered.

16.1 Introduction

The localization of low-lying eigenfunctions for Schrödinger operators $L = -\Delta + V$ with certain random potentials V is a well-established phenomenon, going back to Anderson [A]; however, understanding where these low-energy eigenfunctions will localize in space without explicitly computing them has been difficult.

In 2012, Filoche and Mayboroda [FM] introduced the *landscape function* u, which is the solution to the equation $Lu = 1$ in Ω with 0 Dirichlet boundary conditions, and provided a simple computational method that strongly suggests where the localization of the low-energy eigenfunctions can occur, based on the locations of the critical points of u. Let us be a bit more detailed: consider the graph of u over the domain as a landscape with many "peaks" and "valleys"; then the valleys may be understood to induce a partition on the domain. There is strong numerical evidence [FM] that the low-energy eigenfunctions localize in one or at most a few elements of this partition induced by the valleys of the graph of u. The numerical results further suggest that the eigenfunctions undergo exponential decay when crossing from one element of the partition to another.

The present paper [S] aims to refine these observations. In this paper,

- a localized variation estimate for eigenfunctions is shown,

- it is studied how the landscape function u fits into this framework,
- some possible refinements of the landscape function are discussed.

16.2 A variation estimate for eigenfunctions

Let us state the main results more precisely now. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $V : \Omega \to \mathbb{R}$ a non-negative function. Assume furthermore that $V \in C^2(\Omega) \cap C(\overline{\Omega})$; although this qualitative assumption technically excludes "block potentials" which are merely L^∞, the first few eigenfunctions do not change too much if V is replaced by a suitable mollification of V. Now let ϕ be an eigenfunction of $L = -\Delta + V$, so that

$$\begin{cases} -\Delta\phi + V\phi = \lambda\phi, & \text{in } \Omega \\ \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

for some $\lambda > 0$. Under our assumptions, we have that $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$, and the identity $L\phi = \lambda\phi$ is true pointwise in Ω.

Theorem 1. *Let Ω, V, ϕ be as above. There exists a universal constant c_n (depending only on dimension) so that the following holds: if, for some $c > 0$,*

$$V - \lambda \geq c, \qquad \text{or} \qquad V - \lambda \leq -c \tag{1}$$

uniformly in the ball

$$B = B\Big(x_0, \frac{c_n}{\sqrt{c}}\sqrt{\frac{\lambda}{c} + \log\Big(c_n \frac{\|\phi\|_{L^\infty}}{|\phi(x_0)|}\Big)}\Big) \subset \Omega,$$

then we have

$$\frac{\sup_{x\in B}|\phi(x)|}{\inf_{x\in B}|\phi(x)|} \geq 2. \tag{2}$$

If in (2) the $\geq$ sign is replaced by $\leq$, such an estimate is known as a *doubling estimate*, which holds in certain situations where $V \equiv 0$. Thus Theorem 1 gives an opposite result to a doubling estimate, suggesting that across the ball B, the eigenfunction ought to *at least* double.

A main tool in the proof of Theorem 1 is a probabilistic description of the eigenfunction which we now derive, using the *Feynman-Kac formula* which describes the action of the semigroup e^{-tL} in terms of Brownian motion. For an arbitrary function f, the Feynman-Kac formula states that

$$e^{-tL}f(x) = \mathbb{E}_x\Big(f\big(\omega(t)\big)e^{-\int_0^t V(\omega(s))\,ds}\Big), \qquad \text{for } t \geq 0 \text{ and } x \in \Omega,$$

where the expectation $\mathbb{E}_x$ is taken with respect to the Brownian motion $\omega(\cdot)$ started at x, running for time t and stopped upon impact with the boundary. On the other hand, if ϕ is an eigenfunction of L, then

$$e^{-tL}\phi = e^{-\lambda t}\phi, \qquad \text{for each } t \geq 0,$$

since eigenfunctions diagonalize the semigroup. We now combine these last two equations for $f \equiv \phi$ to obtain the identity

$$\phi(x) = \mathbb{E}_x\Big(\phi\big(\omega(t)\big)e^{\lambda t-\int_0^t V(\omega(s))\,ds}\Big), \qquad \text{for each } t \geq 0.$$

The proof of Theorem 1 is based on this identity. Theorem 1 can be improved in two ways: first, by weakening the assumption (1) so that the inequalities are assumed only 'with respect to path integrals', and second, by rephrasing the conclusion into the following nonlocal formulation (assuming that $V \geq \lambda$):

$$\phi \text{ varies locally by a constant factor on the scale } \sim \sqrt{t_x}, \text{ where}$$

$$t_x = \inf\Big\{t > 0 : \mathbb{E}_x\big(e^{\lambda t-\int_0^t V(\omega(s))\,ds}\big) \leq \frac{1}{2}\Big\}.$$

A similar result can be obtained if instead it is assumed that $V \leq \lambda$.

16.2.1 Relation between the landscape function and the variation estimate

Let us see how the variation estimate may be used to implicitly define a landscape. Pick a sequence of points $\{x_j\} \subset \Omega$ and for each x_j, compute t_{x_j} and draw balls around the points x_j with radius $\sqrt{t_{x_j}}$. Then, by the variation estimate, we see that smaller balls correspond to faster growth/decay. We may thus interpret that the "valleys" in the landscape correspond to regions with smaller balls $B(x_j, \sqrt{t_{x_j}})$, while the "peaks" of the landscape correspond to regions with larger balls $B(x_j, \sqrt{t_{x_j}})$. Since both $\sqrt{t_{x_j}}$ and u may be seen as mollifications of $1/V$, it follows that the landscape generated in this way will be very close to that of the landscape function of u described before.

16.3 New landscape functions

The paper [S] also shows a way to obtain new landscape functions based on an interation procedure. The Filoche-Mayboroda landscape function satisfies the key estimate

$$|\phi(x)| \leq \lambda u(x) \|\phi\|_{L^\infty}, \tag{3}$$

where ϕ is any eigenfunction of L with eigenvalue λ. The following theorem gives a way to iterate this estimate to find new landscape functions satisfying similar confining properties to the classical landscape function u.

Theorem 2. *Suppose* $(-\Delta + V)\phi = \lambda\phi$ *and* $h(x)$ *satisfies*

$$|\phi(x)| \leq h(x) \|\phi\|_{L^\infty},$$

then the same inequality holds for $h(x)$ *replaced by*

$$h_1(x) = \inf_{t \geq 0} \mathbb{E}_x \Big(h(\omega(t)) e^{\lambda t - \int_0^t V(\omega(s))\, ds} \Big) = \inf_{t \geq 0} e^{\lambda t} e^{-tL} h(x)$$

Although Theorem 2 is quite general, computing $e^{-tL}u$ may be equally difficult (or even more difficult) to computing the eigenfunction ϕ directly. Thus new landscape functions which are computationally effective are also considered in [S] for small eigenvalues, essentially by taking a simple cut-off of λu at 1 and then bootstrapping (3).

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17 Localization of eigenfunctions via an effective potential

After D. Arnold, G. David, M. Filoche, D. Jerison and S. Mayboroda [A+2016]

A summary written by Jaume de Dios Pont

Abstract

This work considers the localization of eigenfunctions for the operator $L = -\Delta + V$ on a Lipschitz domain Ω and, more generally, on manifolds with and without boundary. The main tool is the ability of the landscape, defined as the solution to $Lu = 1$, to predict the location of the localized eigenfunctions. Informally, the paper shows that eigenfunctions of eigenvalue $\leq \lambda$ will be (essentially) supported in a connected component of $U^{-1} \leq \lambda + \delta$.

17.1 Introduction

Consider a quantum particle in a potential, evolving with the Schrodinger evoluiton equation

$$i\partial_t \psi = L\psi = -\Delta\psi + V\psi.$$

Where (note the unusual notation) L represents the Hamiltonian of the system. The phenomenenon of *Anderson localization* in physics shows an absence of diffusion of solutions to the Schrodinger evolution equation in disordered media (random V). An essentially equivalent fact is the spatial localization of eigenfunctions of $L := -\Delta + V$.

A key example to keep in mind is the case when V is constant on squares of the form $[k, k+1] \times [l, l+1]$, taking values 0 or V with probability p. If p is small, one could expect the quantum particles to diffuse over time, since the set $V = 0$ has a large connected component that percolates the whole space. This, however, is not the case in the presence of disorder.

Classical approaches to study these phenomena involve a simultaneous understanding of the randomness in V and the evolution equation. This work takes a different direction, showing a way to deterministically construct an effective potential that characterizes the localization of eigenfunctions of L.

17.1.1 The landscape function

The introduction of the landscape function will allow us to disentangle the spectral and random parts of the problem (See Theorems 3 and 4 below for the actual statements). We can informally sumarize the main results of the paper as follows:

Theorem 1 ([A+2016, Theorems 2.1,2.2], **Informal statement**). *Let $L = -\Delta + V$ be a Schrodinger operator with a nonnegative potential, and u be the landscape function, the solution to $L(u) = 1$. Let ϕ_λ be an eigenfunction of L with energy (eigenvalue) λ. Then*

- *The function ϕ_λ is essentially supported on the set $W_{\leq\lambda} := \{x : u^{-1}(x) \leq \lambda\}$ and decays exponentially fast outside of this set.*
- *The function ϕ_λ is in fact essentially supported on a conected component of $V_{\leq\lambda}$ unless there's anotherother eigenvalue λ' very close to λ (a ressonance).*

In other words, the function $w = u^{-1}$ acts like an effective potential barrier that states with lower energy cannot cross.

17.1.2 Generalization to manifolds

While, for simplicity, the results in this summary (and all but the last section of [A+2016]) are written for the manifold $M = \mathbb{R}^n/(K\mathbb{Z})^n$ (with bounds independent of the value of K), they can be extended to C^1 manifolds (See Section 6 of [A+2016])

17.2 Intuition behind the Landscape Function

17.2.1 An informal physics-inspired interpretation

The landscape function is able to capture the following two insights at once:

Physics Insight 1: *Quantum systems should have similar system to their classical counterparts. In particular, most of the mass of an eigenfunction of energy $\leq \lambda$ should be contained on the set $\{x : V(x) \leq \lambda\}$.* This fact can be formally stated and proven, but is not strong enough to show that localization occurs in systems like the random Bernoulli potential described in the introduction. There, the set $\{x : V(x) \leq \lambda\}$ percolates to the whole space for any $\lambda \geq 0$ but the eigenfunctions are localized.

Physics Insight 2: *Low energy eigenfunctions, by the uncertainty principle, must be delocalized in space. In particular, an eigenfunction of energy λ should be roughly constant at scale $\lambda^{-1/2}$, and should therefore* see *a blurred version of V at scale at least $\lambda^{-1/2}$.* The effective potential $w = u^{-1}$ associated to the landscape function solves $Lw^{-1} = 1$ and is (by integrating by parts) formally the solution to the optimization problem:

$$w = \operatorname{argmin}_{\omega^{-1} \in H^1} \int_M \left(\frac{V}{\omega} - 1 \right)^2 dx + \int_M \omega^{-4} |\nabla \omega|^2 dx$$

The first term in the minimization problem *forces* V and w to be as similar as possible, while the second term *forces* the w to be smoother, especially at low energy values (because of the w^{-4} term).

This smoothing turns out to be the key advantage of the Landscape function. Suitable variations of the theorems below still hold (with even easier arguments) for $\tilde{u}^{-1} = \tilde{w} = V$. The advantage of u^{-1} over V is that it can create disconnected potential basins as sub-level sets even when the sub-level set of the whole space is connected.

