Functional Analysis & PDEs

Oct 11, 2019

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Due: Oct 18, 2019

Problem Set 1

Problem 1: C^k and $\mathrm{C}^{k,\alpha}$ are Banach **4 + 6 = 10 marks**

Let $\Omega \subset \mathbb{R}^n$ be open. Moreover, let $k \in \mathbb{N}$ and $0 < \alpha < 1$. We put

$$\mathrm{C}^k(\overline{\Omega}) := \left\{ u\colon \Omega \to \mathbb{R}\colon \begin{array}{c} u \text{ is } k\text{-times differentiable and} \\ \partial^\beta u \in \mathrm{C}(\overline{\Omega}) \text{ for all } |\beta| \leq k \end{array} \right\}$$

and define for $u \in \mathrm{C}^k(\Omega)$

$$\|u\|_{\mathrm{C}^k(\overline{\Omega})} := \sum_{|\beta|\leq k} \|\partial^\beta u\|_{\sup},$$

where $\|\cdot\|_{\sup}$ is the supremum norm as usual.

(a) Prove that $\|\cdot\|_{\mathrm{C}^k(\overline{\Omega})}$ is a norm on $\mathrm{C}^k(\overline{\Omega})$, making $(\mathrm{C}^k(\overline{\Omega}), \|\cdot\|_{\mathrm{C}^k(\overline{\Omega})})$ a Banach space.

Now define, for $v\colon \Omega \to \mathbb{R}$, its α-th *Hölder seminorm* by

$$[v]_{\mathrm{C}^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|v(x)-v(y)|}{|x-y|^\alpha}$$

and consider the *Hölder space*

$$\mathrm{C}^{k,\alpha}(\overline{\Omega}) := \left\{ u \in \mathrm{C}^k(\overline{\Omega})\colon\ \|u\|_{\mathrm{C}^{k,\alpha}(\overline{\Omega})} := \|u\|_{\mathrm{C}^k(\overline{\Omega})} + \sum_{|\beta|=k} [\partial^\beta u]_{\mathrm{C}^{0,\alpha}(\overline{\Omega})} < \infty \right\}.$$

(b) Prove that $\|\cdot\|_{\mathrm{C}^{k,\alpha}(\overline{\Omega})}$ is a norm on $\mathrm{C}^{k,\alpha}(\overline{\Omega})$, making $(\mathrm{C}^{k,\alpha}(\overline{\Omega}), \|\cdot\|_{\mathrm{C}^{k,\alpha}(\overline{\Omega})})$ a Banach space.

Problem 2: A characterisation of Banach spaces **4 + 6 = 10 marks**

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$ or $\mathbb{C}$. Prove that $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series converges.

Terminology: We say that a series $\sum_{j\in\mathbb{N}} x_j$ with $x_1, x_2, \ldots \in X$ *converges* provided there exists $x \in X$ such that

$$\lim_{n\to\infty} \|x - \sum_{j=1}^n x_j\| = 0,$$

and *converges absolutely* provided $\sum_{j\in\mathbb{N}} \|x_j\| < \infty$.

Problem 3: Bounded linear operators **2.5 + 2.5 + 2.5 + 2.5 = 10 marks**

Prove or disprove that the following maps are bounded linear operators $T_i\colon (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$; in case T_i is a bounded linear operator, compute its operator norm.

(a) $X, Y = \mathrm{C}([0,1])$ endowed with $\|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_{\sup}$, and $T_1(f)$ is defined by

$$(T_1 f)(x) := \int_0^x k(x,y) f(y)\,\mathrm{d}y, \qquad x \in [0,1],$$

where $k \in \mathrm{C}([0,1] \times [0,1])$.

(b) Let $1 \le p < \infty$ and let $X = Y = \ell^2(\mathbb{N})$ be endowed with the ℓ^2-norm. Now consider T_2 defined by

$$T_2((x_j)_{j\in\mathbb{N}}) := \Big(\frac{x_j}{j+1}\Big)_{j\in\mathbb{N}}.$$

(c) $X = \mathrm{C}([0,1])$ endowed with $\|\cdot\|_{\sup}$ and $Y = \mathbb{R}$, and $T_3(f) := f(0)$ for $f \in \mathrm{C}([0,1])$.

(d) $X = \mathrm{C}([0,1])$ endowed with $\|\cdot\|_{\mathrm{L}^1([0,1])}$ and $Y = \mathbb{R}$, and $T_4(f) := f(0)$ for $f \in \mathrm{C}([0,1])$.

Problem 4: Riesz for $\ell^p(\mathbb{N})$ **3 + 3 + 3 + 1 = 10 marks**

We aim to establish a characterisation of the dual space of $\ell^p(\mathbb{N})$, $1 \le p < \infty$:

Theorem. *Let $1 \le p < \infty$ and define*

$$p' := \begin{cases} +\infty & \textit{if } p = 1, \\ \frac{p}{p-1} & \textit{otherwise.} \end{cases}$$

Then for any $\Phi \in (\ell^p(\mathbb{N}))'$ there exists a unique element $y^\Phi = (y_j^\Phi) \in \ell^{p'}(\mathbb{N})$ such that

$$\Phi((x_j)) = \sum_{j=0}^{\infty} x_j y_j^\Phi \qquad \textit{for all } (x_j) \in \ell^p(\mathbb{N}).$$

Moreover, the map $T\colon \Phi \mapsto (y_j^\Phi)$ is a linear, bijective isometry $(\ell^p(\mathbb{N}))' \to \ell^{p'}(\mathbb{N})$.

To prove the theorem, proceed as follows:

(a) Let $1 < p < \infty$ first and let $e_i := (0, ..., 0, 1, 0, ...) \in \ell^p(\mathbb{N})$ (with non-trivial entry in the i-th position) be the i-th unit vector in $\ell^p(\mathbb{N})$. Given $\Phi \in (\ell^p(\mathbb{N}))'$, consider the sequence $(y_j^\Phi) := (\Phi(e_j))$. Establish that

$$\Phi((x_j)) = \sum_{j=0}^{\infty} x_j y_j^\Phi \qquad \text{for all } (x_j) \in \ell^p(\mathbb{N}).$$

(b) Given $k \in \mathbb{N}$, consider the specific choice $(x_j) \in \ell^p(\mathbb{N})$ with

$$x_j := \begin{cases} |y_j^\Phi|^{p'}/y_j^\Phi & \text{if } j \le k \text{ and } y_j^\Phi \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Φ to this particular choice (x_j), conclude that $\|(y_j^\Phi)\|_{\ell^{p'}(\mathbb{N})} \le \|\Phi\|_{(\ell^p(\mathbb{N}))'}$.

(c) Using Hölder's inequality, conclude that $T\colon (\ell^p(\mathbb{N}))' \ni \Phi \mapsto (y_j^\Phi) \in \ell^{p'}(\mathbb{N})$ is a linear isometry and establish the above theorem. In particular, give the details for $p = 1$, too.

(d) Which parts of the proof do not work for $p = \infty$? *In fact, the dual of $\ell^\infty(\mathbb{N})$ is not isometrically isomorphic to $\ell^1(\mathbb{N})$.*