Functional Analysis & PDEs

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Problem Set 3

Projections in Hilbert spaces

Problem 1: L^2-theory of conditional expectation **3 + 1 + 3 + 3 = 10 marks**

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space (consisting of a set Ω, a σ-algebra $\mathcal{A}$ on Ω and a probability measure μ on $\mathcal{A}$). We denote $\mathrm{L}^2(\Omega, \mathcal{A}, \mu)$ the space of $\mathcal{A}$-measurable maps $\Omega \to \mathbb{R}$ which are, in addition, square integrable for μ (and employ the usual quotient procedure to make $\mathrm{L}^2(\Omega, \mathcal{A}, \mu)$ a Banach space).

(a) Let $\mathcal{H}$ be a Hilbert space and $\{0\} \neq C \subset \mathcal{H}$ a closed subspace; call the corresponding operator $p \colon \mathcal{H} \ni x \mapsto p(x) \in C$ from Lemma 2.10 the *orthogonal projection onto C*. Prove that the orthogonal projection onto C is a bounded linear operator with $\|p\| = 1$.

(b) Let $\mathcal{F} \subset \mathcal{A}$ be a sub-σ-algebra. Demonstrate that $\mathrm{L}^2(\Omega, \mathcal{F}, \mu)$ is closed in $\mathrm{L}^2(\Omega, \mathcal{A}, \mu)$.

(c) Define $\mathbb{E}[\cdot|\mathcal{F}]$ as the $\mathrm{L}^2(\Omega, \mathcal{A}, \mu)$-orthogonal projection onto $\mathrm{L}^2(\Omega, \mathcal{F}, \mu)$. Explain why this is a well-posed definition and prove that

- for all $X \in \mathrm{L}^2(\Omega, \mathcal{A}, \mu)$ and $Y \in \mathrm{L}^2(\Omega, \mathcal{F}, \mu)$ there holds
$$\mathbb{E}[XY] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{F}]].$$
  Here, $\mathbb{E}[X] := \int_\Omega X \,\mathrm{d}\mu$ as usual.

The linear operator $\mathbb{E}[\cdot|\mathcal{F}]$ is called the *conditional expecation given $\mathcal{F}$*.

(d) Let $X \in \mathrm{L}^2([0,1], \mathcal{B}(([0,1])), \mathcal{L}^1)$ and let $\mathcal{A} := \sigma(\{[0, \frac{1}{2}]\})$ be the σ-algebra generated by $[0, \frac{1}{2}]$. Give a general formula for $\mathbb{E}[X|\mathcal{A}]$.

Problem 2: Direct sums and closedness **4 + (3 + 3) = 10 marks**

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

(a) Let U, V be closed *orthogonal* subspaces of $\mathcal{H}$, orthogonality meaning that $\langle u, v \rangle = 0$ for all $u \in U, v \in V$. Establish that the sum vector space $U + V$ is a closed subspace of $\mathcal{H}$, too.

(b) In general, we cannot omit orthogonality in the foregoing statement. Specifically, let $\mathcal{H} = \ell^2(\mathbb{N})$ and consider
$$U := \{x = (x_1, x_2, ...) \in \ell^2(\mathbb{N}) \colon\ x_{2j} = 0 \text{ for all } j \in \mathbb{N}\},$$
$$V := \{x = (x_1, x_2, ...) \in \ell^2(\mathbb{N}) \colon\ x_{2j+1} = jx_{2j} \text{ for all } j \in \mathbb{N}\}.$$
Prove that

(i) U, V are closed subspaces of $\mathcal{H}$ with $U \cap V = \{0\}$.

(ii) The sum vector space $U + V \subset \mathcal{H}$ is *dense* but *not closed* in $\mathcal{H}$.

Separability

Problem 3: Separability and bases **6 + 4 = 10 marks**

(a) Let μ be a measure on the Borel σ-algebra on $\mathbb{R}$. Give a necessary and sufficient condition on μ such that $(\mathrm{L}^\infty(\mathbb{R};\mu);\|\cdot\|_{\mathrm{L}^\infty(\mathbb{R};\mu)})$ is separable.