17.2.2 The energy identity

Integrating by parts one can show the following equality for the energy of a quantum state:

$$\int_M |\nabla f|^2 + V f^2 dx = \int_M \frac{1}{u} f^2 + u^2 |\nabla (f/u)|^2 dx \qquad (*)$$

as long as f, u, u^{-1} are in $W^{1,2} \cap L^\infty$. Lemma 4.1 in the paper gives a more general identity. Before stating it, we need to give some notation:

Let $0 < c < m(x) < C$ be an L^∞ density, and A be a bounded measurable uniformly elliptic coefficient matrix. Define $L := -\frac{1}{m}\operatorname{div} mA\nabla + V$ the associated elliptic operator. Let $\nabla_A := A^{1/2}\nabla$ so that, formally, in $L^2(m\,dx)$ we have $-\frac{1}{m}\mathrm{div}\ mA\nabla = \nabla_A^*\nabla_A$.

Lemma 2 ([A+2016, Lemma 4.1])**.** *Assume that f and u (with $L(u) = 1$) belong to $W^{1,2}(M)$, that V, f, and $1/u$ belong to $L^\infty(M)$, and that u satisfies $Lu = 1$ weakly on M. Then*

$$\int_M (|\nabla_A f|^2 + Vf^2)\,m\,dx = \int_M \left(u^2\,|\nabla_A(f/u)|^2 + \frac{1}{u}f^2\right)\,m\,dx.$$

From this one already deduces that if f is an eigenfunction of eigenvalue λ (normalized in $L^2(m\cdot dx)$) of L then

$$\int_M u^{-1}f^2 m dx \leq \lambda$$

and by Markov's inequality most of the mass of f^2 must be contained in the set where $w \lesssim \lambda$. This, however, is still much weaker than the statements of Theorem 1.

17.3 Localization Estimates I: Agmon distance

Given a non-negative continuous weight w on M, and a continuous elliptic matrix we define a (possibly degenerate) distance on M via the degenerate Riemannian metric $g = wA^{-1}$, by

$$\rho_w(x,y) := \inf_{\gamma(x\to y)} \int_0^1 (w(\gamma(t)\langle\gamma'(t), A(\gamma(t))^{-1}\gamma'(t)\rangle)^{\frac{1}{2}}dt$$

where $\inf_{\gamma(x\to y)}$ denotes the infimum over all paths joining x to y. The main property of ρ_w that will be needed is that the function

$$\rho_w(x,E) = \inf_{e\in E}\rho(x,e)$$

is Lipschitz and $|\nabla_A\rho(x,E)| \leq w$. This is in general true for any function that is 1−Lipschitz with respect to the ρ_w metric. Our weight $w(x)$ will be the "energy defect" at x, that is

$$w_\lambda(x) = \min(1/u - \lambda, 0)$$

We shall also specialize the set E to the case $E = E(\lambda) = \{x : V(x) \leq \lambda\}$. With this specializations, one can show the following result:

Theorem 3 ([A+2016, Theorem 2.1]). *Let ϕ be an eigenfunction of L, $L\phi = \lambda\phi$ on M. Let*

$$h(x) = \rho_{w_{\lambda+\delta}}(x, E(\lambda+\delta))$$

be the effective distance to the sub-level set (the Agmon distance), and let $\overline{V} = \|V\|_\infty$ Then

$$\int_{h\geq 1} e^h \left(|\nabla_A \phi|^2 + \overline{V}\phi^2\right) m\,dx \leq 18e\frac{\overline{V}^2}{\delta}\int_M \phi^2\, m\,dx. \tag{1}$$

17.4 Localization Estimates II: Projection to energy wells approximately diagonalizes

The analysis on the previous section does not prevent eigenfunctions from having their mass in multiple connected components of $E(\lambda+\delta)$. If, in the most extreme example, there is a high multiplicity eigenvalue, any linear combination of eigenfunctions will give rise to (potentially delocalized) eigenfunctions. This turns out to be (essentially) the only possibility:

Assume $E(\overline{\mu}+\delta) = \bigcup_l E_l$, with Agmon distance between E_i and E_j ($i \neq j$) at least $\overline{S}$. Let Ω_l be the $\overline{S}/2$-neighbourhood of E_l (again, in the Agmon distance).

Let $\Pi_{(a,b)}$ be the spectral projection for L in $L^2(M)$, and $\Pi^{loc}_{(a,b)}$ be spectral projection for L in $L^2(\bigcup_l E_l) = \bigoplus_l L^2(E_l)$ (with Dirichlet boundary conditions). Then, the main result states that:

Theorem 4. *If ϕ is an eigenfunction of L with eigenvalue λ on M and $\lambda \leq \overline{\mu} - \delta$, then*

$$\|\phi - \Pi^{loc}_{(\lambda-\delta,\lambda+\delta)}\phi\|_2^2 \leq 300\left(\frac{\overline{V}}{\delta}\right)^3 e^{-\overline{S}/2}\|\phi\|_2^2. \tag{2}$$

Similarly, if $\psi = \psi_{l,j}$ is a localized eigenfunction of L restricted to E_j with eigenvalue $\mu = \mu_{l,j} \leq \overline{\mu} - \delta$, then

$$\|\psi - \Pi_{(\mu-\delta,\mu+\delta)}\psi\|^2 \leq 300\left(\frac{\overline{V}}{\delta}\right)^3 e^{-\overline{S}/2}\|\psi\|^2.$$

In other words, that eigenfunctions of L in all of M are linear combinations of localized eigenfunctions with similar eigenvalues, and viceversa. If the spectral gap at λ is large enough, this shows that ϕ is supported essentially in a single component of $E(\overline{\mu}+\delta)$.

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18 On fundamental solutions of generalized Schrödinger operators

After Z. Shen [S99]

A summary written by Cole Jeznach

Abstract

We highlight some elements of the proof of the exponential decay (in the Agmon metric) of the fundamental solution of Schrödinger operators of the form $-\Delta + \mu$. As a consequence, one obtains the boundedness of the operators $\nabla(-\Delta+\mu)^{-1/2}$ and $(-\Delta+\mu)^{i\gamma}$ for $\gamma \in \mathbb{R}$ on $L^p(R^n, dx)$, $p \in (1, \infty)$.

18.1 Introduction and the Main Theorem

We consider the Schrödinger operator

$$-\Delta + \mu, \quad \text{in } \mathbb{R}^n, \quad n \geq 3 \tag{1}$$

where μ is a non-negative measure Radon for which there exists uniform constants C_0, C_1 and $\delta > 0$ so that

$$\mu(B(x,r)) \leq C_0 \left(\frac{r}{R}\right)^{n-2+\delta} \mu(B(x,R)), \tag{2}$$

$$\mu(B(x,2r)) \leq C_1 \left\{\mu(B(x,r)) + r^{n-2}\right\}, \tag{3}$$

holds for $x \in \mathbb{R}^n$ and $0 < r < R$. Examples of such measures include $d\mu(x) = d\mathcal{H}^{n-1}|_\Gamma(x)$ where Γ is the graph of a Lipschitz function defined over $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, or $d\mu(x) = V(x)dx$, where $V \in (\mathrm{RH})_{n/2}$, i.e., the Reverse-Hölder class with exponent $n/2$.

With such a measure μ, one can define the Fefferman-Phong maximal function

$$m(x,\mu)^{-1} := \sup\left\{r > 0 \ : \ \mu(B(x,r)) \leq C_1 r^{n-2}\right\} \tag{4}$$

where C_1 is the same constant from (3). Finally, with the metric

$$d(x,y,\mu) := \inf_\gamma \int_0^1 m(\gamma(t),\mu)|\gamma'(t)|\, dt$$

where the infimum above is taken over all $\gamma : [0,1] \to \mathbb{R}^n$ absolutely continuous with $\gamma(0) = x, \gamma(1) = y$, we state the main Theorem.

Theorem 1. *Let μ be a non-negative Radon measure satisfying* (2) *and* (3). *Then the fundamental solution $\Gamma_\mu(x,y)$ of the equation* (1) *exists and satisfies*

$$\frac{ce^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \leq \Gamma_\mu(x,y) \leq \frac{Ce^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}} \tag{5}$$

for some constants $C, c, \varepsilon_1, \varepsilon_2 > 0$ depending only on n and the constants C_0, C_1 from (2), (3). *Moreover, one also has*

$$|\nabla_x \Gamma_\mu(x,y)| \leq \frac{Ce^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-1}}. \tag{6}$$

As a consequence of the exponential decay from Theorem 1, one can deduce via the usual functional calculus that for $\gamma \in \mathbb{R}$, the operators $T_1 = (-\Delta+\mu)^{i\gamma}$ and $T_2 = \nabla(-\Delta+\mu)^{-1/2}$ are singular integral operators of convolution type with associated kernels

$$K_1(x,y) = c_\gamma \int_0^\infty \lambda^{i\gamma}\Gamma_{\mu+\lambda}(x,y)\, d\lambda,$$
$$K_2(x,y) = c \int_0^\infty \lambda^{-1/2}\nabla_x\Gamma_{\mu+\lambda}(x,y)\, d\lambda,$$

respectively. Here $\Gamma_{\mu+\lambda}$ is the fundamental solution associated to the measure $d\mu(x) + \lambda dx$. With some additional work, one can show that the exponential decay estimates (5) and (6) thus give the following result.

Theorem 2. *Let μ be a non-negative Radon measure satisfying* (2) *and* (3). *Then the operators $(-\Delta+\mu)^{i\gamma}$ and $\nabla(-\Delta+\mu)^{-1/2}$ are Calderón-Zygmund operators, and thus bounded on $L^p(\mathbb{R}^n, dx)$ for each $p \in (1,\infty)$.*

Let us turn to describing some key elements of the proof of Theorem 1. For brevity, we focus on the upper bound, since the other arguments, while similar in spirit, are slightly more involved. We assume (2) and (3) throughout.

18.2 The Poincaré inequality and weak formulation

The main tool in the exponential decay of (5) is the following Poincaré inequality, which replaces the Fefferman-Phong inequality from [F83] that was used in [S95] to prove results for Schrödinger operators with absolutely continuous non-negative potentials $d\mu(x) = V(x)\,dx$ where $V \in (\mathrm{RH})_{n/2}$.

Lemma 3. *There is a constant $C > 0$ depending only on n and C_0 so that for any ball $B \subset \mathbb{R}^n$ and any $\psi \in C^1(B)$, one has*

$$\int_B \int_B |\psi(x) - \psi(y)|^2 \, d\mu(y) \, dx \leq C rad(B)^2 \mu(3B) \int_B |\nabla \psi(x)|^2 \, dx. \quad (7)$$

The inequality (7) also serves as a starting point for the usual functional-analytic techniques used to study the existence of weak solutions to the operator (1). In particular, it implies that

$$\int_B |\psi(x)|^2 \, d\mu(x) \leq C_B \left\{ \int_B |\nabla \psi(x)|^2 \, dx + \int_B |\psi(x)|^2 \, dx \right\}$$

so that $W^{1,2}(B) \subset L^2(B, d\mu)$, and in fact, this imbedding can be shown to be compact. Moreover, this imbedding makes legitimate the following definition of a (weak) solution of the operator (1).

