Harmonic Analysis, Problem set 4

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Problem 1 (Hölder's inequality via tensor power trick)**.** Let $1 < p_1, \dots, p_k < \infty$ with $\sum_i 1/p_i = 1$ and $f_i \in \ell^{p_i}$ be sequences with values in $[0, \infty)$.

(a) Show that $a_1 \cdots a_k \leq \max_i a_i^{p_i} \leq \sum_i a_i^{p_i}$ for any numbers $a_1, \dots, a_k \geq 0$.

(b) Using part a show that

$$\sum_x \prod_{i=1}^k f_i(x) \leq k \prod_i \|f_i\|_{p_i}.$$

Hint: consider first the case $\|f_i\|_{p_i} = 1$ for all i.

(c) Using part b show that

$$\sum_x \prod_{i=1}^k f_i(x) \leq \prod_i \|f_i\|_{p_i}.$$

Hint: apply part b to the seqences $f_i^{\otimes m}(x_1, \dots, x_m) = f_i(x_1) \cdots f_i(x_m)$.

Problem 2 (Carathéodory measurability)**.** Let (X, σ) be an outer measure space. A set $E \subset X$ is called *Carathéodory measurable* if for every $A \subset X$ we have $\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$ (here μ denotes the outer measure generated by σ).

(a) Let E_1, E_2 be Carathéodory measurable sets. Show that $E_1 \cup E_2$ is also Carathéodory measurable.

(b) If in addition $E_1 \cap E_2 = \emptyset$, show that $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

(c) Consider the outer measure on $[0, \infty)$ generated by $\sigma(I) = |I|$ for dyadic intervals $|I|$. Show that dyadic intervals are Carathéodory measurable.

(d) Let $\mathcal{D}$ be the set of dyadic intervals in $[0, \infty)$ with the outer measure generated by $\sigma(T_J) = |J|$, $T_J = \{I \subset J\}$. Show that the only Carathéodory measurable sets are $\emptyset, \mathcal{D}$.