(b) Let $(X,\|\cdot\|)$ be a Banach space and let $(b_j)_{j\in\mathbb{N}}$ be a sequence in X. We say that $(b_j)_{j\in\mathbb{N}}$ is a *Schauder basis* of X provided for each $x\in X$ there exists a unique sequence $(x_j)\subset\mathbb{R}$ such that $x=\sum_j x_j b_j$, convergence of the series being understood with respect to $\|\cdot\|$. Show that if $(X,\|\cdot\|)$ has a Schauder basis, then $(X,\|\cdot\|)$ separable. Conclude that $(\mathrm{L}^\infty(\mathbb{R}^n);\|\cdot\|_{\mathrm{L}^\infty(\mathbb{R}^n)})$ does not possess a Schauder basis.

Remark. The converse implication, namely that separable Banach spaces must have a Schauder basis, is *wrong*. This had been a longstanding open problem, eventually solved by ENFLO in 1972, see here:

- P. Enflo: A counterexample to the approximation problem in Banach spaces. Acta Mathematica (130), No. 1, July 1973, 309–317.

Problem 4: Separability in metric spaces I **10 marks**

Let (X,d) be a metric space. Prove the following: If (X,d) is separable and $A\subset X$, then (A,d) is separable.

Problem 5: Separability in metric spaces II **6 + 4 = 10 marks**

Let (X,d) be a separable metric space.

(a) Prove that there exists an isometric map $\ell\colon (X,d)\to(\ell^\infty(\mathbb{N}),\|\cdot\|_{\ell^\infty(\mathbb{N})})$. *In this sense, every separable metric space is 'contained' in* $(\ell^\infty(\mathbb{N}),\|\cdot\|_{\ell^\infty(\mathbb{N})})$.

(b) Consider $\widetilde{\ell}^2(\mathbb{N})$, the space of real sequences $x=(x_j)$ such that

$$\|x\|_{\widetilde{\ell}^2(\mathbb{N})} := \Big(\sum_j \frac{|x_j|^2}{j^2}\Big)^{\frac{1}{2}} < \infty.$$

Show that $(\widetilde{\ell}^2(\mathbb{N}),\|\cdot\|_{\widetilde{\ell}^2(\mathbb{N})})$ is a separable Banach space which contains $\ell^\infty(\mathbb{N})$ as a *proper subspace*. This seems to be an apparent contradiction to (a) – clarify.

Operators on Hilbert spaces

Problem 6: Hilbert space adjoints **5 + 5 = 10 marks**

Let $k\in\mathrm{L}^2([0,1]\times[0,1];\mathbb{C})$ and $T_k\colon\ \mathrm{L}^2([0,1])\to\mathrm{L}^2([0,1];\mathbb{C})$ be given by

$$T_k f(x) := \int_0^1 k(x,y)f(y)\,\mathrm{d}y, \qquad \text{for } \mathcal{L}^1\text{-a.e. } x\in[0,1].$$

(a) Prove that $T_k\colon\ \mathrm{L}^2([0,1];\mathbb{C})\to\mathrm{L}^2([0,1];\mathbb{C})$ is a bounded linear operator with operator norm not exceeding $\|k\|_{\mathrm{L}^2([0,1]^2)}$.

(b) Show that T_k^* is of the form $T_{\widetilde{k}}$ for some $\widetilde{k}\in\mathrm{L}^2([0,1]\times[0,1];\mathbb{C})$ and determine $\widetilde{k}$. Give a sharp condition on k for T_k to be selfadjoint.

Sobolev spaces and Sturm-Liouville problems

Problem 7: Sobolev spaces in one space dimension **10 marks**

Define the space H as in Theorem 2.14 from the lectures. Define $\mathrm{W}_0^{1,2}((0,1))$ as the closure of $\mathrm{C}_c^1((0,1))$ in $\mathrm{C}_b([0,1])$ with respect to the norm $\|u\|:=\|u\|_{\mathrm{L}^2([0,1])}+\|u'\|_{\mathrm{L}^2([0,1])}$. Establish that $H=\mathrm{W}_0^{1,2}((0,1))$.

Problem 8: Sturm-Liouville problems and eigenvalues **10 marks**

Let $q\in\mathrm{C}([0,1])$ be given and suppose that $u\in\mathrm{C}^2([0,1])$ is such that $-u''+qu=\lambda u$. Show that λ necessarily is real. Moreover, show that if $\lambda\neq\lambda'$, then the solutions u and v of $-u''+qu=\lambda u$ or $-v''+qv=\lambda' v$, respectively, are L^2-orthogonal.