Definition 4. *Suppose that $f \in L^1_{\text{loc}}(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$. Then $u \in W^{1,2}_{\text{loc}}(\Omega)$ is said to be a solution to the equation $(-\Delta + \mu)u = f$ in Ω provided that for all $\phi \in C^1_0(\Omega)$, one has*

$$\int_\Omega \langle \nabla u, \nabla \phi \rangle \, dx + \int_\Omega \langle u, \phi \rangle \, d\mu = \int_\Omega \langle f, \phi \rangle \, dx.$$

Beyond just making possible the definition above, the inequality is used to show that for $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, d\mu)$,

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^n} |u(x)|^2 \, d\mu(x) \simeq \int_{\mathbb{R}^n} |\nabla u(x)|^2 + m(x, \mu)^2 |u(x)|^2 \, dx, \quad (8)$$

and, moreover, that the space

$$\mathscr{H} := \{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \; : \; \nabla u \in L^2(\mathbb{R}^n, dx), \; m(\cdot, \mu) \, u \in L^2(\mathbb{R}^n, dx) \} \quad (9)$$

is a Hilbert space with inner product given by

$$\langle u, v \rangle_{\mathscr{H}} := \int_{\mathbb{R}^n} \langle \nabla u(x), \nabla v(x) \rangle + m(x, \mu)^2 \langle u(x), v(x) \rangle \, dx.$$

In particular, using the Lax-Milgram Theorem, one has basic existence and uniqueness of solutions as follows.

Proposition 5. *For each* $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n, dx)$ *with* $m(\cdot,\mu)^{-1} f \in L^2(\mathbb{R}^n, dx)$, *there is a unique weak solution* $u_f \in \mathscr{H}$ *to* $(-\Delta + \mu)u = f$ *in* $\mathbb{R}^n$. *Moreover, there is a unique, symmetric kernel function* $\Gamma_\mu(x, y)$ *so that when* $f \in L^2_{\mathrm{c}}(\mathbb{R}^n, dx)$, *then* u_f *is given by*

$$u_f(x) = \int_{\mathbb{R}^n} \Gamma_\mu(x, y) f(y)\, dy.$$

$\Gamma_\mu(x, y)$ *is called the fundamental solution of* $-\Delta + \mu$ *in* $\mathbb{R}^n$.

18.3 A sketch of the upper bound of (5)

With the precise definition of the fundamental solution in mind, we move on to the upper bound in (5). First, we need some preliminary results on $m(\cdot, \mu)$ and Γ_μ.

Lemma 6. *The function* $m(x, \mu)$ *satisfies the following properties:*

if $|x - y| \leq C/m(x, \mu)$, *then* $m(x, \mu) \simeq_C m(y, \mu)$, (10)

there is some $k_0 \in \mathbb{N}$ *so that if* $|x - y| m(x, \mu) \geq 2$, *then* $c|x - y| m(x, \mu) \leq d(x, y, \mu)^{k_0}$. (11)

Lemma 7. *The fundamental solution* $\Gamma_\mu(x, y)$ *satisfies the following properties:*

$0 \leq \Gamma_\mu(x, y) \leq C|x - y|^{-n+2}$ *for all* $x \neq y \in \mathbb{R}^n$, (12)

for each x_0, $\Gamma_\mu(\cdot, x_0)$ *is a solution to* $(-\Delta + \mu)u = 0$ *(and thus subharmonic) in* $\mathbb{R}^n \setminus \{x_0\}$. (13)

We move on to the main steps in proving the upper bound in (5).

sketch of proof (upper bound) of (5).

Let $x_0 \neq y_0 \in \mathbb{R}^n$, and assume without loss of generality that $y_0 = 0$. Set $u(x) := \Gamma_\mu(x, 0)$. From (10), we see that $d(x, y, \mu) \leq C$ whenever $|x_0| m(x_0, \mu) \leq 4$ or $|x_0| m(0, \mu) \leq 4$, and thus the desired conclusion follows from (12). Thus we may assume that $|x_0| m(x_0, \mu) > 4$, $|x_0| m(0, \mu) > 4$, and thus

$$B(0, 2/m(0, \mu)) \cap B(x_0, 2/m(x_0, \mu)) = \emptyset. \tag{14}$$

Step 1: one shows that if $g \in C^1(\mathbb{R}^n \setminus \{0\})$ is non-negative, $\phi \in C^1_c(\mathbb{R}^n \setminus \{0\})$, and $|\nabla g(x)| \leq C_2 m(x,\mu)$, then for ε_1 sufficiently small,

$$\int_{\mathbb{R}^n} m(x,\mu)^2 |u\phi|^2 e^{2\varepsilon_1 g}\, dx \leq C \int_{\mathbb{R}^n} |u|^2 |\nabla \phi|^2 e^{2\varepsilon_1 g}\, dx. \tag{15}$$

where ε_1 depends only on C_0, C_1, C_2 and n. Setting $\psi = u\phi e^{\varepsilon_1 g}$ and $f = \phi e^{\varepsilon_1 g}$, this follows from the bounds on ∇g by applying (8) to the function ϕ, re-writing $|\nabla \psi|^2 = \langle \nabla u, \nabla(u|f|^2)\rangle + |u|^2 |\nabla f|^2$, and using the fact that $(-\Delta + \mu)u = 0$ $\mathbb{R}^n \setminus \{0\}$ by (13).

Step 2: by a suitable approximation argument, one may take $g^* = d(x,0,\mu)$ in Step 1 and assume that $\|g^*\|_\infty \leq 1$ as long as no estimates depend on this bound. Set $r = m(0,\mu)^{-1}$. By taking ϕ to be a suitable C^1 bump function that is supported in $B(0,2M) \setminus B(0,r)$ that is identically 1 on $B(0,M) \setminus B(0,2r)$, one sees from $g^* \leq C$ on $B(0,2r)$ (by (10)) that,

$$\int_{2r \leq |x| \leq M} m(x,\mu)^2 |u|^2 e^{2\varepsilon_1 g^*}\, dx \leq C \left(r^{-2} \int_{r \leq |x| \leq 2r} |u|^2 + e^{\|g^*\|_\infty} M^{-2} \int_{M \leq |x| \leq 2M} |u|^2\, dx \right).$$

Since $|u(x)| \leq C|x|^{-n+2}$, sending $M \to \infty$ gives

$$\int_{|x| \geq 2r} m(x,\mu)^2 |u|^2 e^{2\varepsilon_1 g^*}\, dx \leq C r^{-n+2}, \tag{16}$$

with constant C independent of $\|g^*\|_\infty$.

Step 3: using that u is subharmonic, the values of g^* are comparable in $B(x_0,R)$, $B(x_0,R) \subset B(0,2r)^c$ from (14), and Step 2, we see that

$$\begin{aligned} |u(x_0)| &\leq \left(\fint_{B(x_0,R)} |u(x)|^2 \right)^{1/2} \\ &\leq R^{-n/2} \left(\int_{|x| \geq 2r} |u(x)|^2\, dx \right)^{1/2} \\ &\leq C \frac{e^{-\varepsilon_1 g^*(x_0)}}{(Rr)^{(n-2)/2}} \end{aligned}$$

Finally, (11) implies that $|x_0| m(x_0,\mu) + |x_0| m(0,\mu) \leq C_\delta e^{\delta g^*(x_0)}$ for any $\delta > 0$. Choosing $\delta > 0$ sufficiently small (depending on k_0) and recalling the definitions of r and R gives then that

$$\Gamma_\mu(x_0,0) = |u(x_0)| \leq C \frac{e^{-\varepsilon_1 d(x_0,0,\mu)/2}}{|x_0|^{n-2}}$$

as desired.

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19 The landscape law for the integrated density of states

After G. David, M. Filoche and S. Mayboroda [DFM]

A summary written by Alex Bergman

Abstract

We summarize the article "The landscape law for integrated density of states". The main result is a non-asymptotic estimate from above and below on the integrated density of states of the Schrödinger operator $L = -\Delta + V$ using the solution to the equation $Lu = 1$.

19.1 Introduction

Consider a domain $\Omega \subset \mathbb{R}^d$ and a Schrödinger operator $L = -\Delta + V$ on Ω. For definiteness we shall consider the case where Ω is a cube of side-length R_0, however, our estimates will not depend on the side-length and hence it will (often) be possible to pass to other regions via exhaustion. We shall assume the potential V is nonnegative and locally integrable. Given periodic boundary conditions for L on $\partial\Omega$ the (normalized) integrated density of states N is defined by

$$N(\mu) = \frac{1}{|\Omega|} \{\text{the number of eigenvalues } \lambda \text{ such that } \lambda \leq \mu\},$$

here eigenvalues are counted with multiplicity. It is implicit in the definition that L has discrete spectrum consisting of positive eigenvalues (this follows from the nonnegativity of V and periodicity of the boundary conditions). The integrated density of states has the following asymptotics, known as the Weyl law

$$N(\mu) \sim (2\pi)^{-d} |\Omega|^{-1} \int_{|\xi|^2 + V(x) < \mu} dx d\xi, \text{ as } \mu \to \infty.$$

The purpose of this article is to present estimates for $N(\mu)$ which hold uniformly in μ based on the concept of the landscape function. The landscape function of L is the solution to the equation $Lu = 1$. It can be shown that u is positive and bounded.

To motivate the introduction of the Landscape function u we note that the operator L has the same spectrum as the conjugated operator

$$-\frac{1}{u^2}\mathrm{div} u^2\nabla + \frac{1}{u},$$

which was shown in [ADFJM] and brings up $1/u$ as an effective potential. Numerical experiments (with Anderson-type potentials) show that at the bottom of the spectrum the eigenvalues are essentially dimensional multiples of the local minima of $1/u$ and the following version of the Weyl law

$$N(\mu) \sim (2\pi)^{-d}|\Omega|^{-1}\int_{|\xi|^2+\frac{1}{u(x)}<\mu} dx d\xi,$$

giving an approximation in the whole spectrum. We stress that these results are only motivated by numerical experiments and not theorems! Indeed they cannot hold as near identities as simple examples show.

Motivated by this let for $r > 0$, such that R_0 is an integer multiple of r, $\{Q\}_r$ be a collection of disjoint cubes of side-length r, such that each Q_r is a subset of Ω and

$$\cup_{Q\in\{Q\}_r}\bar{Q} = \bar{\Omega}.$$

We define the counting function of the minima of $1/u$ via the formula

$$N_u(\mu) = \frac{1}{|\Omega|}\left\{\text{the number of cubes Q } \in \{Q\}_{\kappa\mu^{-1/2}} \text{ such that } \min_Q 1/u \le \mu\right\},$$

where $\kappa = \kappa(\mu)$ is taken to be the smallest number such that R_0 is an integer multiple of $k\mu^{-1/2}$.

Theorem 1. *(The Landscape law). With the above definitions there exists constants $C_i > 0$, $i = 1, 2, 3, 4$, depending only on the dimension, such that*

$$C_1\alpha^d N_u(C_2\alpha^{d+2}\mu) - C_3 N_u(C_2\alpha^{d+4}\mu) \le N(\mu) \le N_u(C_4\mu),$$

for every $\alpha < 2^{-4}$ and $\mu > 0$.

The advantage of the above result is that it is not asymptotic, does not depend on the L^∞ norm of V and is independent of the side-length R_0. The obvious blemish is the left hand side. This can be fixed by assuming u^2 is a doubling weight at small scales.

