Functional Analysis & PDEs

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Problem Set 11

Problem 1: Weyl **10 marks**

Suppose that $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear map which satisfies $\langle Ay, y\rangle \geq |y|^2$ for all $y \in \mathbb{R}^n$. Furthermore, suppose that $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ is such that

$$\int_{\mathbb{R}^n} \langle A\nabla u, \nabla\varphi\rangle \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n).$$

By considering $\varphi := \Delta^-_{s,h}(\rho^2 \Delta^+_{s,h} u)$ for $s \in \{1, ..., n\}$ and $h > 0$, where

$$\Delta^\pm_{s,h} u(x) := \frac{u(x \pm he_i) - u(x)}{h},$$

establish that $u \in \mathrm{C}^\infty(\mathbb{R}^n)$.

Hint: Difference quotient characterisation of $\mathrm{W}^{1,2}$.

Problem 2: Hahn-Banach reloaded **5 marks**

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $M \subset X$ and $x \in X$. Establish that the following are equivalent:

(a) $x \in \overline{\mathrm{span}(M)}$,

(b) every bounded linear functional $f \in X^*$ with $f|_M = 0$ satisfies $f(x) = 0$.

Problem 3: Hahn-Banach reloaded **10 marks**

Let $(X, \|\cdot\|)$ be a normed space and $U \leq X$ be a closed subspace with $U \subset X$. Let $x \in X \setminus U$. Then there exists $x^* \in X^*$ with $x^*|_U = 0$, $\|x^*\| \leq 1$ and $x^*(x) = \mathrm{dist}(x, U)$.

Problem 4: Banach limits **10 + 5 = 15 marks**

Let $(x_j) \in X := \ell^\infty(\mathbb{N})$ (the bounded sequences with real entries) and define the *left shift* $T((x_j)) := (x_{j+1})$.

A linear functional $l\colon X \to \mathbb{R}$ is called *Banach limit* provided the following hold:

(a) l is *positive*: If $x_1, x_2, ... \geq 0$, then $l((x_j)) \geq 0$.

(b) $l \circ T = l$.

(c) $l((1, 1, 1, 1, 1, 1, ...)) = 1$.

(I) Establish that a Banach limit satisfies each of the following:

(a) If $x_j \leq y_j$ for all $j \in \mathbb{N}$, then $l((x_1, x_2, ...)) \leq l((y_1, y_2, ...))$.

(b) l is bounded with norm at most 1.

(c) For all $x = (x_j) \in \ell^\infty(\mathbb{N})$ there holds $\liminf_{j\to\infty} x_j \leq lx \leq \limsup_{j\to\infty} x_j$.

(d) l is not multiplicative: There exist $x = (x_j), y = (y_j) \in \ell^\infty(\mathbb{N})$ such that $l((x_1y_1, x_2y_2, ...)) \neq l(x)l(y)$.

(II) Establish the existence of a Banach limit. *Hint:* Hahn-Banach.