Commuting vector fields and the Frobenius theorem

$U \subset \mathbb{R}^n$ open, $X_1 \ldots X_N$ commuting vector fields

$$[X_i, X_j] = 0 \qquad 1 \le i, j \le N.$$

Flows. $\phi_j = \phi_{X_j}$ exchange

$$\underbrace{\phi(t_1 \ldots t_N, x)}: = \phi_1(t_1, \phi_2(t_2, \phi \ldots \qquad \phi_N(t_N, x)))$$

- does not depend on the order

2.3 Theorem (Frobenius). Let X_j $1 \le j \le N$ be smooth vector fields which are linearly independent at every point. Then the following is equivalent:

1. The commutators at every pt are in the span of the vector fields:
$$[X_j, X_k] = \sum_{l=1}^{N} a_{jkl}(x) X_l$$
2. There is a foliation of a neighborhood of any pt by smooth submanifolds of dimension N, i.e there exist coordinates so that the vector fields are contained in $\underline{\mathbb{R}^N} \times \{0\} \subset \mathbb{R}^n$.

Assume 1.

Proof: (1) Fix $x_0 = 0$. After a linear change of coordinates:

$$X_j(0) = e_j \quad \leftarrow \text{j-th unit vector.}$$

(2) $X_j(x) = \sum_{i=1}^{n} c_{ij}(x) e_i$, ~~$(c_{ij}(0))$ is invertible~~

$C = (c_{ij}(x))_{1 \le i,j \le N}$, $C(0)$ invertible.

$$\tilde{X}_j(x) = \sum_k (C(x))^{-1}_{jk} X_k$$

$$\tilde{X}_j(x) = e_j + \sum_{i=N+1}^{n} b_i e_i$$

Assumption 1 for X_j and $\tilde{X}_j$ is equivalent.

$$[\tilde{X}_j, \tilde{X}_k] = \sum_{l=N+1}^{n} b_{jkl} e_l \overset{1.}{=} \underbrace{\sum_{l=1}^{N} a_{jkl} X_l}$$

$\Rightarrow$ $a_{jkl} = 0$, $b_{jkl} = 0$

$\Rightarrow$ $(\tilde{X}_j)_{1 \le j \le N}$ commute.

$$\mathbb{R}^n = \underset{t}{\underset{\in}{\mathbb{R}^N}} \times \underset{y}{\underset{\in}{\mathbb{R}^{n-N}}}$$

$(t, y) \longrightarrow \phi\left(t, \begin{pmatrix} 0 \\ y \end{pmatrix}\right)$ flow defined by $\tilde{X}_j$

- Diffeomorphism.
- in (t,y) variable the vector fields $\tilde{X}_j$ are e_j. □

2. $\Rightarrow$ 1. trivial.

2.2 The symplectic structure.

2.2.1 Symplectic vector spaces

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, E d-dimensional $\mathbb{K}$ vector space.

2.4 Lemma. Let $\omega : E \times E \to E$ be a skew symmetric bilinear map. There exists a basis e_j so that for some $N \le d/2$

$$\omega\Big(\sum a_j e_j, \sum b_j e_j\Big) = \sum_{j=1}^{N} a_j b_{j+N} - a_{j+N} b_j$$

<u>Proof.</u> Let $e_1, \tilde{e}_2$ be vectors so that $\omega(e_1, \tilde{e}_2) \neq 0$.

$$e_2 := \frac{1}{\omega(e_1, \tilde{e}_2)} \tilde{e}_2 \quad \to \quad \omega(e_1, e_2) = 1$$

$$\omega(e_1, e_1) = \omega(e_2, e_2) = 0.$$

If there are no such vectors then $\omega = 0$, $N = 0$.

Recursion: $(e_j)_{1 \le j \le 2J}$, $\omega\Big(\sum_{j=1}^{2J} a_j e_j, \sum_{j=1}^{2J} b_j e_j\Big)$

$$= \sum_{j=1}^{J} a_{2j-1} b_{2j} - a_{2j} b_{2j-1}.$$

Suppose $\exists$ $f, \tilde{f}$ with $\omega(f, e_j) = \omega(\tilde{f}, e_j) = 0$, $\omega(f, \tilde{f}) \neq 0$. Define $e_{2J+1} = f$, $e_{2J+2} = \frac{1}{\omega(f, \tilde{f})} \tilde{f}$.