Theorem 2. *(The doubling case). Retain the definitions of Theorem 1. Suppose also there exists a constant $C_D \geq 1$, such that*

$$\int_{Q_{2s}} u^2 dx \leq C_D \left(\int_{Q_s} u^2 dx + s^{d+4} \right),$$

for every cube Q_s of side-length $s > 0$, then

$$N_u(C_2' \mu) \leq N(\mu) \leq N_u(C_4 \mu),$$

where C_4 is as in Theorem 1 and C_2' depends only on C_D and the dimension.

We shall work exclusively with periodic functions and hence we assume our functions are extended periodically when necessary (in particular we assume this in the doubling condition above).

19.2 Proofs of the main results

We begin with the proof of the upper bound valid for both Theorem 1 and 2. In this section we let H be the space of all periodic functions in $W^{1,2}(\Omega)$.

Proof. (of upper bound).

The estimate $|\Omega| N(\mu) \leq N$ will follow if we can find a subspace H_N of H of codimension N, such that

$$\frac{\langle Lv, v\rangle_2}{\|v\|_2^2} = \frac{\int_\Omega |\nabla v|^2 + V v^2 dx}{\int_\Omega v^2 dx} > \mu, \text{ for } v \in H_N \setminus \{0\}.$$

To this end denote

$$\mathcal{F} = \left\{ Q \in \{Q\}_{\kappa\mu^{-1/2}} \text{ such that } \inf_Q 1/u \leq C_4 \mu \right\},$$

with C_4 to be fixed later and $1 \leq \kappa < 2$ is the smallest number, such that R_0 is an integer multiple of $\kappa\mu^{-1/2}$. Let H_N be the set of $v \in H$, such that $\int_Q v dx = 0$ for all $Q \in \mathcal{F}$. Since the cubes are disjoint the space H_N has codimension equal to the cardinality of $\mathcal{F}$. We shall need the following estimate from [ADFJM]

$$\int_\Omega |\nabla v|^2 + V v^2 dx \geq \int_\Omega \frac{1}{u} v^2 dx, \text{ for all } v \in H.$$

From the above estimate it follows that

$$2\int_{\Omega}|\nabla v|^2 + Vv^2dx \geq \int_{\Omega}|\nabla v|^2 + \frac{1}{u}v^2dx, \text{ for all } v \in H,$$

and so it suffices to prove

$$\int_{\Omega}|\nabla v|^2 + \frac{1}{u}v^2dx > 2\mu\int_{\Omega}v^2dx, \text{ for all } v \in H_N \setminus \{0\}.$$

For cubes $Q \in \{Q\}_{\kappa(C_4\mu)^{-1/2}}$, such that $Q \notin \mathcal{F}$ it follows if $C_4 > 2$ since $\inf_Q 1/u \geq C_4\mu$ on any such cube. Conversely, if $Q \in \mathcal{F}$ an application of Poincare's inequality gives

$$\int_Q|\nabla v|^2 \geq C_PC_4\mu\int_Q|v - v_Q|^2dx = C_PC_4\mu\int_Q v^2dx,$$

where the constant C_P arising from the Poincare inequality depends only on the dimension. Choosing C_4 large enough such that $C_4C_P > 2$ gives the desired result for each cube Q. Since the cubes are disjoint the proof is done. □

We now turn to the proof of the lower bound.

Proof. (of the lower bound in the doubling case). In order to prove $M \leq |\Omega|N(\mu)$ we must find a subspace H_M of dimension M, such that

$$\frac{\langle Lv, v\rangle}{\|v\|_2^2} = \frac{\int_{\Omega}|\nabla v|^2 + Vv^2dx}{\int_{\Omega}v^2dx} \leq \mu, \text{ for all } v \in H_M \setminus \{0\}.$$

To this end let

$$\mathcal{F}' = \left\{Q \in \{Q\}_{\kappa(C_2\mu)^{-1/2}} \text{ such that } \inf_Q 1/u \leq C_2\mu\right\},$$

where C_2 is to be fixed later. Let H_M be the linear span of functions of the form $u\chi_Q$, $Q \in \mathcal{F}'$, where χ_Q is smooth with compact support in Q, $\chi_Q = 1$ on $Q/2$, $0 \leq \chi_Q \leq 1$ on Q and $|\nabla\chi_Q| \leq 4\ell(Q)^{-1}$. Since $-\Delta u \leq 1$ a variant of Harnack's inequality (Theorem 4.14. in [HL]) implies

$$\sup_Q u \leq C_H\left(\frac{1}{|Q|}\int_{2Q}u^2dx\right)^{1/2} + C_H\ell(Q)^2,$$

where the constant C_H depends only on the dimension. Applying the doubling condition three times leads to

$$\sup_Q u \leq C_H \left(\frac{1}{|Q|} (C_D^3 \int_{Q/4} u^2 dx + \ell(Q/4)^{d+4} + \ell(Q/2)^{d+4} + \ell(Q)^{d+4}) \right)^{1/2} + C_H \ell(Q)^2$$
$$\leq C_H C_D^{3/2} \sup_{Q/4} u + C' \ell(Q)^2,$$

where C' depends only on C_D, C_H and the dimension. Now a computation based on the above inequality yields

$$\frac{\langle L(u\chi_Q), u\chi_Q \rangle}{\|u\chi_Q\|_2^2} \leq \frac{4^{d+2}\ell(Q)^{-2} \sup_Q u^2 + 4^d \sup_Q u}{\left(\frac{\sup_Q u}{C_H^2 C_D^{3/2}} - (\frac{\ell(Q)^2}{16} + \frac{C'\ell(Q)^2}{C_H^2 C_D^{3/2}}) \right)^2}. \qquad (1)$$

Modifying κ to be small (in terms of C_D and C_H) we can assume

$$\frac{1}{C_H C_D^{3/2}} \sup_Q u \geq \left(\frac{1}{16} + \frac{C'}{C_H^2 C_D^{3/2}} \right) \ell(Q)^2.$$

Applying this to equation (1) gives

$$\frac{\langle L(u\chi_Q), u\chi_Q \rangle}{\|u\chi_Q\|_2^2} \leq C'_{d,5} \ell(Q)^{-2} + C''_{d,5} \frac{1}{\sup_Q u} \leq C_{d,5} C_2 \mu,$$

where the constants $C'_{d,5}$, $C''_{d,5}$ and $C_{d,5}$ depend on C_H, C_D and the dimension. Choosing C_2, such that $C_2 C_{d,5} = 1$ gives the desired estimate for the basis elements of H_M and hence since the cubes are disjoint for all elements. Also κ may be outside of the range $1 \leq \kappa < 2$, however, increasing κ decreases the cardinality of $\mathcal{F}'$ and so the result follows.

□

The proof of the lower bound in the non-doubling case is similar and is omitted for reasons of space. We stress that the important features of the above estimates are that they are not asymptotic, do not depend on $\|V\|_\infty$ or on the side-length of the cube R_0.

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20 Applications of the landscape function for Schrödinger operators with singular potentials and irregular magnetic fields

After B. Poggi [P]

A summary written by Alioune Seye

Abstract

We extend the definition of the Filoche-Mayboroda landscape function to unbounded domains and singular potentials, and recover the uncertainty principle and decay estimates. We also show the pointwise equivalence between the landscape function and the Fefferman-Phong-Shen maximal function.

20.1 Introduction

Let us consider on $\Omega \subset \mathbb{R}^n$ a Schrödinger operator

$$L = -\mathrm{div} A\nabla + V \tag{1}$$

with A satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j} A_{ij}\xi_i\xi_j \ , \ ||A||_{L^\infty(\Omega)} \leq \frac{1}{\lambda} \tag{2}$$

for all $\xi \in \mathbb{R}^n$ and for some $\lambda \in (0,1)$. We want to extend the definition of the Filoche-Mayboroda landscape function to the case of unbounded domains Ω and non-negative singular potentials $V \in L^1_{loc}(\Omega)$. The original definition [FM] was given for bounded domains and positive bounded potentials as the solution u to the equation

$$Lu = 1. \tag{3}$$

In this setting the following inequality was derived:

$$\int_\Omega \frac{1}{u} f^2 \leq \int_\Omega \left[A\nabla f \nabla f + V f^2\right] \tag{4}$$

which is similar to the Fefferman-Phong-Shen uncertainty principle [S94]. This allows to prove several decay estimates e.g. for eigenfunctions via an

Agmon distance with weight $1/u$. The intuitive understanding of such estimates is that $1/u$ acts as an effective potential and looks like a smoothed version of the potential at a locally appropriate scale. A low energy wavefunction sees this average rather than the true potential, especially when the potential varies faster than the wavelenght. This averaging of the potential is in fact the very idea behind the Fefferman-Phong-Shen uncertainty principle with the maximal function, m.

In the next section, we extend the definition of the landscape function to unbounded domains and singular potentials, and recover the uncertainty principle and the decay estimates. Then we show the pointwise equivalence between $1/u$ and m^2

20.2 A landscape function for unbounded domains

Let us start by defining the generalization of the landscape function:

Theorem 1. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be an open connected set, with empty or Lipschitz $(n-1)$-dimensional boundary, let A be a not necessarily symmetric matrix of bounded measurable coefficients verifying the uniform ellipticity condition (2), and $0 \leq V \in L^1_{loc}(\Omega)$ such that $\int_\Omega V > 0$. Denote $L = -\mathrm{div} A\nabla + V$. The following statements hold.*

(i) Fix $x_0 \in \Omega$ and for each $R \in \mathbb{N}$, let $\Omega_R := \Omega \cap B(x_0, R)$. Consider the landscape function $u_R := L^{-1}_{\Omega_R} \mathbb{1}_{\Omega_R}$ on the domain Ω_R with zero Dirichlet boundary conditions, and extend it by 0 on $\Omega \backslash \Omega_R$. Then the sequence $(u_R)_{R=1}^\infty$ is pointwise non-decreasing, and the limit

$$u := \limsup_{R\to\infty} u_R \tag{5}$$

exists as a measurable non-negative function on Ω (whose values may be $+\infty$ everywhere).

(ii) The function u of (i) verifies that $u \in W^{1,2}_{loc}(\Omega) \cap L^2_{loc}(\Omega, V dx)$ and solves the equation $Lu = 1$ in the weak sense in Ω if and only if there exists $q > 0$ such that

$$\int_\Omega G(x, y)\, dy \in L^q_{loc}(\Omega, dx). \tag{6}$$

We have $u(x) = \int_\Omega G(x, y)\, dy$ a.e. in Ω.

The proof of the first statement comes from the weak maximum principle. The argument relies on the fact that if $\Omega_R \subset \Omega_{R'}$ then $u_R \leq u'_R$. The same argument implies that $\Omega_{R,x_0} \subset \Omega_{R',y_0} \Rightarrow u_{R,x_0} \leq u_{R',y_0}$. We can then show with appropriate subsequences that $u_{x_0} \leq u_{y_0}$, so u does not depend on x_0.

The second uses the De Giorgi-Nash-Moser estimate, Caccioppoli inequality and the dominated convergence theorem.

We now give the uncertainty principle. Let $\mathcal{D}(\Omega)$ be the completion of $\mathcal{C}_c^\infty(\Omega)$ under the norm $\mathcal{I}(f) := \sqrt{\int_\Omega [A\nabla f \nabla f + V f^2]}$.

