Functional Analysis & PDEs

Dec 05, 2019
PROF. DR. H. KOCH
DR. F. GMEINEDER
Due: Dec 13, 2019

universitätbonn

Problem Set 8

Problem 1: Around the Baire theorem **5 + 5 = 10 marks**

Let X, Y, Z be real Banach spaces.

(a) Let $\beta\colon X \times Y \to \mathbb{R}$ be a bilinear map. Suppose that β is *partially continuous*, i.e., for all $x \in X$ the map $Y \ni y \mapsto \beta(x,y) \in \mathbb{R}$ is continuous and for all $y \in Y$ the map $X \ni x \mapsto \beta(x,y) \in \mathbb{R}$ is continuous. Prove that β is continuous.

(b) Let $T\colon X \to Y$ be linear, $J\colon Y \to Z$ linear, injective and bounded such that $JT\colon X \to Z$ is bounded, too. Prove that T is bounded, too.

Problem 2: Distributions I **3 + 4 + 3 = 10 marks**

Prove or disprove whether the following maps define elements $T \in \mathcal{D}'(\Omega)$:

(a) $\Omega = (0,1)$, $T\varphi := \sum_{n=2}^{\infty} \varphi^{(n)}(\frac{1}{n})$,

(b) $\Omega = \mathbb{R}$, $T\varphi := \sum_{n=1}^{\infty} \varphi^{(n)}(\frac{1}{n})$,

(c) $\Omega = \mathbb{R}^2$, $T\varphi := \int_0^{2\pi} \varphi(\cos(\alpha), \sin(\alpha))\,\mathrm{d}\alpha$,

where $\varphi \in \mathcal{D}(\Omega)$ in each of the cases.

Problem 3: Distributions II **5 + 5 = 10 marks**

Let $T \in \mathcal{D}'(\mathbb{R})$. Establish the following:

(a) For all compact sets $K \subset \mathbb{R}$ there exist $f \in \mathrm{C}(\mathbb{R})$ and a number $k \in \mathbb{N}_0$ such that

$$T\varphi = \int_{\mathbb{R}} f(x) \frac{\mathrm{d}^k \varphi}{\mathrm{d}x^k}(x)\,\mathrm{d}x \tag{3.1}$$

holds for all $\varphi \in \mathcal{D}(\mathbb{R})$ with support in K.

(b) Consider $T\colon \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ given by $T\varphi := \sum_{n=0}^{\infty} \varphi^{(n)}(n)$. Then $T \in \mathcal{D}'(\mathbb{R})$, but there are no f and k as in (a) such that (3.1) holds for all $\varphi \in \mathcal{D}(\mathbb{R})$ (!).

Problem 4: Distributions III **10 marks**

Exhibit a constant $c \in \mathbb{R}$ such that

$$\Box \frac{1}{\sqrt{t^2 - |x|^2}} = c\delta_0$$

holds in $\mathcal{D}(\mathbb{R} \times \mathbb{R}^2)$ (where $(t,x) \in \mathbb{R} \times \mathbb{R}^2$).