Integrable systems and nonlinear Fourier transforms

1. Introduction

Integrable systems connect many different areas.
The notion is only vaguely defined.

1. Liouville integrability. Hamiltonian ODEs.
   - general solutions can expressed in terms of integrals, implicit fcts and algebraic manipulations.
   - precise, not explicit
   - Somewhat stable KAM (Kolmogorov, Arnold, Moser) theory.

     Solar systems: Sun, planets
   - neglect attraction of planets: Sum of Kepler problems two body problems.
   - with attraction of planets; KAM: somewhat stable.

     difficult question: Consequences for the solar system.

2. Integrable systems: . Solvability by formulas'
   - rare, but important examples

Nonlinear Schrödinger equation $i\underline{\partial_t u} + \underline{u_{xx}} = \underline{\pm} 2|u|^2 u$

$(t,x) \in \mathbb{R}\times\mathbb{R}$, $u: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, $+$ defocusing
$-$ focusing.

Korteweg-de Vries equation: $u_t + u_{xxx} - 6uu_x = 0$

- Russel: Water waves, soliton, traveling.

· universal asymptotic equations for wave propagation.

3. Integrable systems are exceptional

but: there not many different such nonlinear structures

One feature: Lax pair structure

KdV $\qquad u_t + u_{xxx} - 6 u u_x = 0, \qquad u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

Schrödinger operator $\qquad L\psi = -\psi_{xx} + u\psi \qquad$ | u given

$$L = -\partial^2 + u$$

$$B\psi = -4\psi''' + 3\left(u\psi' + (u\psi)'\right)$$

$$B \doteq -4\partial^3 + 3(u\partial + \partial u) \quad \text{skew symmetric}$$

Then $\quad L_t = [B, L] \quad$ is equivalent to KdV

Operator identity: $\quad L_t = \partial_t u \quad$, multiplication.

$$[B, L] = -\partial^3_{xxx} u + 6 u u_x, \quad \text{multiplication}$$

$[B, L]$ commutator. $\quad [B, L] = BL - LB$

$$[-4\partial^3 + 3(\partial u + u\partial), -\partial^2 + u] = 4[-\partial^3, u] + 3[\partial u + u\partial, -\partial^2 + u]$$

($\partial u + u\partial$ " $2u\partial + u'$)

$$= -4u^{(3)} - 12 u^{(2)}\partial - 12 u'\partial^{(2)} + 6u''\partial + 12 u'\partial^2 + 6uu' + 3u^{(3)} \ldots$$

$$= -u^{(3)} + 6uu'.$$

$$L_t = [B, L] \iff u_t + u_{xxx} - 6uu_x = 0$$

Consider $\partial_t \psi = B(u(t))\,\psi$

Assume: Initial value problem is nicely solvable: $\psi(0)=\psi_0$

$$\mathcal{U}(t,0)\,\psi_0 = \psi(t) \qquad \text{linear evolution operator.}$$

Claim: $L(u(t)) = -\partial^2 + u(t) = \mathcal{U}(t,0)\, L(u(0))\,(\mathcal{U}(t,0))^{-1}$

Consequence: · $L(u(t))$ is similar to $L(u(0))$

- same up to a choice of basis
- same spectrum.

Proof of claim.

$$\underbrace{\frac{d}{dt}\overbrace{\left(L(u(t)) - \mathcal{U}(t,0)\, L(u(0))\,(\mathcal{U}(t,0))^{-1}\right)}^{f(t)}}$$

$$= [B(u(t)), L(u(t))] - B(u(t))\,\mathcal{U}(t,0)\, L(u(0))\,(\mathcal{U}(t,0))^{-1} + \mathcal{U}(t,0)\, L(u(0))\,(\mathcal{U}(t,0))^{-1}\, B(u(t))$$