Theorem 2. *Under the same assumptions as in Thm.1, suppose that eq.(6) holds, if u is the landscape function defined in Thm.1, then $1/u \in L^1_{loc}(\Omega)$, $\nabla \log u \in L^2_{loc}(\Omega)$, and for each $f \in \mathcal{D}(\Omega)$, each of the following integrands lies in $L^1(\Omega)$, and we have that*

$$\int_\Omega \frac{1}{u} f^2 \leq \int_\Omega \frac{1}{\lambda^4} A\nabla f \nabla f + \int_\Omega V f^2. \tag{7}$$

The proof of this theorem is basically the same as for bounded domain and potential, with the use of Fatou's lemma to go to the limit $(V_N) = \min(V, N) \to V$ and $\Omega_R \to \Omega$

With this uncertainty principle, we can now prove exponential decay estimates for the eigenfunctions of L. Let us first define the Agmon distance we will use. We consider a non-negative continuous function f on Ω and associate the potentially degenerate metric

$$ds^2 = f(x) \sum_{i,j} (A^{-1})_{ij}(x) dx_i dx_j. \tag{8}$$

We then denote $\rho_A(x, y, f)$ the distance of the shortest path between x and y for this metric:

$$\rho_A(x, y, f) = \inf_\gamma \int_0^1 f(\gamma(t)) \sum_{i,j} (A^{-1})_{ij}(\gamma(t)) \dot\gamma(t)_i \dot\gamma(t)_j \, dt \tag{9}$$

where the infimum is taken over smooth paths $\gamma : [0, 1] \to \Omega$ going from x to y. We have the following decay estimate for the eigenfunctions of L:

Theorem 3. *Retain the setting of Thm.1, and moreover, assume that eq.(6) holds, that* $\rho_A(x, y, \frac{1}{u}) \longrightarrow 0$ *as* $|x-y| \longrightarrow 0$, *and that* A *is symmetric and continuous. Let* L *be the operator in (1) with domain* $D(L) \in L^2(\Omega)$. *Suppose that there exist* $\mu > 0$ *and* $\psi \in D(L)$ *such that* $L\psi = \mu\psi$. *Let*

$$w(x) := \left(\frac{1}{u(x)} - \mu\right)_+ = \max\left(0, \frac{1}{u(x)} - \mu\right) \quad \text{and} \quad E := \left\{x \in \Omega\,,\, \frac{1}{u(x)} \leq \mu\right\}.$$

Then for each $\alpha/in(0, 1/4)$, *we have that*

$$\int_\Omega u^2 A\nabla\left(\frac{e^{\alpha\rho_A(\cdot,E,w)}\psi}{u}\right)\nabla\left(\frac{e^{\alpha\rho_A(\cdot,E,w)}\psi}{u}\right) + \int_\Omega \left(\frac{1}{u} - \mu\right) e^{2\alpha\rho_A(\cdot,E,w)}|\psi|^2 \leq \frac{1}{1-\alpha^2}\int_E \left(\mu - \frac{1}{u}\right)_+ |\psi|^2 \quad (10)$$

For the proof we refer to [P].

20.3 Equivalence between the landscape function and the Fefferman-Phong-Shen maximal function

Now that we have an uncertainty principle with the landscape function, we seek to make a link with the Fefferman-Phong-Shen maximal function m. For a weight function w, the maximal function is define by

$$\frac{1}{m(x,w)} = \sup\left\{r\,,\, \frac{1}{r^{n-2}}\int_{B(x,r)} w(y)\,dy \leq 1\right\}. \quad (11)$$

In the Fefferman-Phong-Shen uncertainty principle, the potential V in L is used as the weight function. Under certain conditions on V, which include the class of polynomial potentials, we prove the pointwise equivalence between u and m^2.

Theorem 4. *Retain the setting of Thm.1, and moreover, assume that* $n \geq 3$, *that* $\Omega = \mathbb{R}^n$, *and suppose that* V *verifies:*

(V1) (Scale-invariant Kato Condition). There exist positive constants C_0 *and* δ *such that for all* $x \in \mathbb{R}^n$ *and all* $0 < r < R$,

$$\int_{B(x,r)} V(y)\,dy \leq C_0\left(\frac{r}{R}\right)^{n-2+\delta}\int_{B(x,R)} V(y)\,dy$$

(V2) (Doubling on balls with high mass). There exists a positive constant C_1 such that for all $x \in \mathbb{R}^n$ and all $r > 0$,

$$\int_{B(x,2r)} V(y)\,dy \le C_1 \left[\int_{B(x,r)} V(y)\,dy + r^{n-2}\right]$$

Then eq.(6) holds, and there exists $C > 0$, depending only on n, λ, C_0, C_1, δ, such that

$$\frac{1}{C}\frac{1}{m(x,V)^2} \le u(x) \le C\frac{1}{m(x,V)^2}, \tag{12}$$

for each $x \in \mathbb{R}^n$, where $m(\cdot,V)$ is the maximal function from (11), and u is the landscape function defined in Thm.1.

The potentials satisfying conditions (V1) and (V2) are called Shen potentials. The main steps of the proof are as follows (we drop the V in the argument of $m(\cdot,V)$). The left inequality is derived using [S99]. For $x \in \mathbb{R}^n$, we have:

$$G(x,y) \ge \frac{c_n}{|x-y|^{n-2}} \tag{13}$$

for each $y \in B(x, \frac{c}{m(x)})$, where c_n depends only on n and A. Hence:

$$u(x) = \int_{\mathbb{R}^n} G(x,y)\,dy \ge \int_{B(x,\frac{c}{m(x)})} G(x,y)\,dy \ge \int_0^{c/m(x)} \frac{c_n' r^{n-1}}{r^{n-2}} \ge \frac{C}{m(x)^2} \tag{14}$$

For the right inequality, let us fix $x \in \mathbb{R}^n$. Let A_k be annuli $A_k := B\left(x, \frac{2^k}{m(x)}\right) \backslash B\left(x, \frac{2^{k-1}}{m(x)}\right)$, $f_k := \mathbb{1}_{A_k}$ and $u_k := L^{-1} f_k$. Prove that $u = \sum_{k=0}^{\infty} u_k$. With the help of [MP] Thm. 4.16 and the Moser estimate, and using properties of the maximal function, we prove first $u_0(x) \le \frac{C}{m(x)^2}$ and for each $k \le 1$:

$$u_k(x) \le \frac{C}{m(x)^2} 2^{kn} e^{-\epsilon 2^{\frac{k}{k_0+1}}}.$$

The result follows. We leave technical details.

Eventually, we can prove the following properties of the landscape function for Shen potentials:

Theorem 5. *Retain the setting and assumptions of Thm.4. Then there exist* $C > 0$ *and* $k_0 \in \mathbb{N}$*, which depend only on* n*,* λ*,* C_0*,* C_1*, and* δ*, such that the following statements hold.*

(i) ($1/u$ *encodes* V*). For each* $x \in \mathbb{R}^n$*, we have that*

$$\frac{1}{C}\frac{1}{u(x)} \leq \int_{B(x,\sqrt{u(x)})} V(y)\,dy \leq C\frac{1}{u(x)} \tag{15}$$

(ii) (Scale-invariant Harnack inequality). For each $x \in \mathbb{R}^n$*, if* $y \in B(x, \sqrt{u(x)})$*, then*

$$\frac{1}{C}\frac{1}{u(x)} \leq u(y) \leq C\frac{1}{u(x)}. \tag{16}$$

All these properties follow from the equivalence theorem and the similar properties of the maximal function.

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21 Hölder regularity and area distortion of quasiconformal mappings in the plane

After Astala, Iwaniec, and Martin [AIM09]

A summary written by Ignasi Guillén-Mola

Abstract

We give a brief introduction to quasiconformal mappings. We start by giving some equivalent definitions, and we conclude by proving the Hölder regularity and the area distortion.

21.1 Quasiconformal mappings

Definition 1. *An orientation-preserving*[1] *homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal, $1 \leq K < \infty$, if $f \in W^{1,2}_{loc}(\Omega)$ and*

$$\sup_{\alpha\in[0,2\pi)} |\partial_\alpha f(z)| \leq K \inf_{\alpha\in[0,2\pi)} |\partial_\alpha f(z)| \quad \text{for almost every } z \in \Omega. \qquad (1)^2$$

Remark 2. *A mapping is 1-quasiconformal if and only if it is conformal.*

Remark 3. *The inequality in (1) is equivalent to*

$$|Df(z)|^2 \leq KJ(z,f) \text{ for almost every } z \in \Omega, \qquad (2)$$

$$|\partial_z f| + |\partial_{\bar z} f| \leq K\left(|\partial_z f| - |\partial_{\bar z} f|\right) \text{ for almost every } z \in \Omega, \text{ or}$$

$$|\partial_{\bar z} f| \leq k|\partial_z f| \text{ for almost every } z \in \Omega$$

for $k = (K-1)/(K+1)$.

Here we state some fundamental properties of quasiconformal mappings.

Theorem 4. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping from $\Omega \subset \mathbb{C}$ onto $\Omega' \subset \mathbb{C}$ and let $g : \Omega' \to \mathbb{C}$ be a K'-quasiconformal mapping. Then*

- *$f^{-1} : \Omega' \to \Omega$ is K-quasiconformal.*
- *$g \circ f : \Omega \to \mathbb{C}$ is KK'-quasiconformal.*
- *For all measurable sets $E \subset \Omega$, $|E| = 0$ if and only if $|f(E)| = 0$.*
- *The Jacobian determinant $J(z,f) > 0$ almost everywhere in Ω.*

[1] If $J(z,f) = |\det Df(z)| = |\partial_z f|^2 - |\partial_{\bar z} f|^2 \geq 0$ almost everywhere.

[2] $\partial_\alpha f(z) = Df(z)e^{i\alpha} = \cos(\alpha)\partial_x f(z) + \sin(\alpha)\partial_y f(z) = \lim_{r\to 0} \frac{f(z+re^{i\alpha})-f(z)}{r}$.

21.1.1 Equivalent definition via Beltrami equation

Quasiconformal mappings are related to solutions of the Beltrami equation (3). Write $\mu(z) = \partial_{\bar{z}} f(z)/\partial_z f(z)$ where $\partial_z f(z) \neq 0$, and $\mu(z) = 0$ otherwise. With this, we have the following equivalence.

Theorem 5. *Suppose $f : \Omega \to \Omega'$ is a homeomorphic $W^{1,2}_{loc}(\Omega)$-mapping. Then f is K-quasiconformal if and only if*

$$\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z) \text{ for almost every } z \in \Omega, \tag{3}$$

where μ, called the Beltrami coefficient of f, is a bounded measurable function satisfying

$$\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1.$$

For compactly supported Beltrami coefficient μ, we call the *principal solution* the function $f \in W^{1,2}_{loc}(\mathbb{C})$ with $\partial_{\bar{z}} f = \mu \partial_z f$ normalized by the condition $f(z) = z + \mathcal{O}(1/z)$ near infinity. The measurable Riemann mapping theorem ensures the existence and uniqueness of the principal solution.

Theorem 6 (Measurable Riemann mapping theorem)**.** *Let $|\mu| \leq k < 1$ be compactly supported and defined on $\mathbb{C}$. Then there is a unique principal solution to the Beltrami equation*

$$\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z) \text{ for almost every } z \in \mathbb{C},$$

and the solution $f \in W^{1,2}_{loc}(\mathbb{C})$ is a K-quasiconformal homeomorphism of $\mathbb{C}$.