If not: $J = N$ done. (choose a basis in the set $\omega(f, e_j) = 0$, $1 \le j \le 2J$.) $\square$

2.5 Definition. We call ω symplectic if $2N = n$ in the lemma above.

Let ω be skew symmetric. We define

$$B: E \to E^*$$

by $B(x)(y) = \omega(x, y)$.

ω is symplectic $\Leftrightarrow$ B is invertible.

We define $J = B^{-1}$ in that case, $J: E^* \to E$.

On $\mathbb{R}^{2N} \ni (p, q)$ there is a canonical symplectic form

$$\omega_0((p_1, q_1), (p_2, q_2)) = (p_1, q_2)_{\mathbb{R}^n} - (p_2, q_1)_{\mathbb{R}^n}.$$

Then

$$J = \begin{pmatrix} 0 & 1_{\mathbb{R}^n} \\ -1_{\mathbb{R}^n} & 0 \end{pmatrix}, \quad J^{-1} = \begin{pmatrix} 0 & -1_{\mathbb{R}^n} \\ 1_{\mathbb{R}^n} & 0 \end{pmatrix}$$

2.6 Definition. Let (E, ω) be a $2N$ dimensional symplectic vector space. A subvector space $F \subset E$ of dimension N is called Lagrangian if $\omega|_{F \times F} = 0$.

Remark: $\mathbb{R}^N \times \{0\} \subset \mathbb{R}^{2N}$ is Lagrangian in $(\mathbb{R}^{2N}, \omega_0)$.

2.7 Definition. Let (E, ω) and $(F, \tilde{\omega})$ be symplectic vector spaces of dim $2N$. A linear map $A: E \to F$ is called symplectic, if

$$\omega(z_1, z_2) = \tilde{\omega}(Az_1, Az_2)$$

Remark: Symplectic $\Rightarrow$ invertible.

The symplectic group is the group of symplectic linear maps $(\mathbb{K}^{2N}, \omega_0) \to (\mathbb{K}^{2N}, \omega_0)$.

2.2.2 Symplectic manifolds

- submanifolds of $\mathbb{R}^d$
- argue always locally, work in open subsets of $\mathbb{R}^n$.

Primer on differential forms

Skew k linear maps $(\mathbb{K}^n)^k \to \mathbb{K}$, denote by Λ^k.

$T \in \Lambda^k$, $(v_1, \ldots, v_k) \to T(v_1 \ldots v_k)$

- changes sign if we exchange two vectors.

A basis is given by $dx_{j_1} \wedge dx_{j_2} \ldots \wedge dx_{j_k}$

with $1 \leq j_1 < j_2 \ldots < j_k \leq n$.

$$dx_{j_1} \wedge \ldots dx_{j_k} (e_{i_1}, \ldots e_{i_k}) = \begin{cases} \pm 1 & \text{if } (i_k) \text{ is a permutation of } (j_k) \\ 0 & \text{otherwise.} \end{cases}$$

Smash product $\underbrace{dx_i \wedge dx_j = dx_j \wedge dx_i}$

$$X \in \mathbb{R}^n, \quad \alpha \in \Lambda^k, \quad \underbrace{i_X \omega (X_2 \ldots X_n)}_{\Lambda^{k-1}} = \omega(X_1 \ldots X_n)$$

A differential k-form on $U \subset \mathbb{R}^n$, open, is a (cont. smooth) map $U \to \Lambda^k$

- $\wedge$ product, i_X pointwise
- exterior derivative:

$$d(f\, dx_{i_1} \wedge \quad \wedge dx_{i_k}) = df \wedge dx_{i_1} \quad \wedge dx_{i_k}.$$

$\phi : \underset{\mathbb{R}^n}{\underset{\cap}{U}} \to \underset{\mathbb{R}^n}{\underset{\cap}{V}}$ is a C^1 map we define the pull back of the k-form ω on V by

$$\phi^* \omega(x, X_1 \quad , X_k) = \omega(\phi(x))(D\phi X_1 \qquad D\phi X_k).$$

Rule:
$$\phi^*(\omega \wedge \nu) = \phi^*(\omega) \wedge \phi^*(\nu)$$
$$d\, \phi^* \omega = \phi^* d\omega$$

X vector field, ϕ_t flow. The Lie derivative of a k form ω is defined by

$$\mathcal{L}_X \omega = \frac{d}{dt} \phi^*(t)\, \omega \Big|_{t=0}$$

Rule $\boxed{\mathcal{L}_X \omega = i_X d\omega + d(i_X \omega)}$

2.2.4 Definition of symplectic manifolds and first properties

- Formulate everything in local coordinates.