Lax-pair, u solves KdV

$$= \underbrace{\left[B(u(t)),\ L(u(t)) - \mathcal{U}(t,0)\, L(u(0))\,(\mathcal{U}(t,0))^{-1} \right]}$$

$$|\quad| \leq C\, |f(t)| \qquad \text{(bound).}$$

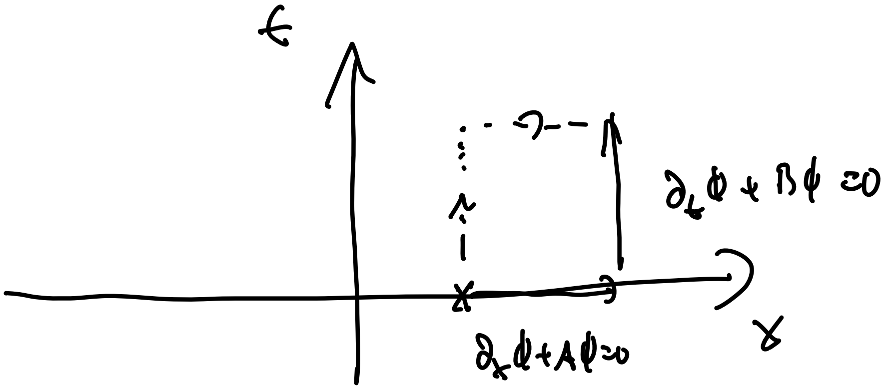
$$f(0)=0, \qquad |f'(t)| \leq C\, |f(t)| \qquad \text{Gronwall: } f \equiv 0.$$

Consider $L\psi = z^2\psi$, $\frac{\partial}{\partial t}\psi = B\psi$ $\quad z \in \mathbb{C}$, u as usual

Rewrite as 2×2 system:

$$\underbrace{\partial_x \phi + A(z)\,\phi = 0, \qquad \partial_t \phi + B(z)\,\phi}$$
$B(z)$: $(u, u_x, \dots)$.

- Initial value problem is solvable by linear ODEs.



Do we get the same?

Answer: Yes if

$$0 = [\partial_x + A, \partial_t + B] = B_x - A_t + [A, B]$$

<u>Interpretation:</u> Curvature of connection on a vector bundle vanishes.

Iff: u satisfies KdV.

- Close relation between KdV and the study of the spectrum of the Schrödinger.

<u>Outline</u>

1. Symplectic structures and Hamiltonian ODEs (Liouville integrability)
2. Examples of integrable ODEs
3. Korteweg - de Vries hierarchy.
4. KdV at H^{-1} after Killip & Visan

5. If time permits: Global solutions to the defocusing Nonlinear Schrödinger Equation after Ifrim and Tataru.

2. Hamiltonian equations

2.1 Vector fields and flows

2.1 Definition (Lie algebra). A Lie algebra is a vector space E with a skew symmetric bilinear map called Lie bracket

$$E \times E \ni (a,b) \to [a,b] \in E$$

which satisfies

$$[a,b] + [b,a] = 0 \qquad \text{skew symmetry}$$

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0 \qquad \text{Jacobi identity.}$$

Examples:

1. $(n \times n)$ matrices, commutator.

$$[A,B] + [B,A] = AB - BA + (BA - AB) = 0$$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$= A\cancel{B}C - A\cancel{C}B - B\cancel{C}A + C\cancel{B}A$$
$$+ B\cancel{C}A - C\cancel{A}B - B\cancel{A}C + A\cancel{C}B$$
$$+ C\cancel{A}B - C\cancel{B}A - A\cancel{B}C + B\cancel{A}C = 0$$

2. Bounded operators on Banach spaces.

3. Smooth vector fields on an open set $\mathcal{U}$ in $\mathbb{R}^n$.

- $C^\infty(\mathcal{U}, \mathbb{R}^n) \ni X = (a_j(x))_{1\le j\le n}$, $Y = (b_j(x))$
- Identify with differential operators

$$C^\infty(\mathcal{U}) \ni f \to f_X = \partial_X f = \sum_{j=1}^n a_j \partial_j f.$$

Commutator $[X, Y] f := \partial_X \partial_Y f - \partial_Y \partial_X f$

$$= \sum_{j,k=1}^d (a_j \partial_j b_k - b_j \partial_j a_k) \partial_k f$$

$[X, Y] \in C^\infty(\mathcal{U}, \mathbb{R}^n)$

skew symmetric, satisfies the Jacobi identity.

