Functional Analysis & PDEs

Oct 18, 2019
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Due: Oct 25, 2019

Problem Set 2

Problem 1: Normed spaces and their unit balls **5 + 5 = 10 marks**

(i) Let X be a real vector space and $p\colon X \to [0,\infty)$ be a map which satisfies each of the following:

(a) $p(x) = 0 \Leftrightarrow x = 0$,

(b) $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

Demonstrate that p is a norm if and only if the corresponding unit ball $\{x \in X\colon\ p(x) \leq 1\}$ is convex.

(ii) As explained in the lectures, if K is a non-empty, closed and convex subset of a Hilbert space $\mathcal{H}$, then for any $x \in \mathcal{H}$ there exists a unique closest point $p(x) \in K$ of x. Considering $\mathbb{R}^2$ equipped with norms

$$\|x\|_1 := |x_1| + |x_2| \quad \text{or} \quad \|x\|_\infty := \max\{|x_1|, |x_2|\}, \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

demonstrate that the analogous statement does *not extend* to general Banach spaces. Give an explanation by drawing the corresponding unit balls for these norms.

Problem 2: Hardy spaces **2+2+2+2+2 = 10 marks**

Denote $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$ the complex unit disk and let $1 < p < \infty$. We define

$$\mathcal{H}^p(\mathbb{D}) := \left\{ f\colon \mathbb{D} \to \mathbb{C} \text{ holomorphic}\colon\ \|u\|_{\mathcal{H}^p(\mathbb{D})} := \Big(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\mathrm{i}\,t})|^p \,\mathrm{d}t \Big)^{\frac{1}{p}} < \infty \right\}$$

(a) Prove that $(\mathcal{H}^p(\mathbb{D}), \|\cdot\|_{\mathcal{H}^p(\mathbb{D})})$ is a Banach space.

(b) Prove that $(\mathcal{H}^p(\mathbb{D}), \|\cdot\|_{\mathcal{H}^p(\mathbb{D})})$ is a Hilbert space if and only if $p = 2$, in which case the inner product on $\mathcal{H}^2(\mathbb{D})$ is given by

$$\langle f, g\rangle_{\mathcal{H}^2(\mathbb{D})} := \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{\mathrm{i}\,t})\overline{g(re^{\mathrm{i}\,t})}\,\mathrm{d}t, \quad f, g \in \mathcal{H}^2(\mathbb{D}).$$

(c) Prove that an analytic function $f(z) = \sum_{k=0}^\infty a_k(f) z^k$ belongs to $\mathcal{H}^2(\mathbb{D})$ if and only if $(a_k(f))_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$.

(d) Demonstrate that, setting $e_k(z) := z^k$ for $k \in \mathbb{N}$, (e_k) is an orthonormal basis for $\mathcal{H}^2(\mathbb{D})$.

(e) Let $z_0 \in \mathbb{D}$ and consider the map $\Phi\colon \mathcal{H}^2(\mathbb{D}) \ni f \mapsto f(z_0)$. Prove that $\Phi \in (\mathcal{H}^2(\mathbb{D}))^*$.

Problem 3: A minimum problem in Hilbert spaces **10 marks**

Let $\mathcal{H}$ be a real Hilbert space and K be a non-empty, closed and convex subset of $\mathcal{H}$. Moreover, let $A\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be an *bounded, coercive bilinear form*: There exist $\mu_1, \mu_2 > 0$ such that A satisfies

$$|A(u,v)| \leq \mu_1 \|u\| \, \|v\| \qquad \text{for all } u, v \in \mathcal{H}$$

(boundedness) and

$$A(u,u) \geq \mu_2 \|u\|^2 \qquad \text{for all } u \in \mathcal{H}$$

(coercivity). Establish that there exists a unique minimiser $u \in K$ of the functional $\mathcal{E}\colon K \ni x \mapsto A(x,x) \in \mathbb{R}$.

Problem 4: Elliptic variational inequalities **3 + 4 + 3 = 10 marks**

Let $\mathcal{H}$ be a real Hilbert space and $K \subset \mathcal{H}$ non-empty, convex and closed. Let $A\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bounded, coercive bilinear form (in the sense as specified as in Problem 3) which, in addition, is symmetric. Let $f \in \mathcal{H}^*$ and consider the variational inequality

$$\begin{cases} u \in K, \\ A(u, v-u) \geq f(v-u) \quad \text{for all } v \in K. \end{cases} \tag{4.1}$$

We wish to prove the existence of some $u \in K$ solving (4.1). To do so, proceed as follows:

(a) Apply the Riesz representation theorem on the Hilbert space $(\mathcal{H}, A)$ to find an element $w \in \mathcal{H}$ such that $A(w,v) = f(v)$ holds for all $v \in \mathcal{H}$.

(b) Consider the projection $u := P_K(w)$ of w onto K. Show that u solves (4.1).

(c) Give a functional $\mathcal{J}\colon K \to \mathbb{R}$ such that if $u \in K$ is a minimiser for $\mathcal{J}$, then u solves (4.1). *In this sense,* (4.1) *can be interpreted as a Euler-Lagrange-inequality for the functional $\mathcal{J}$ on K.*