Harmonic Analysis, Problem set 6

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The first problem establishes equivalence between classical and outer L^p spaces. In particular, this shows that the Marcinkiewicz interpolation theorem can be used with any combinations of outer and classical L^p spaces. We denote

$$\mu\{f > \lambda\} = \mu(\{x : f(x) > \lambda\}), \quad \mu(Sf > \lambda) = \inf_{\mathbf{E} \subset \mathcal{E} : \sup_{E' \in \mathcal{E}} S(f 1_{(\cup \mathbf{E})^c})(E') \leq \lambda} \sum_{E \in \mathbf{E}} \sigma(E).$$

(in the lecture notes the latter quantity is sometimes denoted by $\mu(\{Sf > \lambda\})$, but this is not intended and will hopefully be corrected soon).

Problem 1. Let $(X, \mathcal{E}, \sigma)$ be an outer measure space. It is known that the class $\mathcal{X}$ of Carathéodory measurable sets is a σ-algebra and the outer measure μ is σ-additive on $\mathcal{X}$. Assume that μ is also σ-finite (so that Fubini's theorem applies) and that $\mathcal{E} \subset \mathcal{X}$. Let $\mathcal{B}$ denote the set of $\mathcal{X}$-measurable functions from X to $\mathbb{R}_+$.

(a) For $f \in \mathcal{B}$ show that

$$\int_X f(x)^p \mathrm{d}\nu(x) = p \int_0^\infty \lambda^{p-1} \mu\{f > \lambda\} \mathrm{d}\lambda.$$

Hint: write $f(x)^p = \int_0^{f(x)} p\lambda^{p-1} \mathrm{d}\lambda$.

(b) Consider the size

$$S_\infty f(E) := \inf_{A : \mu(A) = 0} \sup_{E \setminus A} f.$$

Show that $\mu\{f > \lambda\} = \mu(S_\infty f > \lambda)$ for all $f \in \mathcal{B}$ and $0 < \lambda < \infty$.

(c) Suppose additionally that $\sigma(E) < \infty$ for all $E \in \mathcal{E}$ and consider the size

$$S_1 f(E) := \sigma(E)^{-1} \int_E f \mathrm{d}\mu.$$

Show that $\mu\{f > \lambda\} = \mu(S_1 f > \lambda)$ for all $f \in \mathcal{B}$ and $0 < \lambda < \infty$.

Problem 2 (Hilbert transform)**.** It has been sketched in the lecture that

$$\int_{-\infty}^{+\infty} (f + iHf)^k = 0$$

for all smooth compactly supported real-valued functions f and integers $k \geq 2$, where H denotes the Hilbert transform (in particular Hf is also real-valued).

(a) Let $k \geq 2$ be an even integer. Show that $\|Hf\|_{L^k} \leq C_k \|f\|_{L^k}$ for some C_k that does not depend on k. Hint: expand the real part of the above identity and use Hölder's inequality to separate L^k norms of f and Hf.

(b) Let $2 \leq p < \infty$. Show that $\|Hf\|_{L^p} \leq C_k \|f\|_{L^p}$.

(c) Let $\tilde{f}(x) = f(-x)$. Show that for all smooth compactly supported functions $f, g \in C_0^\infty(\mathbb{R})$ we have $\int (Hf) g = \int (H\tilde{g}) \tilde{f}$. Use duality between L^p and $L^{p'}$, $1/p + 1/p' + 1$, and the fact that C_0^∞ is dense in $L^{p'}$ to show that $\|Hf\|_{L^p} \leq C_k \|f\|_{L^p}$, $1 < p < 2$ (here again $f \in C_0^\infty(\mathbb{R})$).