Summer school

Brascamp–Lieb inequalities

Kopp, September 26 – October 01, 2021

Organized by Christoph Thiele and Pavel Zorin-Kranich
Funded by the Hausdorff Center for Mathematics

Topics

Unless specified otherwise, all results should be presented in the "geometric" case.

Heat flow

1. Geometric BL inequalities, motivation and special cases (Hölder, Young, Loomis-Whitney). Gaussian extremizers via heat flow [BCCT08, Section 3]. Alternative source: [BB19]. (*Ferrante*)

2. Reverse BL and Ehrhard inequalities via heat flow [BH09, Theorems 1 and 4]. (*Mnatsakanyan*)

3. Prékopa-Leindler and Shannon-Stam inequalities as limits of sharp direct/reverse Young [Gar02]. Also deduce Brunn-Minkowski and isoperimetric inequalities. Gaussian isoperimetric, Bobkov, and Gross log-Sobolev inequalities as limits of Ehrhard's inequality [Lat02]. The aim is to expose relations between the various inequalities and to compare the Euclidean and the Gaussian cases. (*Dosidis*)

4. Dimension conditions for non-geometric BL inequalities: [BCCT08, Section 5], using Gaussian extremizers, and [Mal19] using Hölder and induction on dimension. (*Bulj*)

Mass transport

5. Fenchel-Rockafellar duality, Kantorovich duality, Knott-Smith criterion [Vil03, Section 1 and Theorem 2.12] (*Bilz*)

6. Construction and regularity of Brenier's map [McC95], [ADM99, Theorem 1.3], used in the articles below. For Rockafellar's cyclical monotonicity theorem, see also [Vil03, Theorem 2.27] (*Costa de Sousa*)

7. BL and reverse BL via mass transport [Bar98]. (*Govindan Sheri*)

8. Gaussian extremizability of forward-reverse BL inequalities [BW18, Section 4]. Ingnore Section 5 and later, where it is studied under which conditions the inequalities hold for Gaussians. (*Ciccone*)

9. Near-extremizers for the isoperimetric inequality [FMP10, Theorem 1.1]. For simplicity, pretend that all sets have smooth boundaries. (*Nastasi*)

Stochastic representation formulas

10. Boué-Dupuis formula using any of the references [BD98; Bor00; Leh13] and background from stochastic analysis. (*Negro*)

11. BL and reverse BL via Boué-Dupuis formula [Leh14, up to Section 3]. Also explain relation between Gaussian and Euclidean case. (*Olmos*)

12. Forward-reverse geometric BL via Boué-Dupuis formula [CL21, Section 2]. (*de Dios*)

Gaussian inequalities

13. Functional Ehrhard inequality [Bor03, inequality (3)], [Bor07, Theorem 1.1]. (*Mauth*)
14. Functional Ehrhard inequality via stochastic minimax [vHan18]. (*I. Lee*)
15. Unification of Ehrhard and Prékopa-Leindler inequalities [Iva19]. (*Saari*)

Geometric invariant theory

16. Equivalence of capacity and rank non-decreasing property for positive operators, iterative procedure for finding a doubly stochastic scaling [Gur04, Lemma 4.5 and Theorem 4.6(1)]. (*Denson*)
17. Dimensional condition for BL inequalities via operator scaling [GGOW18, Sections 3 and 4]. (*F. Gonçalves*)
18. Real Kempf-Ness theorem [BL17, Theorem 1.1(i)]. If this fits in, upgrade Corollary 5.5 to the algebraic (separation by polynomials) version [MFK94, Corollary 1.2]. (*Brocchi*)
19. Polynomial certificate for finiteness of BL constants [Gre20, Section 3]. (*Srivastava*)

Further extensions

20. Non-commutative BL inequality [BSW19] (*Sovine*)
21. Nonlinear BL inequality [BBBCF20; Dun21] (*Lin*)

Entropy and fractional/subset inequalities

22. Subset entropy power inequality [MB07; MG19] and its relation to a conjectural fractional Young inequality [BMW11]. (*Weigt*)
23. Subadditivity of constants in Poincaré inequalities and the corresponding central limit theorem [Cou20]. (*Duncan*)
24. BL version of entropy power inequality [AJN19] (*Gonçalves Ramos*)

Schedule

	Mo	Tu	We	Th	Fr
9:00	1	7	13	16	22
10:05	2	8	14	17	23
11:10	3	9	15	18	24
Lunch					
14:30	4	10		19	
15:35	5	11		20	
16:40	6	12		21	

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