Functional Analysis & PDEs

Nov 15, 2019
PROF. DR. H. KOCH
DR. F. GMEINEDER
Due: Nov 22, 2019

Problem Set 5

Problem 1: Ascoli-Arzelá I **3 + 7 = 10 marks**

Let $\Omega \subset \mathbb{R}^n$ open and bounded. Given $0 < \alpha \leq 1$, define for a function $u \colon \Omega \to \mathbb{R}$

$$[u]_{\alpha,\Omega} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

Denote $\mathrm{C}^{0,\alpha}(\Omega)$ the set of all $u \in \mathrm{C}_b(\Omega)$ such that

$$\|u\|_{\mathrm{C}^{0,\alpha}(\Omega)} := \|u\|_{\sup} + [u]_{\alpha,\Omega} < \infty.$$

(a) Prove that $0 < \alpha \leq \beta \leq 1 \Rightarrow \mathrm{C}^{0,\beta}(\Omega) \subset \mathrm{C}^{0,\alpha}(\Omega)$.

(b) Give a necessary and sufficient condition on $\alpha, \beta \in (0,1]$ such that the closed unit ball in $(\mathrm{C}^{0,\beta}(\Omega), \|\cdot\|_{\mathrm{C}^{0,\beta}(\Omega)})$ is a compact subset of $(\mathrm{C}^{0,\alpha}(\Omega), \|\cdot\|_{\mathrm{C}^{0,\alpha}(\Omega)})$.

Problem 2: Ascoli-Arzelá II **3 + 3 + 4 = 10 marks**

Recall the following terminology: A subset A of a metric space (X,d) is a called *relatively compact* provided its closure $\overline{A}$ is compact in (X,d).

Consider the following subsets of $\mathrm{C}([0,1])$ and decide (with proof) whether their closures are relatively compact in $\mathrm{C}([0,1])$ (for $\|\cdot\|_{\sup}$):

(a) $\mathcal{F}_1 := \{x \mapsto \sin(kx) \colon \; k \in \mathbb{N}\}$.

(b) $\mathcal{F}_2 := \{x \mapsto \frac{1}{\lambda}\cos(\lambda x) \colon \; \lambda > 0\}$.

(c) $\mathcal{F}_3 := \{x \mapsto g(x-k) \colon \; k \in \mathbb{N}\}$, where $g \colon \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(x) = 0$ for $x \notin [0,1]$.

In case $\mathcal{F}_i$, $i = 1,2,3$, is *not relatively compact* in $\mathrm{C}([0,1])$, explain carefully where the application of the Arzelá-Ascoli theorem fails.

Problem 3: Kolmogorov-Riesz compactness criterion, revisited **10 marks**

Let $1 \leq p < \infty$. Recall the Kolmogorov-Riesz compactness characterisation (Thm. 3.37 from the lectures): A closed subset $C \subset \mathrm{L}^p(\mathbb{R}^n)$ is compact if and only if

(i) C is bounded.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|h| < \delta$ and all $f \in C$ there holds

$$\|f(\cdot + h) - f\|_{\mathrm{L}^p(\mathbb{R}^n)} < \varepsilon.$$

(iii) For every $\varepsilon > 0$ there exists $R > 0$ such that for all $f \in C$ there holds

$$\|\chi_{\mathrm{B}_R^{\complement}(0)} f\|_{\mathrm{L}^p(\mathbb{R}^n)} < \varepsilon.$$

The main objective now is to get a better understanding of (i), (ii) and (iii) and their interplay.

(a) Prove that (ii) and (iii) already imply (i). *In this sense, the Kolmogorov-Riesz compactness characterisation is redundant; this was proved by* SUDAKOV *first.*

(b) Give explicit examples of closed sets $C \subset \mathrm{L}^p(\mathbb{R}^n)$ such that, whenever (ii) or (iii) are dropped as conditions on C, C is not compact in $\mathrm{L}^p(\mathbb{R}^n)$.

Problem 4: BV functions in one dimension **4 + 6 = 10 marks**

An summable function $u\colon (a,b) \to \mathbb{R}$ is said to be of *bounded variation* if and only if

$$V_a^b(u) := \sup\left\{\sum_{i=1}^{n-2} |f(x_{i+1}) - f(x_i)|\colon\ n \in \mathbb{N},\ a = x_0 < x_1 < ... < x_{n-1} < x_n = b\right\} < \infty.$$

(a) Prove or disprove whether $(0,1) \ni x \mapsto \sin(\frac{1}{x})$ is of bounded variation on $(0,1)$. Give an example of a discontinuous function of bounded variation $u\colon (0,1) \to \mathbb{R}$.

(b) Prove that if $u\colon (a,b) \to \mathbb{R}$ is of bounded variation, then it arises as the difference of two non-decreasing functions $f, g\colon (a,b) \to \mathbb{R}$: $u = f - g$. *Hint:* Put $f(x) := V_a^x u + \frac{u(x)}{2}$.

We will continue with this exercise on the next sheet.