Functional Analysis & PDEs

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Problem Set 9

Problem 1: Sobolev spaces **(1 + 2 + 2 + 2) + 3 = 10 marks**

Given $n \in \mathbb{N}$ and subsets $\Omega \subset \mathbb{R}^n$, exhibit the maximal interval $I \subset [1, \infty]$ such that the following functions belongs to $\mathrm{W}^{1,p}(\Omega)$ for all $p \in I$:

$$\begin{aligned} &\text{(a)} \quad f(x) = |x|^\alpha, \quad \Omega = \mathrm{B}_1(0), \\ &\text{(b)} \quad g(x) = |x|^\alpha, \quad \Omega = \mathbb{R}^n \setminus \mathrm{B}_1(0), \\ &\text{(c)} \quad h(x) = \log(\log\Big(1 + \frac{1}{|x|}\Big)), \quad \Omega = \mathrm{B}_1(0), \\ &\text{(d)} \quad i(x) = |1 - |x|^2|, \quad \Omega = \mathrm{B}_2(0). \end{aligned}$$

depending on $\alpha \in \mathbb{R}$.
Finally, exhibit the smallest $n \in \mathbb{N}$ for which it is possible to find $u \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that for any open $U \subset \mathbb{R}^n$ there holds $u|_U \notin \mathrm{L}^\infty(U)$. For such n, construct an explicit element $u \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with this property.

Problem 2: Weak derivatives **10 marks**

Let $\Omega \subset \mathbb{R}^n$ be an open and connected set. Let $u \in \mathrm{W}^{1,1}(\Omega)$ be such that its weak gradient satisfies $Du = 0$ m^n-a.e. in Ω. Prove that u is constant.

Problem 3: Traces **8 + 2 = 10 marks**

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. Prove that there is no bounded linear operator $T\colon \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\partial\Omega)$ which satisfies

$$Tu = u|_{\partial\Omega} \ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \text{ for all } u \in \mathrm{C}(\overline{\Omega}).$$

Discuss this negative result in view of the trace operator on Sobolev spaces.

Problem 4: **10 marks**

Let $u \in \mathrm{C}^1(\overline{\mathrm{B}_1(0)})$ and $v \in \mathrm{C}^1(\overline{\mathrm{B}_2(0) \setminus \overline{\mathrm{B}_1(0)}})$. Let w be defined m^n-a.e. on $\mathrm{B}_2(0)$ by

$$w := \begin{cases} u & \text{in } \mathrm{B}_1(0), \\ v & \text{in } \mathrm{B}_2(0) \setminus \overline{\mathrm{B}_1(0)}. \end{cases}$$

Give necessary and sufficient conditions on u and v such that w belongs to $\mathrm{W}^{1,1}(\mathrm{B}_2(0))$.