Harmonic Analysis, Problem set 13

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Problem 1 (Dini criterion for pointwise convergence of Fourier series)**.** To fix notation, we denote Fourier series by $\hat{f}(n) := \int_0^1 e^{-2\pi i n x} f(x) \mathrm{d}x$.

(a) Assume that $g \in L^1[0,1]$. Show that $\hat{g}(n) \to 0$ as $|n| \to \infty$. (This is known as the *Riemann–Lebesgue lemma*). Hint: consider first differentiable functions g.

(b) Let $f : [0,1] \to \mathbb{C}$ be a function such that the function $g(x) := (f(x) + f(1-x))/(1 - e^{2\pi i x})$ is integrable on $[0,1]$. Show that

$$\sum_{n=-N}^{N} \hat{f}(n) \to 0 \text{ as } N \to \infty.$$

Hint: write $\hat{f}(n) + \hat{f}(-n)$ in terms of Fourier coefficients of g.

Remark. The Dini criterion applies to functions f that decay as x^α, $\alpha > 0$, near zero, but not to functions that decay as $1/|\log x|$.

Problem 2. The *Hilbert transform* is the operator

$$Hf(x) := \lim_{\epsilon \to 0} \int_{\epsilon < |t| < 1/\epsilon} f(x-t) \frac{\mathrm{d}t}{t}.$$

It is known that for $f \in L^2(\mathbb{R})$ this limit exists almost everywhere and that the operator H so defined is bounded on $L^2(\mathbb{R})$.

(a) Let a be a $(1,2)$-atom. Show that $\|Ha\|_{L^1} \lesssim 1$. Hint: if I is the interval associated to the atom a, consider Ha separately on $3I$ (the interval with the same center as I but thee times its length) and on $\mathbb{R} \setminus 3I$.

(b) Let f be a bounded function. Show that $\|Hf\|_{\mathrm{BMO}} \lesssim \|f\|_\infty$. Hint: to estimate the mean oscillation on an interval I split the function f into a part supported on $3I$ and a part supported on $\mathbb{R} \setminus 3I$.