Functional Analysis & PDEs

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Problem Set 7

Problem 1: Baire **4 + 6 = 10 marks**

Prove the following:

(a) In an infinite dimensional normed space $(X, \|\cdot\|)$ any finite dimensional subspace is closed and has empty interior.

(b) Consider the space $\mathcal{P}(\mathrm{B}(0,1))$ of real polynomials $p\colon \mathrm{B}(0,1) \to \mathbb{R}$, where $\mathrm{B}(0,1) \subset \mathbb{R}^n$ denotes the open euclidean unit ball as usual. There is no norm $\|\cdot\|$ such that $(\mathcal{P}(\mathrm{B}(0,1)), \|\cdot\|)$ is a Banach space.

Problem 2: Baire **2.5 + 2.5 + 2.5 + 2.5 = 10 marks**

Given $j \in \mathbb{N}$, put

$$M_j := \Big\{ f \in \mathrm{L}^2([0,1])\colon \int_0^1 |f_j|^2 \,\mathrm{d}x \leq j \Big\}.$$

(a) Establish that $\mathrm{L}^2([0,1]) = \bigcup_{j \in \mathbb{N}} M_j$.

(b) Show that each M_n is a closed subset in $\mathrm{L}^1([0,1])$ (!).

(c) Show that the interior of each M_j in the norm topology of $\mathrm{L}^1([0,1])$ is empty.

(d) From (a)–(c) it appears that $\mathrm{L}^2([0,1])$ is the countable union of sets with empty interior. Explain why this does not contradict Baire's theorem.

Problem 3: Banach-Steinhaus **5 + 5 = 10 marks**

Prove the following:

(a) Let $x = (x_i)$ be a real sequence such that for any $y = (y_i) \in c_0(\mathbb{N})$ the series $\sum_i x_i y_i$ converges. Then $x \in \ell^1(\mathbb{N})$.

(b) Let $(X, \|\cdot\|)$ be a Banach space and let (x_j) be a convergent sequence in X; put $x := \lim_{j\to\infty} x_j$. Let $x^* \in X^*$ and let (x_j^*) be a sequence in X^* such that for all $y \in X$ there holds $x_j^*(y) \to x^*(y)$ as $j \to \infty$. Then there holds $\sup_j \|x_j^*\|_{X^*} < \infty$ and $x_j^*(x_j) \to x^*(x)$.

Problem 4: **6 + 4 = 10 marks**

(a) Consider, for $f \in \mathcal{S}(\mathbb{R})$, in one dimension the equation

$$-u'' + u = f,$$

where $u \in \mathcal{S}(\mathbb{R})$, too (Schwartz space). Establish, for $1 \leq p \leq \infty$, bounds of the form

$$c_1\|u\|_{\mathrm{L}^s(\mathbb{R})} + c_2\|u'\|_{\mathrm{L}^q(\mathbb{R})} + c_3\|u''\|_{\mathrm{L}^p(\mathbb{R})} \leq c_4\|f\|_{\mathrm{L}^p(\mathbb{R})},$$

where $c_i > 0$, $i = 1, 2, 3, 4$, and $1 \leq s, q \leq \infty$ are suitable numbers, each of them being independent of u, u', u'' and f. What can you assert if u, f are a priori known to be supported in $[0, 1]$? Try to find the optimal values of s and q.
Hint: Try to express u as $I(f)$, where I is a suitable integral operator.

(b) Let $n \geq 2$. Consider the equation $\Delta u = f$, where $u \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We recall from EPDE that

$$\int_{\mathbb{R}^n} |D^2 u|^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} |f|^2 \, \mathrm{d}x.$$

Argue whether, for $p \in \{1, \infty\}$, there exists a constant $c = c(n, p) > 0$ such that

$$\|D^2 u\|_{\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^{n\times n})} \leq c\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Compare your findings with (a).