Functional Analysis & PDEs

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Problem Set 12

Problem 1: **10 marks**

Let K, L be non-empty, disjoint, closed and convex subsets of a normed space $(X, \|\cdot\|)$ such that one of the two sets is compact. Prove that there exists $x^* \in X^*$ such that $\sup_{x \in K} \mathrm{Re}(x^*(x)) < \inf_{v \in L} \mathrm{Re}(x^*(x))$. Does this inequality remain valid without the compactness assumption?

Problem 2:

Let $\ell \in (\ell^\infty)^*$. Establish that ℓ can be decomposed as $\ell = \ell_1 + \ell_2$, where $\ell_1((s_j)) = \sum_{j=1}^{\infty} s_j t_j$ for some suitable (t_j) and $\ell_2|_{c_0} \equiv 0$ *and that such a decomposition is unique.* Moreover, establish that $\|\ell\| = \|\ell_1\| + \|\ell_2\|$. Finally, conclude that any bounded linear functional on c_0 can be uniquely extended to a bounded linear functional on ℓ^∞ with the same norm.

Problem 3: Weak convergence I **10 marks**

Let (x_j) be a sequence in a finite dimensional normed space $(X, \|\cdot\|)$ which converges weakly to some $x \in X$. Prove that (x_j) converges to x with respect to $\|\cdot\|$.

Problem 4: Weak convergence II **10 marks**

Let (x_j) be a bounded sequence in a normed space $(X, \|\cdot\|)$. Prove that (x_j) converges weakly to some $x \in X$ if and only if there exists a subset $D \subset X^*$ such that the closure of its span is X^* and $\lim_{j \to \infty} x^*(x_j) = x^*(x)$ for all $x^* \in D$.