Harmonic Analysis, Problem set 2

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Winter term 2016/17

Due on Thursday, 2016-11-03

The set of harmonic functions on the upper half-plane from the lectures 1 and 2 is denoted by $\mathcal{M}$ here, and the set of their primitives by $\mathcal{PM}$.

Problem 1. Let $F \in \mathcal{PM}$. Show that $\lim_{t\to 0} F(x,t) = (f_l(x) + f_r(x))/2$ for every $x \in \mathbb{R}$. You may use the bijections between $\mathcal{PM}, \mathcal{M}, \mathcal{PB}', \mathcal{B}'$ from the lectures.

Problem 2. Define *adjacent systems of dyadic intervals* by

$$\mathcal{D}^\alpha = \{2^{-k}([0,1) + m + (-1)^k \alpha/3), m, k \in \mathbb{Z}\},$$

where $\alpha = 0, 1, 2$. Not that $\mathcal{D}^0$ is the usual system of dyadic intervals.

(a) Show that each $\mathcal{D}^\alpha$ is nested in the sense that for $I, J \in \mathcal{D}^\alpha$ we have $I \cap J \in \{I, J, \emptyset\}$.

(b) Show that for every interval $I \subset \mathbb{R}$ there exists $\alpha \in \{0,1,2\}$ and $J \in \mathcal{D}^\alpha$ such that $I \subset J$ and $|J| \le 4|I|$.

(c) Let $f \in \mathcal{B}$. The (continuous) Hardy–Littlewood maximal function is defined by

$$Mf(x) := \sup_{x \in I} |I|^{-1} \int_I f,$$

where the supremum is taken over all intervals containing x. Show that $Mf < \infty$ almost everywhere.

(d) Let $F \in \mathcal{M}$ and $MF(x) := \sup_{t>0} F(x,t)$. Show that $MF < \infty$ almost everywhere. Hint: use part c.