Functional Analysis & PDEs

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Problem Set 10

You can achieve up to 40 bonus marks on this sheet.

Problem 1: Hardy's inequality **10 marks**

Prove that for any $n \geq 3$ there exists $c = c(n) > 0$ such that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x \leq c \int_{\mathbb{R}^n} |Du(x)|^2\,\mathrm{d}x \qquad \text{for all } u \in \mathrm{W}^{1,2}(\mathbb{R}^n).$$

To do so, observe that $|Du + \lambda\frac{x}{|x|^s}u(x)|^2 \geq 0$ for all $\lambda \in \mathbb{R}$ and $s > 0$.

Problem 2: Fourier characterisation of $\mathrm{W}^{k,2}(\mathbb{R}^n)$ **10 marks**

Let $k \in \mathbb{N}_{\geq 1}$. Establish that the following are equivalent for $u \in \mathrm{L}^2(\mathbb{R}^n)$:

(a) $u \in \mathrm{W}^{k,2}(\mathbb{R}^n)$.

(b) There holds

$$\|u\|_{H^k(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\widehat{u}(\xi)|^2\,\mathrm{d}\xi\Big)^{\frac{1}{2}} < \infty.$$

Problem 3: Embeddings into $\mathrm{L}^\infty(\mathbb{R}^n)$, revisited **10 marks**

Let $k, n \in \mathbb{N}$ be such that $k > \frac{n}{2}$. Use the foregoing exercise to establish that $\mathrm{W}^{k,2}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^\infty(\mathbb{R}^n)$. More precisely, establish that there exists a constant $c = c(k,n) > 0$ such that

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \leq c\|u\|_{\mathrm{W}^{k,2}(\mathbb{R}^n)} \qquad \text{for all } u \in \mathrm{W}^{k,2}(\mathbb{R}^n).$$

Deduce from this estimate that every $u \in \mathrm{W}^{k,2}(\mathbb{R}^n)$ has a continuous representative and conclude that

$$\bigcap_{k\in\mathbb{N}} \mathrm{W}^{k,p}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^\infty(\mathbb{R}^n)$$

for any $1 \leq p \leq \infty$.

Problem 4: Removing points **10 marks**

Let $\mathrm{B}_1(0) \subset \mathbb{R}^2$ be the unit ball in two dimensions. Establish that $\mathrm{W}_0^{1,2}(\mathrm{B}_1(0)) = \mathrm{W}_0^{1,2}(\mathrm{B}_1(0) \setminus \{0\})$.

Problem 5: Negative Sobolev spaces **10 marks**

Let $\Omega \subset \mathbb{R}^n$ be open and let $\ell \Subset \Omega$ be a non-trivial line segment. Determine (depending on n) all numbers $m \in \mathbb{N}$ such that the functional

$$F_\ell(\varphi) := \int_\ell \varphi \,\mathrm{d}\mathcal{H}^1, \qquad \varphi \in \mathrm{C}_c^\infty(\Omega)$$

extends to an element $\mathrm{W}^{-m,2}(\Omega)$.

Problem 6: Embeddings on domains **10 marks**

Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 < x_1 < 1,\ 0 < x_2 < x_1^2\}$. Prove that there exists $p > 2$ such that there exists no constant $c > 0$ such that $\|\varphi\|_{\mathrm{L}^\infty(\Omega)} \leq c\|\varphi\|_{\mathrm{W}^{1,p}(\Omega)}$ for all $\varphi \in \mathrm{W}^{1,p}(\Omega)$. Discuss this in view of Morrey's embedding known from the lectures.

Conclude that there exists $p > 2$ such that Ω is *not* an extension domain for $\mathrm{W}^{1,p}$. By this we understand that there is no bounded linear operator $E\colon\ \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^n)$ such that $Eu = u$ $\mathcal{L}^n$-a.e. on Ω for all $u \in \mathrm{W}^{1,p}(\Omega)$.

Problem 7: p-energies **5 + 5 = 10 marks**

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Given $1 < p < \infty$, consider the energy

$$\mathcal{J}_p[u] := \int_\Omega \frac{1}{p}|\nabla u|^p - fu \,\mathrm{d}x, \qquad u \in \mathrm{W}_0^{1,p}(\Omega),$$

where $f \in \mathrm{L}^{p'}(\Omega)$ is given.

(a) Establish that $m_p := \inf_{\mathrm{W}_0^{1,p}(\Omega)} \mathcal{J}_p[u] \in (-\infty, \infty)$.

(b) Pick a sequence (u_j) in $\mathrm{W}_0^{1,p}(\Omega)$ such that $\mathcal{J}_p[u_j] \to m_p$ as $j \to \infty$. Establish that (u_j) is a Cauchy sequence in $\mathrm{W}^{1,p}(\Omega)$ and hereafter the existence of a minimiser of $\mathcal{J}_p$ – that is, an element $u \in \mathrm{W}_0^{1,p}(\Omega)$ such that $\mathcal{J}_p[u] \leq \mathcal{J}_p[v]$ for all $v \in \mathrm{W}_0^{1,p}(\Omega)$.
Hint: Uniform convexity of L^p-spaces – use ideas known from the proof of separability of Sobolev spaces to conclude the uniform convexity of the relevant Sobolev spaces.

Problem 8: p-Laplacean **10 marks**

In the situation of problem 7, establish that the minimiser u from 7(b) is unique and satisfies $-\Delta_p u = f$ in $\mathcal{D}'(\Omega)$, i.e., for all $\varphi \in \mathrm{C}_c^\infty(\Omega)$ there holds

$$\int_\Omega |Du|^{p-2}Du \cdot D\varphi = \int_\Omega f\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}_c^\infty(\Omega).$$

Merry Christmas & a Happy New Year 2020 –
see you in the next decade!