- Vector fields are derivations (Leibniz rule)

$$\partial_X (fg) = f \partial_X g + g \partial_X f.$$

- bilinear map $C^\infty(\mathcal{U}) \times C^\infty(\mathcal{U}, \mathbb{R}^n) \ni f \times X \to fX \in C^\infty(\mathcal{U}, \mathbb{R}^n)$

4. <u>Virasoro algebra</u>. Smooth vector fields on the unit circle

$X = f\partial$. Virasoro algebra is $C^\infty(S^1, \mathbb{C}) \times \underline{\mathbb{C}}$

with the Lie bracket

$$\Big[(\underset{\in C^\infty(S^1,\mathbb{R})}{f\partial}, \underset{\in \mathbb{C}}{\alpha}), (g\partial, \underset{\in \mathbb{C}}{\beta}) \Big] = \Big([f\partial, g\partial], \underbrace{\frac{1}{2\pi} \int_0^{2\pi} f' g'' \, dx}_{} \Big)$$

5. $\mathbb{R}^n$, $[a,b]=0$ <u>Lie algebra</u>

<u>Vector fields define flows.</u> X vector field,

$$\frac{d}{dt}x = \dot{x} = X(x) \qquad \text{system of ODEs},$$

$$x(0) = x_0$$

- This Cauchy problem has a unique maximal local solution.
- The dependence of initial data and parameters is smooth.

Denote $\phi_X(t, x_0) = x(t)$ $\qquad \phi_X$ flow.

$$\boxed{\frac{\partial}{\partial t}\phi_X(t, x_0) = X(\phi_X(t, x(t))}$$

$$\phi_X(0, x_0) = x_0$$

- $x_0, t \to \phi_X(t, x_0)$ is smooth.

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_0^j}\phi(t, x_0)\right) = \frac{\partial}{\partial x_0^j}\left(\partial_t \phi(t, x_0)\right)$$

chain rule

$$= \frac{\partial}{\partial x_0^j} X(\phi(t, x_0)) = DX(\phi(t, x_0))\left(\frac{\partial}{\partial x_0^j} D\phi(t, x_0)\right)$$

<u>2.2 Lemma.</u> The flows commute, i.e.

$$\phi_X(t, \phi_Y(s, x)) = \phi_Y(s, \phi_X(t, x)) \qquad (\forall x, |s|, |t| \text{ small})$$

iff the vector fields commute, $[X, Y] = 0$.

Proof. Suppose the flows commute.

$$0 = \frac{d}{dt}\Big(\phi_X(t, \phi_Y(s,x)) - \phi_Y(s, \phi_X(t,x))\Big)$$

$$= X(\phi_X(t, \phi_Y(s,x))) - D\phi_Y(s, \phi_X(t,x))\,[X(\phi_X(t,x))]$$

· evaluate at $t=0$, differential with respect to s:

$$0 = DX(\phi_Y(s,x))\, Y(\phi(s,x)) - DY(\phi_Y(s,x))\, D_x\phi(s,x)\cdot X(x)$$

evaluate at $s=0$

$$[X,Y] = \underbrace{DX(x)}\cdot Y(x) - DY(x)\cdot X(x) = 0$$

Suppose $[X,Y]=0$. $\qquad f(0)=0$

$$\frac{d}{ds}\Big[\overbrace{X(\phi_Y(s,x)) - D_x\phi(s,x)\,X(x)}^{f(s)}\Big]$$

$$= DX(\phi_Y(s,x))\,Y(\phi_Y(s,x)) - DY(\phi_Y(s,x))\,D_x\phi(s,x)\cdot X(x)$$

$$\overset{[X,Y]=0}{=} DY(\phi_Y(s,x))\Big(X(\phi_Y(s,x)) - D_x\phi(s,\phi_X(t,x))\cdot X(x)\Big)$$

$$= DY(\phi_Y(s,x))\cdot f(s)$$

$$\Rightarrow \left|\frac{d}{ds} f(s)\right| \le C\cdot|f(s)| \qquad C = \sup|DY|$$

$$\Rightarrow f = 0.$$

$$\boxed{X(\phi_Y(s,x)) = D_x\phi(s,x)\,X(x).}$$

Fix s,

$$\frac{d}{dt}\Big[\overbrace{\phi_X(t, \phi_Y(s,x)) - \phi_Y(s, \phi_X(t,x))}^{f(t)}\Big]$$

$$= X(\phi_X(t, \phi_Y(s,x)) - D_x \phi_Y(s, \phi_X(t,x))\, X(\phi_X(t,x))$$

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$$= \underbrace{X(\phi_X(t, \phi_Y(s,x))) - X(\phi_Y(s, \phi_X(t,x)))}$$

- $f(0) = 0$, $\quad |f'(t)| \leq C\, |f(t)|$

Lipschitz constant

Gronwall: $\quad f \equiv 0$

$$\Rightarrow \quad \phi_X(t, \phi_Y(s,x)) = \phi_Y(s, \phi_X(t,x))$$

flows commute.