21.1.2 Quasisymmetric mappings

Definition 7. *Let $\eta : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism[3], $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$ a mapping. We say f is η-quasisymmetric if for each triple $z_0, z_1, z_2 \in A$ we have*

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right).$$

Should f be defined on an open set, we will assume that it is orientation-preserving and further, we say f is quasisymmetric if there is some η as above for which f is η-quasisymmetric.

[3] It implies $\eta(0) = 0$ and $\eta(t) \to \infty$ as $t \to \infty$

Remark 8. *Let $f : \Omega \to \Omega'$ be an η-quasisymmetric mapping onto. Then $f^{-1} : \Omega' \to \Omega$ is $\{1/\eta^{-1}(1/\cdot)\}$-quasisymmetric.*

From the definition, it follows that quasisymmetric mappings are injective and continuous. In particular, quasisymmetric mappings are homeomorphism onto their image by the previous remark.

21.1.3 Relation between quasiconformality and quasisymmetry

Quasiconformal maps and quasisymmetric maps are related to each other. More precisely, the following two theorems specify the relation.

Theorem 9. *Suppose that $f : \Omega \to \Omega'$ is an η-quasisymmetric mapping. Then f is quasiconformal. In particular, $f \in W^{1,2}_{loc}(\Omega)$.*

Theorem 10. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal homeomorphism. Then f is η-quasisymmetric, where η depends only on K.*

Note that for the quasiconformal to quasisymmetric direction, we need global quasiconformal. For a local equivalence, see [AIM09, Theorem 3.6.2].

21.2 Hölder regularity

The key points to prove the Hölder regularity of quasiconformal mappings (Theorem 11 below) are:

- Global quasiconformal maps are quasisymmetric.
- [4]Isoperimetric inequality $|\Omega| \le \frac{1}{4\pi}[\mathcal{H}^1(\partial\Omega)]^2$.

Theorem 11 (Mori's theorem)**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(0) = 0$ and $f(1) = 1$. Then there exists a constant C_K depending only on K such that*

$$|f(z)| \le C_K |z|^{1/K}, \quad 0 \le |z| \le 1.$$

The K-quasiconformal mapping $f(z) = z|z|^{1/K-1}$ shows that the Hölder exponent $1/K$ is optimal.

[4]For the precise statement of the isoperimetric inequality and a detailed proof, see [AIM09, Theorem 3.10.1].

Sketch of the proof. Let $t > 0$ and $B = B_t(0)$. By means of the isoperimetric inequality, Hölder inequality and K-quasiconformal condition (2),

$$\phi(t) := \int_B J \leq \frac{Kt}{2} \int_{\partial B} J = \phi'(t).$$

Equivalently $\frac{d}{dt}\left(t^{-2/K}\phi(t)\right) \geq 0$ for almost every $t > 0$. Integrating over $[t, 1]$ we obtain $\phi(t) \leq \phi(1)t^{2/K}$ for $0 \leq t \leq 1$.

From this (in second inequality) and quasisymmetry (in first and third inequality) we obtain

$$|f(z)| \leq C_K \left(\pi^{-1}|f(B_r(0))|\right)^{1/2} \leq C_K \left(\pi^{-1}|f(B_1(0))|\right)^{1/2} r^{1/K} \leq C_K^2 r^{1/K}.$$

□

Corollary 12. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal. Then there exists a constant C_K depending only on K such that*

$$|f(z) - f(w)| \leq C_K |f(R + w) - f(w)| \frac{|z - w|^{1/K}}{R^{1/K}}, \quad \textit{for } |z - w| \leq R.$$

Proof. Define $\widetilde{f}(\xi) = \frac{f(R\xi+w)-f(w)}{f(R+w)-f(w)} : \mathbb{C} \to \mathbb{C}$, which is K-quasiconformal with $\widetilde{f}(0) = 0$ and $\widetilde{f}(1) = 1$. Apply Theorem 11 to $\widetilde{f}$. □

The Hölder regularity of *global* quasiconformal mappings allows to see that quasiconformal mappings are locally Hölder continuous.

Corollary 13 (Locally Hölder)**.** *Every K-quasiconformal mapping $f : \Omega \to \Omega'$ is locally $\frac{1}{K}$-Hölder continuous. More precisely, if a disk $B \subset 2B \subset \Omega$, then*

$$|f(z) - f(w)| \leq \widetilde{C}_K \mathrm{diam}\,(f(B)) \frac{|z - w|^{1/K}}{(\mathrm{diam} B)^{1/K}}, \quad z, w \in B.$$

Sketch of the proof. Extend f everywhere K-quasiconformally, i.e., construct a K-quasiconformal mapping $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ such that $\widetilde{f}_{|B} = f_{|B}$. Then apply Corollary 12. □

21.3 Distortion of area

Before stating the area distortion theorem (see Theorem 18) we present some preliminary results. They describe simpler situations that will be used in the proof of Theorem 18.

Theorem 14. *Suppose f is a K-quasiconformal principal mapping of $\mathbb{C}$ that it is conformal outside a compact subset E. Then we have $|f(E)| \leq K|E|$.*

Theorem 15 (Weak [AIM09, Theorem 13.1.3])**.** *Suppose f is a principal K-quasiconformal mapping of $\mathbb{C}$ that is conformal outside the unit disk $\mathbb{D}$. Assume also that we are given a measurable set $E \subset \mathbb{D}$. If $f_{|E}$ is conformal, meaning that $f_{\bar{z}}(z) = 0$ for almost every $z \in E$, then*

$$\left(\frac{|E|}{\pi}\right)^{K} \leq \frac{|f(E)|}{\pi} \leq \left(\frac{|E|}{\pi}\right)^{1/K}. \tag{4}$$

Remark 16. *Note that Theorems 14 and 15 describe "opposite" situations. While Theorem 14 requires the mapping to be conformal outside the set, Theorem 15 requires the conformal condition inside the set.*

This will allow us to reduce to the case of Theorems 14 and 15 in the proof of Theorem 17. Note that the conclusion of Theorem 17 is the mixture of the conclusion in Theorem 14 and the right-hand side inequality in (4).

Theorem 17. *Suppose f is a K-quasiconformal principal mapping of $\mathbb{C}$ that is conformal outside the unit disk $\mathbb{D}$. Let $E \subset \mathbb{D}$ be measurable. Then $\frac{|f(E)|}{\pi} \leq K\left(\frac{|E|}{\pi}\right)^{1/K}$.*

Sketch of the proof. Let μ be the Beltrami coefficient of f. Define $\mu_0 = \mu\chi_{\mathbb{C}\setminus E}$ and denote g the principal K-quasiconformal mapping of $\mathbb{C}$ arising from the Beltrami coefficient μ_0.

Then the function $h = f \circ g^{-1}$ is K-quasiconformal in $\mathbb{C}$, conformal outside $g(\overline{E})$ (since $\mu_h = 0$ in $g(\overline{E})$ by [AIM09, Theorem 5.5.6][5]), and normalized by $h(z) = z + \mathcal{O}(1/z)$. Hence $f = h \circ g$.

By Theorem 14 we have $|f(E)| = |h(g(E))| \leq K|g(E)|$. By the choice of μ_0, the K-quasiconformal principal solution g is conformal in $(\mathbb{C} \setminus \mathbb{D}) \cup E$, and hence $\frac{|g(E)|}{\pi} \leq \left(\frac{|g(E)|}{\pi}\right)^{1/K}$ by (4). □

[5][AIM09, Theorem 5.5.6] gives the Beltrami coefficient of $h = f \circ g^{-1}$ in terms of μ_f and μ_g.

Note that we are still in the situation that the map is conformal outside the unit disk. The *area distortion theorem* completes the area distortion for general quasiconformal mappings.

Theorem 18 (Area distortion theorem)**.** *For every $K \geq 1$ there is a constant C_K, depending only on K, such that for any K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, for any disk $B \subset \mathbb{C}$ and for any subset $E \subset B$, we have*

$$\frac{1}{C_K}|f(B)|\left(\frac{|E|}{|B|}\right)^K \leq |f(E)| \leq C_K|f(B)|\left(\frac{|E|}{|B|}\right)^{1/K}$$

Sketch of the proof. The left-hand side inequality follows from the one on the right-hand side using that f is quasisymmetric. To prove the right-hand side inequality it suffices to prove it when B is the unit disc $\mathbb{D}$.

Let ϕ the principal quasiconformal mapping with Beltrami coefficient $\mu_\phi = \mu_f \chi_{\mathbb{D}}$. Then $f = \gamma \circ \phi$ where $\gamma : \phi(\mathbb{D}) \to f(\mathbb{D})$ is conformal (as in the proof of Theorem 17).

The core of the proof is to obtain (via conformal theory) $\frac{|f(E)|}{|f(\mathbb{D})|} = \frac{|\gamma \circ \phi(E)|}{|f(\mathbb{D})|} \leq C_K|\phi(E)|$. At this point apply Theorem 17 to ϕ to get $|\phi(E)| \leq C_K|E|^{1/K}$. □

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22 The Stoïlow factorisation theorem

After Astala, Iwaniec, and Martin [AIM09, §5]

A summary written by Gianmarco Brocchi

Abstract

We give a proof of the Stoïlow factorisation theorem for quasiconformal mappings on the plane. This result is then used to obtain a classification of quasiconformal mappings, and so uniqueness of (normalised) solutions to the Beltrami equation.

22.1 Introduction

The Stoïlow factorisation theorem says that two different solutions to the Beltrami equation are related by a holomorphic function.

22.1.1 Quasiconformal mappings

Let μ be a measurable function on $\Omega \subset \mathbb{C}$ with small L^∞-norm $\|\mu\|_\infty = \varepsilon < 1$. The weak solutions (in $H^1_{\text{loc}}(\Omega)$) to the Beltrami equation

$$\frac{\partial}{\partial \bar{z}} f(z) = \mu(z) \frac{\partial}{\partial z} f(z) \qquad \text{a. e. } z \in \Omega \subset \mathbb{C} \tag{B}$$

that are *homeomorphisms* are called quasiconformal mappings. Geometrically, these are maps of "bounded distorsion" on the plane.

The following result relates a homeomorphic solution to (B) and any other solution.

Theorem 1 (Stoïlow factorization)**.** *Let $\Omega \subset \mathbb{C}$, and let $f, g \in W^{1,2}_{loc}(\Omega)$ be two solutions to the same Beltrami equation* (B) *with f a quasiconformal map. Then there exists a holomorphic map Φ on $f(\Omega)$ such that*

$$g(z) = \Phi(f(z)) \qquad \textit{for all } z \in \Omega.$$

Moreover, for any holomorphic function Φ on $f(\Omega)$, the map $\Phi \circ f$ is a solution of (B)*.*

The Stoïlow factorisation can be used to parameterise solutions to (B) on the whole plane by their value at different points.

Corollary 2. *Let $f, g \in W^{1,2}_{loc}(\Omega)$ be two homeomorphic solutions to* (B) *on* $\mathbb{C}$. *If f and g fix the points* 0 *and* 1, *then $f = g$.*

Proof of the Corollary. By Stoïlow factorisation, there exists an entire function Φ such that $g = \Phi \circ f$. Since both f and g are homeomorphism, Φ has to be injective, so in particular Φ is conformal. Entire conformal maps are similarities (they preserve the ratio of distances), and a similarity which fixes 0 and 1 (and $\Phi(\infty) = \infty$) is the identity. □

We say that a quasiconformal homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is *normalised* if it fixes the origin and the point 1. Then the corollary above implies that normalised solutions to the Beltrami equation are unique.

