Harmonic Analysis, Problem set 12

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Problem 1. Recall the definition of the (dyadic) BMO norm and the dyadic John–Nirenberg inequality from Homework 11.

(a) Show that for every $1 < q \leq \infty$ we have

$$\|f\|_{\mathrm{BMO}_d} \sim \sup_{I \in \mathcal{D}} \big(|I|^{-1} \int_I |f - f_I|^{q'}\big)^{1/q'}, \tag{1}$$

where the supremum is taken over all dyadic intervals.

(b) Show that the (non-dyadic) BMO space is the intersection of 3 dyadic BMO spaces. (Hint: recall Problem 2 from Homework 2.) Conclude that an analogue of (1) holds for the non-dyadic BMO space.

(c) Recall the definition of a $(1, q)$-atom from Homework 9. Show that if a is a $(1, q)$-atom and f a BMO function, then

$$|\int af| \lesssim \|f\|_{\mathrm{BMO}}.$$

(d) Let X be the linear subspace of L^q (algebraically) spanned by the $(1, q)$ atoms (this is just the space of L^q functions with bounded support and vanishing integral). Let $L : X \to \mathbb{C}$ be a linear functional such that $|La| \leq 1$ for each $(1, q)$-atom a. Show that L can be represented by a BMO function f with $\|f\|_{\mathrm{BMO}} \lesssim 1$ in the sense $L(a) = \int af$.

Problem 2. Recall that the *Fejér kernel* is given by

$$F_t(x) = \int_{-t}^{t} (1 - |\xi|/t) e^{2\pi i x \xi} \mathrm{d}\xi = \frac{\sin(\pi t x)^2}{\pi^2 t x^2}, \quad t > 0.$$

Show that for $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have $F_t * f \to f$ as $t \to \infty$ pointwise almost everywhere. Hint: consider first Schwartz functions f and use the Hardy–Littlewood maximal inequality.