2.8 Definition. A symplectic form is a closed 2 form of maximal rank at every point.* A manifold with a symplectic 2-form is called symplectic.

*on an even dim. manifold.

ω closed $\Leftrightarrow$ $d\omega = 0$

ω exact $\Leftrightarrow$ $\exists \alpha$: $\omega = d\alpha$

Poincaré lemma: ω closed on a convex set $\Rightarrow$ ω exact.

Remark: ω symplectic differential form $\Rightarrow$ $\omega(x)$ is a symplectic form on the tangent space.

<u>Question</u>: Can we choose coordinates so that the symplectic form becomes simple? Yes.

2.9 Theorem (Darboux). Let (M^{2N}, ω) be a symplectic manifold. For $z \in M$ $\exists$ neighborhood $\mathcal{U} \subset M$ and a diffeomorphism $\phi: \mathcal{U} \to V \subset \mathbb{R}^{2N}$ so that

$$\omega = \phi^* \omega_0 .$$

Equivalently: We can choose coordinates so that $\omega = \omega_0$.

<u>Proof.</u> (Different proof in Arnold, Section 43 B)

- Suffices to consider a symplectic form ω on $0 \in \mathcal{U} \subset \mathbb{R}^{2n}$.
- Use an argument of J. Moser.

By linear change of coordinates (Lemma 2.4) we may assume that $\omega(0) = \omega_0$

Define $\omega_t = (1-t)\omega_0 + t\omega \qquad 0 \le t \le 1.$

Then $d\omega_t = 0$, in a neighborhood ω_t is a symplectic 2-form.

$\omega_t(0) = \omega_0$

Idea of Moser: Construct vector fields $X(t,x)$ with $X(t,0) = 0$, define an evolution by

$$\varphi(0,x) = x, \qquad \frac{\partial}{\partial t}\varphi(t,x) = X(t, \varphi(t,x))$$

so that $\underline{\varphi^*(t)\,\omega_t = 0}$

Assume we found X, φ. $\qquad V, W \in \mathbb{R}^{2n}$

$$0 = \frac{d}{dt}\omega_0(V,W) = \frac{d}{dt}(\varphi^*\omega_t)(V,W)$$

$$= \varphi^*\left(\frac{d}{dt}\omega_t\right)(V,W) + \varphi^*\mathcal{L}_{X(t)}\omega_t(V,W)$$

$$= \varphi^*\left((\omega - \omega_0)\right)(V,W) + \varphi^*\Big(i_X \underbrace{d\omega_t}_{=0} + d(i_X\omega)\Big)(V,W)$$

$$= \varphi^*\left((\omega - \omega_0) + d(i_X\omega)\right)(V,W)$$

- $\omega - \omega_0$ closed, exact on small ball, $\omega - \omega_0 = d\alpha$

$= \varphi^t(d(\alpha + i_X\omega))(V,W)$.

- Solve $\boxed{\alpha + i_X\omega = 0}$ at every pt.
defines a vector field X, read the identities in the opposite order.

$\boxed{}$ pointwise identity of 1-forms

- $\alpha(x) \in (\mathbb{R}^{2N})^*$
- may assume: $\omega = \omega_0$, $i_X\omega(y) = \omega(X,y)$

$$= J^{-1}(X)(y)$$

$J^{-1}(X)$ is invertible,

$$X(x) = J\alpha(x)$$

$\square$

2.10 Theorem (Darboux–Weinstein thm). Let (M^{2N},ω) be symplectic manifold, $N^d \subset M^{2N}$ a submanifold. Let $\tilde{\omega}$ be a second symplectic diff. form so that $\tilde{\omega}|_N = \omega|_N$.

Then there exist two neighborhoods U_1, U_2 of N and a diffeomorphism $\phi: U_1 \to U_2$ so that $\tilde{\omega} = \phi^*\omega$.

2.11 Definition. Let (M^{2n}, ω) and $(N^{2n}, \tilde{\omega})$ be two symplectic manifolds. A symplectomorphism is diffeomorphism

$\Phi : M^{2n} \to N^{2n}$ such that $\omega = \Phi^* \tilde{\omega}$.