Before going into the proof of the Theorem 1, we recall a few useful facts needed to find solutions to the Beltrami equation.

22.2 Solving the Beltrami equation

To find solutions to (B), we start by assuming that the Beltrami coefficient μ is smooth and compactly supported. Then, consider the inhomogeneous equation:

$$\frac{\partial}{\partial \bar{z}}\sigma = \mu(z)\frac{\partial}{\partial z}\sigma + \varphi \tag{1}$$

where $\varphi \in L^p(\mathbb{C})$ and compactly supported.

22.2.1 Cauchy and Beurling transform

A couple of operators are relevant to us: the Cauchy transform $\mathcal{C} := (\partial/\partial\bar{z})^{-1}$, mapping $\mathcal{C}\colon L^p(\mathbb{C}) \to W^{1,p}(\mathbb{C})$ for $p > 2$, which is the singular integral operator given by

$$\mathcal{C}f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{z-\zeta}d\zeta,$$

and the Beurling transform S, which is $\partial/\partial z \circ \mathcal{C}$, so it is given by

$$Su(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(\zeta)}{(z-\zeta)^2}d\zeta.$$

Remark 3. *The operator S is bounded on $L^p(\mathbb{C})$ and it exchanges the weak derivative $\partial/\partial\bar{z}$ with $\partial/\partial z$, in particular $S(\sigma_{\bar{z}}) = \sigma_z$ for $\sigma \in W^{1,p}(\mathbb{C})$.*

Let $u := (\partial/\partial\bar{z})\sigma$. By using the Beurling transform S, the inhomogeneous equation (1) is equivalent to

$$u = \mu(z)Su + \varphi.$$

The solution can be written as $u = (I - \mu S)^{-1}\varphi$, where we used the inverse of the Beltrami operator $(I - \mu(z)S)$. Indeed, the Neumann series of $(I - \mu(z)S)^{-1}$ converges if $\|\mu\|_\infty \|S\|_{L^p} < 1$. Since $\|\mu\|_\infty = \varepsilon < 1$, the Beltrami operator $I - \mu(z)S$ is invertible in a range of p where $\|S\|_p < 1/\varepsilon$.

Despite the fact that the exact value of $\|S\|_{L^p}$ is still unknown, it is a deep result that the maximal range of invertibility of $I - \mu(z)S$ is

$$I_\varepsilon := \left(1 + \varepsilon, 1 + \frac{1}{\varepsilon}\right).$$

Back to our inhomogeneous problem, a solution to (1) is given by the Cauchy transform of u:

$$\sigma = \left(\frac{\partial}{\partial\bar{z}}\right)^{-1} (I - \mu(z)S)^{-1}\varphi.$$

Given a solution to the inhomogeneous problem (1), a solution to the original Beltrami equation (B) is $f = z + \sigma$, by taking $\varphi = \mu$ in (1).

Now one can show that solution are smooth by a bootstrapping argument: first, taking $\mu = \varphi \in W^{1,p}(\mathbb{C})$ implies $\sigma \in W^{2,p}(\mathbb{C})$. Then the smoothness of μ implies $\sigma \in C^\infty$, and so the smoothness of f.

For the general case (μ only measurable with small L^∞ norm) one can solve (B) with the smooth approximation $\mu_\delta := \mu * \phi_\delta \to \mu$ as $\delta \to 0$, and ϕ smooth. Then exploit the compactness properties of the class of quasiconformal mappings and the boundedness of the Cauchy transform to prove the uniform convergence of the approximate solution f_δ. Details are in [AIM09, §5.3].

We now move to the proof of the factorisation theorem.

22.3 Proof of the Stoïlow factorisation

The idea of the proof is to show that the map

$$\Phi := g \circ f^{-1}$$

is continuous and then that is holomorphic. Note that g is not a priori continuous, but we will see that any $W^{1,2}_{\text{loc}}$-solution to the Beltrami equation (B) is indeed continuous. We will prove this result later. For the time being, we will assume the g is continuous and it has weak derivative in L^2.

Then it follows that Φ is continuous, because f is a homeomorphism. We want to show that $\partial/\partial\bar{w}\Phi \equiv 0$. By the chain rule:

$$\frac{\partial}{\partial \bar{w}}\Phi(w) = (g_z \circ f^{-1})(w)\frac{\partial}{\partial \bar{w}}(f^{-1})(w) + (g_{\bar{z}} \circ f^{-1})(w)\overline{\frac{\partial}{\partial w}(f^{-1})(w)}.$$

Since it can be seen that inverse function f^{-1} satisfies the equation

$$\frac{\partial}{\partial \bar{w}}(f^{-1}) = -\mu(f^{-1}(w))\overline{\frac{\partial}{\partial w}f^{-1}}$$

by rewriting the chain rule above, we have that

$$\frac{\partial}{\partial \bar{w}}\Phi = \overline{\frac{\partial}{\partial w}f^{-1}}\big[- g_z(f^{-1})\mu(f^{-1}) + g_{\bar{z}}(f^{-1})\big] = 0$$

because g satisfies the Beltrami equation.

To conclude, we invoke the Weyl's lemma, which states that weak solution to $\partial/\partial\bar{w}$ in $L^1_{\text{loc}}(\mathbb{C})$ are analytic. This shows that Φ is holomorphic and concludes the proof. □

It is left to show that $W^{1,2}_{\text{loc}}$-solutions to the Beltrami equation are continuous. This result exploits the L^p mapping property of the Beltrami operator presented above. In particular, when $\|\mu\|_\infty < \varepsilon < 1$, the inverse of the Beltrami operator is continuous on L^p for p in the range $I_\varepsilon \coloneqq (1+\varepsilon, 1+1/\varepsilon)$.

22.3.1 Continuity of $W^{1,2}_{\text{loc}}$ solutions

Theorem 4. *Let $\Omega \subset \mathbb{C}$. Let $f \in W^{1,2}_{loc}(\Omega)$ be a solution to* (B) *with $\|\mu\|_\infty = \varepsilon < 1$. Then $f \in W^{1,p}_{loc}(\Omega)$ for all $p \in I_\varepsilon$.*

In particular, $f \in W^{1,2+s}_{\text{loc}}(\Omega)$ for some $s > 0$, so by the Sobolev embedding f is continuous.

Sketch of the proof of Theorem 4. Consider the function ψf, for $\psi \in C^\infty_c(\Omega)$. Since f is a solution to the Beltrami equation, by the chain rule we have

$$(\psi f)_{\bar{z}} - \mu(\psi f)_z = f \cdot (\psi_{\bar{z}} - \mu\psi_z) =: \varphi.$$

By solving the inhomogeneous Beltrami equation for $F = \psi f$

$$F_{\bar{z}} = \mu F_z + \varphi$$

we find expressions for the weak derivative of F, that are

$$\begin{aligned} F_{\bar{z}} &= (I - \mu S)^{-1} \varphi \\ F_z &= S F_{\bar{z}} = S \circ (I - \mu S)^{-1} \varphi. \end{aligned}$$

The Sobolev membership of f then follows from the L^p boundedness of the partial derivative $F_{\bar{z}}, F_z$, which is a consequence of the L^p mapping property of the Beurling transform S and $(I - \mu S)^{-1}$ for p in the range I_ε. □

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23 Sign and area in nodal geometry of Laplace eigenfunctions

After F. Nazarov, L. Polterovich, M. Sodin, [NPS]

A summary written by Georgios Dosidis

Abstract

The paper deals with asymptotic nodal geometry for the Laplace–Beltrami operator on closed surfaces. Given an eigenfunction f corresponding to a large eigenvalue, we study local asymmetry of the distribution of $sign(f)$ with respect to the surface area. It is measured as follows: take any disc centered at the nodal line f = 0, and pick at random a point in this disc. What is the probability that the function assumes a positive value at the chosen point? It is shown that this quantity may decay logarithmically as the eigenvalue goes to infinity, but never faster than that. In other words, only a mild local asymmetry may appear. The proof combines methods due to Donnelly-Fefferman and Nadirashvili with a new result on harmonic functions in the unit disc.

23.1 Introduction

Consider a compact manifold S endowed with a C^∞ Riemannian metric g. Let $\{f_\lambda\}$, $\lambda \nearrow +\infty$, be any sequence of eigenfunctions of the Laplace–Beltrami operator Δ_g:

$$\Delta_g f_\lambda + \lambda f_\lambda = 0.$$

The eigenfunctions f_λ give rise to the nodal sets $L_\lambda = \{f_\lambda = 0\}$. For instance, when S is 2–dimensional, any nodal line L_λ at a singular point p looks like the union of an even number of smooth rays meeting at p at equal angles. In spite of this “infinitesimal simplicity”, the global picture of nodal sets for large λ becomes more and more complicated, partially due to the fact that L_λ is $\sim \frac{1}{\sqrt{\lambda}}$–dense in S.

A nodal domain is a connected component of the set $S \setminus L_\lambda$. All nodal domains can be naturally grouped into two subsets $S_+(\lambda) := \{f_\lambda > 0\}$ and $S_-(\lambda) := \{f_\lambda < 0\}$. Our starting point is two theorems due to Donnelly–Fefferman.

The first one is a "local version" of the Courant nodal domain theorem [DF2]: let $D \subset S$ be a metric ball and let U be any component of $S_+(\lambda) \cap D$ such that

$$U \cap \frac{1}{2} D \neq \varnothing \tag{1}$$

which means that U enters deeply enough into D[6]. Then

$$\frac{Volume(U)}{Volume(D)} > a \cdot \lambda^{-k} \tag{2}$$

where a depends only on the metric g and k only on the dimension of S.

The second result is the following quasi–symmetry theorem that was proven in [DF1] under the extra assumption that the metric g is real analytic. Let $D \subset S$ be a fixed ball. Then there exists Λ depending on the radius of the ball D and the metric g such that for all $\lambda > \Lambda$

$$\frac{Volume(S_+(\lambda) \cap D)}{Volume(D)} > a, \tag{3}$$

where $a > 0$ depends only on the metric g.

From the geometric viewpoint, there is a significant difference between the measurements presented above: the quasi-symmetry theorem (3) deals with a ball of fixed radius and large λ. In contrast to this, the local version of the Courant theorem (2) is valid for all scales and all λ's though the collection of balls depends on λ through the "deepness assumption" (1).

A natural problem arising from this discussion is to explore what remains of quasi–symmetry on all scales and for all λ, provided that the nodal set enters deeply enough into a ball: $L_\lambda \cap \frac{1}{2}D \neq \varnothing$. The main result in [NPS] is that only a mild local assymetry may appear.

Theorem 1. *Let S be a compact connected surface endowed with a smooth Riemannian metric g, and let f_λ, be an eigenfunction of the Laplace–Beltrami operator. Assume that the set $S_+(\lambda) := \{f_\lambda > 0\}$ intersects a metric disc $\frac{1}{2}D$. Then*

$$\frac{Area(S_+ \cap D)}{Area(D)} \geq \frac{a}{\log \lambda} \cdot \frac{1}{\sqrt{\log \log \lambda}}. \tag{4}$$

where the constant $a > 0$ depends only on g.

[6]Here $\frac{1}{2}D$ refers to the ball with the same center and half the radius of D

The optimality of Theorem 1 (up to the double logarithm) is showcased by the sphere $\mathbb{S}^2$.

