Functional Analysis & PDEs

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Problem Set 4

Problem 1: Uniform convexity of Banach spaces **3 + 2 + 4 + 1 = 10 marks**

Let $(X, \|\cdot\|)$ be a Banach space. For $0 < \varepsilon \leq 2$, define the *modulus of convexity* of $\|\cdot\|$ by

$$\delta_X(\varepsilon) := \inf\left\{1 - \|\tfrac{1}{2}(x+y)\| \colon\ \|x\|, \|y\| < 1,\ \|x-y\| > \varepsilon\right\}.$$

We say that $\|\cdot\|$ is *uniformly convex* provided $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. If $\|\cdot\|$ is uniformly convex, we call $(X, \|\cdot\|)$ a *uniformly convex Banach space.*

(a) Let $X = \mathbb{R}^2$ and consider, for $1 \leq p \leq \infty$, the p-norms $|(x_1, x_2)|_p := (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ for $p < \infty$ and $|(x_1, x_2)|_\infty := \max\{|x_1|, |x_2|\}$ for $p = \infty$. Compute $\delta_X(\varepsilon) > 0$ for $0 < \varepsilon \leq 2$ for $(\mathbb{R}^2, |\cdot|_p)$ depending on p. Visualise the modulus of continuity conveniently at the corresponding unit balls.

(b) Let $\Omega \subset \mathbb{R}^n$ be measurable with $m^n(\Omega) > 0$. Show that $(\mathrm{L}^p(\Omega); \|\cdot\|_{\mathrm{L}^p(\Omega)})$ is uniformly convex if and only if $1 < p < \infty$.

(c) Show that the following are equivalent for a Banach space $(X, \|\cdot\|)$:

- $(X, \|\cdot\|)$ is uniformly convex.
- If $(x_j), (y_j) \subset X$ are such that $\lim_{j\to\infty}(2\|x_j\|^2 + 2\|y_j\|^2 - \|x_j + y_j\|^2) = 0$ and (x_j) is bounded, then $\lim_{j\to\infty} \|x_j - y_j\| = 0$.
- If $(x_j), (y_j) \subset X$ are such that $\|x_j\|, \|y_j\| < 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \|x_j + y_j\| = 2$, then $\lim_{j\to\infty} \|x_j - y_j\| = 0$.

(d) *Uniform convexity is not a topological property.* Prove that if $\|\cdot\|$ and $|||\cdot|||$ are two equivalent norms such that $(X, \|\cdot\|)$ is a uniformly convex Banach space, then $(X, |||\cdot|||)$ is not necessarily a uniformly convex Banach space.

Problem 2: Projections in uniformly convex spaces **10 marks**

Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and let $K \subset X$ be a non-empty, closed and convex set. Prove that for each $x \in X$ there exists a unique $\pi_K(x) \in K$ such that $\operatorname{dist}(x, K) = \|x - \pi(x)\|$.
Hint: Exercise 1(c).

Remark. By Sheet 2, Exercise 1, this property does not persist for non-uniformly convex Banach spaces.

Problem 3: Weak Lebesgue spaces **4 + 3 + 3 = 10 marks**

Given $1 \leq p < \infty$ and a measurable set $\Omega \subset \mathbb{R}^n$ with $m^n(\Omega) > 0$, define $\mathrm{L}^p_w(\Omega)$ as the collection of all measurable $u \colon \Omega \to \mathbb{R}$ such that

$$|u|_{\mathrm{L}^p_w(\Omega)} := \sup_{t>0}(t^p m^n(\{x\colon\ |u(x)| > t\})^{\frac{1}{p}} < \infty.$$

The space $\mathrm{L}^p_w(\Omega)$ is called *weak Lebesgue space* (alternatively: Marcinkiewicz space or (p,∞)-Lorentz space).

(a) Establish that $|\cdot|_{\mathrm{L}^p_w(\Omega)}$ is a quasinorm, i.e., there exists $c > 0$ such that for all $f, g \in \mathrm{L}^p_w(\Omega)$ there holds $|f+g|_{\mathrm{L}^p_w(\Omega)} \leq c(|f|_{\mathrm{L}^p_w(\Omega)} + |g|_{\mathrm{L}^p_w(\Omega)})$.

(b) Establish that $\mathrm{L}^p(\Omega) \subset \mathrm{L}^p_w(\Omega)$ and demonstrate that this inclusion is strict throughout.

(c) Is the unit ball $\{f \in \mathrm{L}^p_w(\Omega)\colon\ |f|_{\mathrm{L}^p_w(\Omega)} < 1\}$ a convex set? Justify your answer.

Historical remark. J. Marcinkiewicz (1910–1940) is one of the pivotal personas of the Polish functional analysis school. As many others, he did not survive the Second World War; in the six years prior to his death, he produced fifty papers – many of which have become classical results by now – and was an acclaimed functional analyst. A short biography is available on the Mac Tutor archive site:

http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Marcinkiewicz.html

George G. Lorentz (1910–2006) also had to suffer from the German occupation of Leningrad, however, survived the Second World War; check out his biography:

http://www-history.mcs.st-andrews.ac.uk/Biographies/Lorentz-George.html

Problem 4: The dual of L^1 **10 marks**

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, i.e., there exists a sequence $(A_j) \subset \mathcal{A}$ such that $\mu(A_j) < \infty$ for each $j \in \mathbb{N}$ and $\bigcup_j A_j = \Omega$. Establish that

$$(\mathrm{L}^1(\Omega, \mathcal{A}, \mu))^* \cong \mathrm{L}^\infty(\Omega, \mathcal{A}, \mu).$$

More precisely, prove that for any $f \in (\mathrm{L}^1(\Omega, \mathcal{A}, \mu))^*$ there exists a unique $g_f \in \mathrm{L}^\infty(\Omega, \mathcal{A}, \mu)$ such that

$$f(u) = \int_\Omega u \cdot g_f \, \mathrm{d}\mu \qquad \text{for all } u \in \mathrm{L}^1(\Omega, \mathcal{A}, \mu).$$

Also, show that the map $\varPhi\colon (\mathrm{L}^1(\Omega, \mathcal{A}, \mu))^* \ni f \mapsto g_f \in \mathrm{L}^\infty(\Omega, \mathcal{A}, \mu)$ is a bijective isometry.