Harmonic Analysis, Problem set 8

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Definition. We denote the embeddings that have been introduced it the lecture by

$$Af(x,t) := \int_{\mathbb{R}^d} |f(z)| \frac{t}{(t + \|x-z\|)^{d+1}} \mathrm{d}z,$$

$$Df(x,t) := \sup\Big\{ \Big| \int_{\mathbb{R}^d} f(z)\phi(z)\mathrm{d}z \Big| \ : \ \int \phi = 0,\ |\phi(z)| \leq \frac{t}{(t+\|x-z\|)^{d+1}},\ \|\nabla\phi(z)\| \leq \frac{t}{(t+\|x-z\|)^{d+2}} \Big\}.$$

Problem 1. (a) Show that if $\|x-y\| \lesssim t$, then $|Df(x,t)| \lesssim |Df(y,t)|$.

(b) For $f \in L^p(\mathbb{R}^d)$, $2 < p < \infty$, consider the square function

$$Sf(x) := \Big(\int_0^\infty Df(x,t)^2 \frac{\mathrm{d}t}{t} \Big)^{1/2}.$$

Show that $\|Sf\|_p \lesssim \|f\|_p$ by estimating $\int (Sf)^2 g$, $g \in L^{(p/2)'}(\mathbb{R}^d)$, using the outer Hölder inequality and the embeddings from the lectures.

Problem 2 (Calderón reproducing formula)**.** Let ϕ, ψ be functions on $\mathbb{R}^d$ with $\int \phi = \int \psi = 0$, $|\phi(x)|, |\psi(x)| \lesssim (1+\|x\|)^{-d-1}$, $|\nabla\phi(x)|, |\nabla\psi(x)| \lesssim (1+\|x\|)^{-d-2}$.

(a) Show that the operators

$$T_R f := \int_{\|x\| \leq R} \int_{1/R}^R (f * \psi_t)(y) \phi_t(\cdot - y) \frac{\mathrm{d}t}{t} \mathrm{d}y,$$

where $\phi_t(x) = t^{-d}\phi(x/t)$ and analogously for ψ, are bounded on $L^2(\mathbb{R}^d)$ uniformly in $R > 1$. Hint: use the L^2 almost orthogonality result from the lecture.

(b) Show that $T := \lim_{R\to\infty} T_R$ exists in the weak operator topology (this means that for every $f \in L^2(\mathbb{R}^d)$ there exists $Tf \in L^2(\mathbb{R}^d)$ such that for every $g \in L^2(\mathbb{R}^d)' = L^2(\mathbb{R}^d)$ we have $\lim_{R\to\infty} \int (T_R f - Tf) g = 0$).

(c) Show that T is a bounded linear operator on $L^2(\mathbb{R}^d)$ and $T = \lim_{R\to\infty} T_R$ in the strong operator topology (this means that for every $f \in L^2(\mathbb{R}^d)$ we have $\lim_{R\to\infty} \|T_R f - Tf\|_2 = 0$).

(d) Assuming that $\int_0^\infty \hat\phi(t\xi)\hat\psi(t\xi)\frac{\mathrm{d}t}{t} = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$ show that $Tf = f$.