Theorem 2. *Consider the 2–sphere* $\mathbb{S}^2$ *endowed with the standard metric. There exist a positive numerical constant* C*, a sequence of Laplace–Beltrami eigenfunctions* f_i*,* $i \in \mathbb{N}$ *corresponding to eigenvalues* $\lambda_i \to \infty$*, and a sequence of discs* $D_i \subset \mathbb{S}^2$ *such that each* f_i *vanishes at the center of* D_i *and*

$$\frac{Area(S_+ \cap D_i)}{Area(D_i)} \le \frac{C}{\log \lambda_i}. \tag{5}$$

23.2 The Proof in Five Steps

23.2.1 The Donnelly–Fefferman Bound

For any continuous function f on a closed disc D (in any metric space), define its doubling exponent $\beta(D, f)$ by

$$\beta(D, f) := \log \frac{\max_D |f|}{\max_{\frac{1}{2}D} |f|}.$$

The following fundamental inequality was established in [DF1] in any dimension. For any metric disc $D \subset S$ and any λ,

$$\beta(D, f_\lambda) \le a\sqrt{\lambda}.$$

where the constant a depends only on the metric g.

23.2.2 Reduction to harmonic functions

Assume now that $D \subset S$ is a disc of radius $\sim 1/\sqrt{\lambda}$. It turns out that on this scale the eigenfunction f_λ can be "approximated" by a harmonic function u on the unit disc $\mathbb{D}$. More precisely, the set $\{f_\lambda > 0\}$ can be transformed into the set $\{u > 0\}$ by a K–quasiconformal homeomorphism with a controlled dilation K. Moreover, the doubling exponent of u on $\mathbb{D}$ is essentially the same as that of f_λ in D. This idea originates in Nadirashvili's paper [N].

23.2.3 Topological interpretation of the doubling exponent

Let $u : \mathbb{D} \to \mathbb{R}$ be a non–zero harmonic function. Denote by $\nu(r\mathbb{T}, u)$ the number of sign changes of u on the circle $r\mathbb{T} = \{|z| = r\}$. Then

$$C^{-1}\big(\beta(\frac{1}{4}\mathbb{D}, u) - 1\big) \le \nu(\frac{1}{2}\mathbb{T}, u) \le C\big(\beta(\mathbb{D}, u) + 1\big)$$

where C is a positive numerical constant. This result goes back to Gelfond. We will need the inequality on the right only.

23.2.4 The Nadirashvili constant

Denote by $\mathcal{H}_d$ the class of all non–zero harmonic functions u on $\mathbb{D}$ with $u(0) = 0$ that have no more than d sign changes on the unit circle $\mathbb{T}$. Define the Nadirashvili constant

$$\mathcal{N}_d := \inf_{u \in \mathcal{H}_d} Area(\{u > 0\}).$$

Using an ingenious compactness argument, Nadirashvili [N] showed that $\mathcal{N}_d$ is strictly positive. The following result of [NPS] gives a satisfactory estimate of the Nadirashvili constant

Theorem 3. *There exists a positive numerical constant C such that for each $d > 2$,*

$$\frac{C^{-1}}{\log d} \leq \mathcal{N}_d \leq \frac{C}{\log d}.$$

The proof of Theorem 3 is the main innovation of [NPS].

23.2.5 Arbitrary Disks

The four steps described above yield Theorem in the case when the disc D is small, that is, of radius $\leq \lambda^{-1/2}$. The double logarithm term is the price we pay for the fact that the transition from the eigenfunction f_λ to the approximating harmonic function u is given by a quasiconformal homeomorphism, which in general is only Hölder. The case of an arbitrary (not necessarily small) disc D is based on the following standard argument: The nodal line $L_\lambda = \{f_\lambda = 0\}$ is $\sim 1/\sqrt{\lambda}$–dense in S. Hence every disc D with $L_\lambda \cap \frac{1}{2}D \neq \varnothing$ contains a disjoint union of small discs D_i whose centers lie on L_λ and such that the total area of these discs is $>$ const $\cdot$ Area(D). Since the area bound is already established for each D_i, it extends with a weaker constant to D. This completes the outline of the proof of Theorem 1.

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24 The Landis conjecture on exponential decay

After A. Logunov, E. Malinnikova, N. Nadirashvili, and F. Nazarov[LC]

A summary written by Martin Hsu

Abstract

We show that all "super-exponentially" decaying solutions to the Laplace equation with bounded real potential on $\mathbb{R}^2$ are trivial.

24.1 Introduction

Imagine an enormous hot surface with a relatively weak position-dependent radiative effect on the surface. A snapshot captured by the thermal camera reveals that heat dissipates super-exponentially across the surface. Intuitively speaking, the surface is "too cold" far out, and heat should flow outwards. As a result, the system cannot be in equilibrium at that very moment. To be more precise, we may describe the heat flow $u(x,t)$ by equation $u_t(x,t) = \Delta_x u(x,t) + V(x)u(x,t)$ with a radiative effect controlled by the bounded potential $V(x)$. The moral of the story is that, given a snapshot at time t_0 of the hot surface $u(\cdot,t_0) \not\equiv 0$, we shall expect that:

$$|u(x,t_0)| \lesssim e^{-|x|^{1+\epsilon}} \implies \Delta_x u(\cdot,t_0) + V(\cdot)u(\cdot,t_0) = u_t(\cdot,t_0) \not\equiv 0.$$

In other words, we have:

Conjecture 1 (Weak Landis conjecture[KL][KS]). *Let u be the solution to*

$$\Delta u + Vu = 0, \tag{1}$$

where V is real measurable and $|V| \leq 1$. Suppose, additionally,

$$|u(x)| \lesssim e^{-|x|^{1+\epsilon}}$$

for some $\epsilon > 0$, than $u \equiv 0$.

The paper resolves the above conjecture in dimension two.

24.2 Heuristic ideas

Indeed, by considering $u_\delta(\cdot) := u(\delta\cdot)$, we may zoom-in and see

$$(1) \implies \Delta u_\delta = \underbrace{-\delta^2 V}_{\ll 1} u_\delta \implies u_\delta \text{ is } \textbf{almost} \text{ harmonic}$$
$$\implies u \text{ is } \textbf{almost} \text{ harmonic.}$$

Therefore, we expect that u behaves "almost" like a harmonic function and exhibits certain levels of **rigidity**. If we can utilize that rigidity, as in Liouville's theorem to harmonic function, then Q.E.D.

24.3 Sketch of the proof

To turn those heuristic ideas into rigorous proof, the two concepts need to be made precise:

Step 1. Establish **"almost"** harmonicity:

- **Porous disk/domain**: Fixing a disk, we puncture as many well-chosen-sized separated holes away from the zeros of u on the disk. We now call the $u \neq 0$ part on the porous disk the porous domain. Such domain, due to the geometry of the nodal set[CF], has a small Poincaré constant, and by construction, u doesn't change its sign around each hole.
- **Distortion on the image**: Through an iterative process, the small Poincaré constant allows us to construct a "near 1" auxiliary solution ϕ to (1) on the porous domain. This "near 1" solution is the correct distortion on the image. Namely, after careful analysis, we show that the distorted function $f := \frac{u}{\phi}$ actually solves:

  $$\nabla \cdot (\phi^2 \nabla f) = 0 \text{ on the whole porous disk.} \quad (2)$$

  Since $\phi \sim 1$, this confirms that $u \sim f$ is "almost" harmonic.
- **Distortion on the domain**: On further inspection, (2) tells us that $\phi^2 \nabla f$ is a divergence-free vector field. It seems tempting to use the inverse gradient theorem:

  $$\phi^2 \nabla f = \nabla \times \tilde{f}, \quad (3)$$

rephrase (3) into Beltrami equation by setting $w := f + i\tilde{f}$:

$$\frac{\partial w}{\partial \bar{z}} = \underbrace{\frac{1-\phi^2}{1+\phi^2} \cdot \frac{f_x + i f_y}{f_x - i f_y}}_{=:\mu} \cdot \frac{\partial w}{\partial z}, \tag{4}$$

and then conclude that $f+i\tilde{f}$ is a quasi-conformal map and, therefore, has bounded distortion. However, there's a catch: constructing $\tilde{f}$ requires a simply connected domain. Thus, (3) and (4) can only be formulated locally on the porous disk. Yet, all hope isn't lost. We notice that the Beltrami coefficient μ is globally defined on the porous disk and can be extended by zero to the full complex plane. Via measurable Riemann mapping theorem[EQ], we may reverse engineer a global solution ψ to (4). Now, it becomes natural to compare the two solutions w and ψ. A version of Stoïlow factorization[EQ] states that, locally, we can find a holomorphic function W such that $w = W \circ \psi$. With some rearrangement, we may conclude that:

$$h := f \circ \psi^{-1} = \Re w \circ \psi^{-1} = \Re W \text{ is a harmonic function.} \tag{5}$$

Note that, although (5) is a local statement, since "harmonicity" is also a local property, it automatically implies that h is harmonic on the entire distorted porous disk. Moreover, after some normalization, the distortion of quasi-conformal map ψ can be further characterized by Mori's theorem[AF]. That is, ψ preserves distance in a loose sense. As a result, the distortion doesn't alter the general shape of the porous disk.

In summary, what we've achieved for now is the following:

$$u = \phi \cdot (h \circ \psi) \text{ on the porous disk.}$$

We're left to quantify the rigidity of h on the distorted porous disk.

Step 2. Quantify local **"rigidity"**:

- **Rigidity around circles/holes**: By construction, h inherits the sign-preserving property of u around holes. As a result, when restricted to a circle C around a hole, due to Harnack's inequality, we have the following rigidity condition:

$$|h| \approx \inf_C |h| \gtrsim |\nabla h| \text{ on an annulus around } C. \tag{6}$$

– **Connecting circles/holes**: Lastly, we leverage rigidity around circles/holes to a quantitative estimate across the distorted porous disk. Roughly speaking, the argument, at its core, could be simplified into the following scheme:

$$\inf_{C_j} |h| \underset{(6)}{\gtrsim} \sup_{C_j} |\nabla h| \overset{\textbf{Connecting } C_j\textbf{s and apply F.T.C.}}{\rightsquigarrow} \text{control on } |h| \textbf{ v.s. } \sup_{C_k} |h|.$$

The result is formulated in terms of the doubling index:

Theorem 2 (**Theorem 5.3** in [LC]). *Fix $\lambda < \frac{63}{64}$. Let $\bigsqcup_j D_j$ denote the union of all the holes. Given the a priori estimate:*

$$\frac{\sup_{D(0,R)\setminus \bigsqcup_j D_j} |h|}{\sup_{D(0,\lambda R)\setminus \bigsqcup_j D_j} |h|} \leq e^N,$$

then, for any $r \ll R$, we have the following lower bound:

$$\frac{\sup_{D(0,r)\setminus \bigsqcup_j D_j} |h|}{\sup_{D(0,R)\setminus \bigsqcup_j D_j} |h|} \geq \left(\frac{r}{R}\right)^{C(R+N)},$$

where $C \gg 1$ is an absolute constant.

With a little more effort, we can show that **Theorem 2** implies a stronger version of **Conjecture 1** in dimension two:

Theorem 3 (**Theorem 1.1** in [LC]). *Suppose u satisfies (1) on $\mathbb{R}^2$ with V real measurable and $|V| \leq 1$. There's an absolute constant $C \gg 1$ such that:*

$$|u(x)| \lesssim e^{-C|x|\log^{1/2}|x|}, \text{ for } |x| > 2 \implies u \equiv 0.$$

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