Functional Analysis & PDEs

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Problem Set 6

Problem 1: Conditional expectation, revisited **5 + 5 = 10 marks**

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $\mathcal{F} \subset \mathcal{A}$ be a sub-σ-algebra of $\mathcal{A}$. Let $X \in \mathrm{L}^1(\Omega, \mathcal{A}, \mu)$.

(a) Show that there exists an $\mathcal{F}$-measurable function Y such that

$$\int_F X \,\mathrm{d}\mu = \int_F Y \,\mathrm{d}\mu \qquad \text{for all } F \in \mathcal{F}.$$

Moreover, establish that Y is uniquely determined (modulo μ-nullsets): If $\widetilde{Y}$ is another $\mathcal{F}$-measurable function with $\int_F X \,\mathrm{d}\mu = \int_F Y \,\mathrm{d}\mu$ for all $F \in \mathcal{F}$, then $\mu(\{Y \neq \widetilde{Y}\}) = 0$. *Hint:* Radon-Nikodým.

(b) Show that if $X \in \mathrm{L}^2(\Omega, \mathcal{A}, \mu)$, then $Y = \mathbb{E}[X|\mathcal{F}]$ μ-a.e., where $\mathbb{E}[X|\mathcal{F}]$ is given as in Problem 1(c) of sheet 3.

Problem 2: Riemann-Stieltjes & BV, continued 1 **5 + 5 = 10 marks**

Similarly to the last sheet, we call for $-\infty < a < b < \infty$ a function $u\colon [a,b] \to \mathbb{R}$ a *function of bounded variation* provided

$$V_a^b(u) := \sup_{a=t_0<\ldots<t_\ell=b} \sum_{i=1}^{\ell} |u(t_i) - u(t_{i-1})| < \infty.$$

Put $\mathrm{BV}([a,b]) := \{u\colon [a,b] \to \mathbb{R}\colon\ V_a^b(u) < \infty\}$. Establish the following:

(a) If $u \in \mathrm{BV}([a,b])$, then u has at most countably many discontinuity points.

(b) If $u \in \mathrm{BV}([a,b])$, then for each $t_0 \in (a,b)$ the limits $\lim_{t\searrow t_0} u(t)$ and $\lim_{t\nearrow t_0} u(t)$ exist.

Problem 3: Riemann-Stieltjes & BV, continued 2 **10 marks**

Let $u \in \mathrm{BV}([a,b])$ be left-continuous. Prove that $u \in \mathrm{BV}([a,b])$ if and only if there exists a finite Borel measure λ on $\mathcal{B}([a,b])$ and a λ-measurable function $\sigma\colon [a,b] \to \{-1,1\}$ such that

$$u(y) - u(x) = \int_{[x,y)} u\sigma \,\mathrm{d}\lambda \qquad \text{for all } a \le x < y \le b.$$

To do so, proceed as follows: Consider, for $u \in \mathrm{BV}([a,b])$ and $h \in \mathrm{C}([a,b])$

$$\Lambda_u(h) := \int_a^b h \,\mathrm{d}u := \sup_{a=t_0<\ldots<t_\ell=b} \sum_{i=1}^{\ell} \Big(\inf_{[t_{i-1},t_i]} h\Big) \cdot (u(t_i) - u(t_{i-1})).$$

If existent, $\Lambda_u(h)$ is referred to as *Riemann-Stieltjes integral* of h *with respect to the integrator* u. Use ideas from the last sheet to represent u as the difference of two monotone functions and aim to apply the Riesz representation theorem.

Problem 4: Riemann-Stieltjes & BV, continued 3 **5 + 5 = 10 marks**

In the situation of the previous two exercises, let $u \in \mathrm{BV}([a,b])$ be left-continuous and λ as in Problem 3.

(a) Demonstrate that

$$V_a^b(u) = \int_{[a,b]} \mathrm{d}\lambda.$$

(b) Defining H as in Theorem 2.14, show that $H \subsetneq \mathrm{BV}([0,1])$. Moreover, if $F \in H$ – so there exists $f \in \mathrm{L}^2([0,1])$ such that $\int_0^1 f \,\mathrm{d}x = 0$ and $F(x) = \int_0^x f(y)\,\mathrm{d}y$ for all $0 \leq x \leq 1$ – show that

$$V_0^1(F) = \int_{[0,1]} |f(y)|\,\mathrm{d}y.$$

Historical Remark. On the last sheet, we discussed the Kolmogorov-Riesz compactness theorem. The *Riesz* of this compactness theorem was Marcel Riesz, the younger brother of Frigyes Riesz – who in turn is the name behind the *Riesz representation theorem.* You have already encountered Frigyes Riesz during Analysis 3 – he is the name behind the *Riesz–Fischer theorem.* During your further studies in analysis you will frequently encounter theorems by the *Riesz brothers* – be sure to check out which brother the relevant theorems refer to!